Fundamentals of discrete LTI Systems

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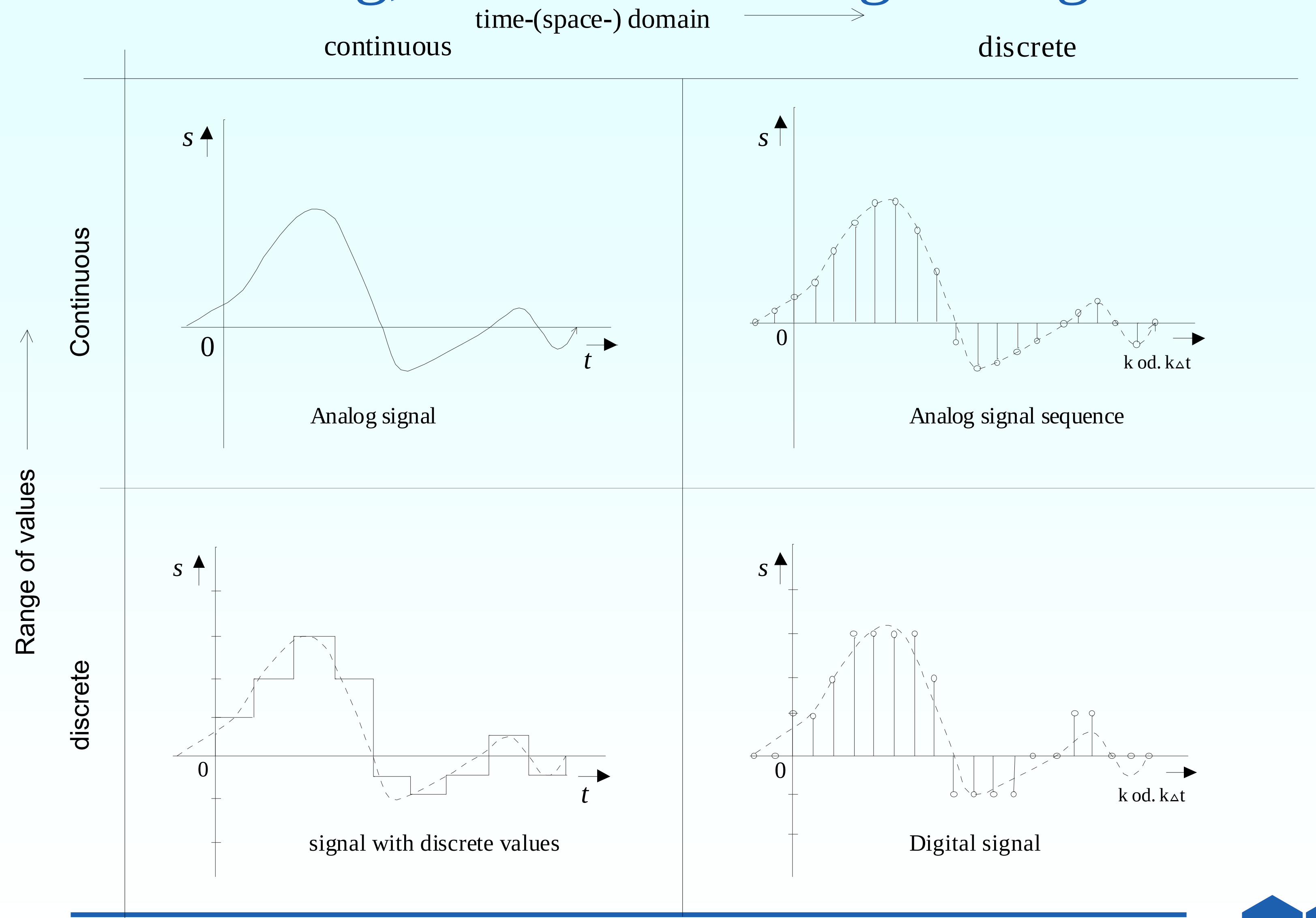
Contents

1 Introduction
Signals and Systems

- **Signals** represent a physical quantity changing over time
- Signal usually contain some information relevant for the observer of the signal

2.1 Analog, Discrete and Digital Signals



2.2 Deterministic Signals in the Time Domain
2.2.1 The Exponential Signal

$$s(t) = e^{j\omega t} = \cos\omega t + j\sin\omega t$$

For voltages it holds:

$$u(t) = \hat{u}\cdot\cos(\omega t + \varphi_u) = \mathrm{Re}\left\{\hat{u}\cdot e^{j(\omega t+\varphi_u)}\right\} = \mathrm{Re}\left\{\underline{u}\cdot e^{j\omega t}\right\} \quad \text{where} \quad \underline{u} = \hat{u}\cdot e^{j\varphi_u}$$

For increasing/decreasing signals:

$$e^{(\sigma+j\omega)t} = e^{\sigma t}\cdot e^{j\omega t} = e^{pt}$$

2.2.2 The Exponential Sequence

$$\{s(k)\} = \{z^k\} \quad \text{for} \quad k \in Z$$

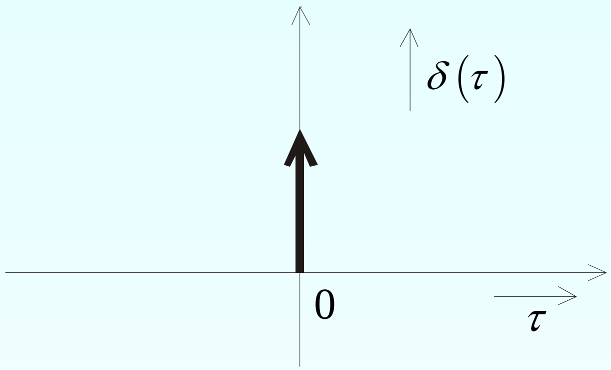
$$\{s(k)\} = \{e^{j\omega T \cdot k}\} = \{\cos(\omega Tk) + j \cdot \sin(\omega Tk)\}$$

$$\{s(k)\} = \{e^{pTk}\} = \{e^{(\sigma + j\omega)k \cdot T}\} = \{e^{\sigma kT} \cdot e^{j\omega kT}\}$$
$$= \{e^{\sigma kT} \cdot \cos(\omega Tk) + j \cdot e^{\sigma k \cdot T} \cdot \sin(\omega Tk)\}$$

2.2.3 The Dirac Function

Approximation:

$$\delta(\tau) = \lim_{T \to 0} \frac{1}{T} rect\left(\frac{t}{T}\right)$$



2.2.3 The Dirac Function

Definition:

$$\Phi(t_0) = \int_{-\infty}^{+\infty} \delta(t - t_0) \cdot \Phi(t) dt \text{ with } \Phi(t) \text{ as an arbitrary signal}$$

Properties:

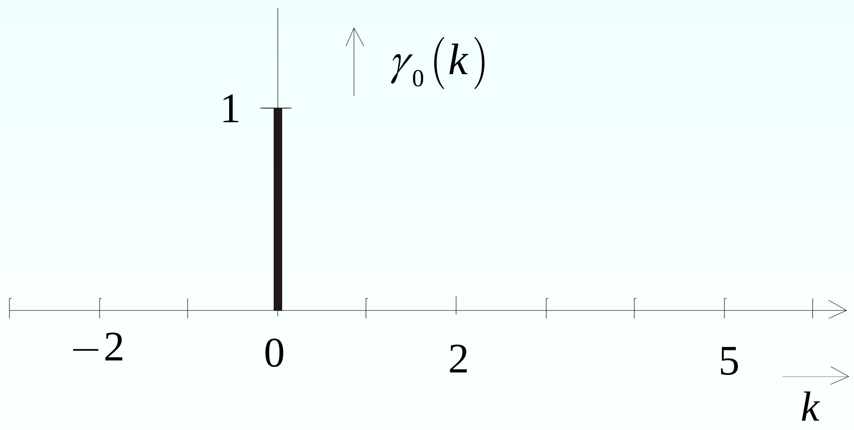
$$\delta(at) = \frac{1}{|a|} \cdot \delta(t)$$

If $a = -1$ then: $\delta(-t) = \delta(t)$

$$s(t) = \int_{-\infty}^{+\infty} \delta(\tau - t) \cdot s(\tau) d\tau$$

2.2.4 The Unit Impulse

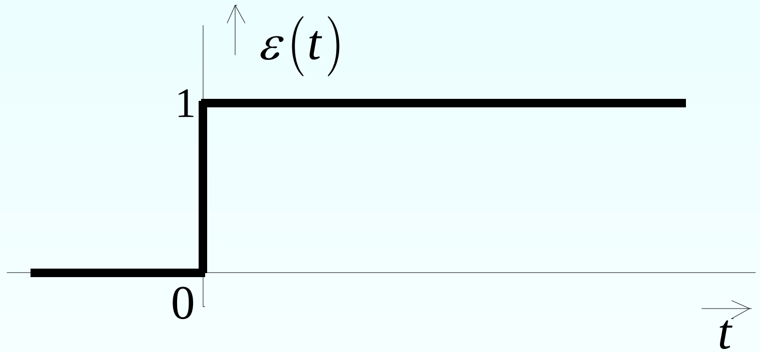
$$\{s(k)\} = \gamma_0(k) = \begin{cases} 1 & \text{for} \quad k = 0 \\ 0 & \text{for} \quad k \neq 0 \end{cases}$$



2.2.5 The Step Function

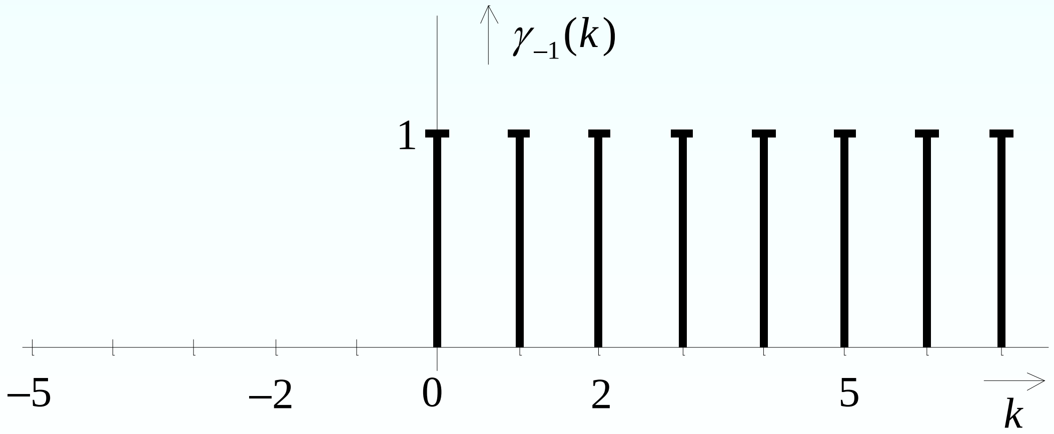
$$\varepsilon(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad t \geq 0 \end{cases}$$

$$\varepsilon(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$



2.2.6 The Step Sequence

$$\gamma_{-1}(k) = \begin{cases} 0 & \text{for} \quad k < 0 \\ 1 & \text{for} \quad k \geq 0 \end{cases}$$



2.2.7 Periodic Signals

General property:

$$s(t) = s(t + nT) \quad \text{where} \quad n = -\infty, \ldots, -1, +1, \ldots, +\infty$$

Transform of impulses into a periodic signal:

$$s_2(t) = \sum_{n=-\infty}^{+\infty} s_1(t - nT_0)$$

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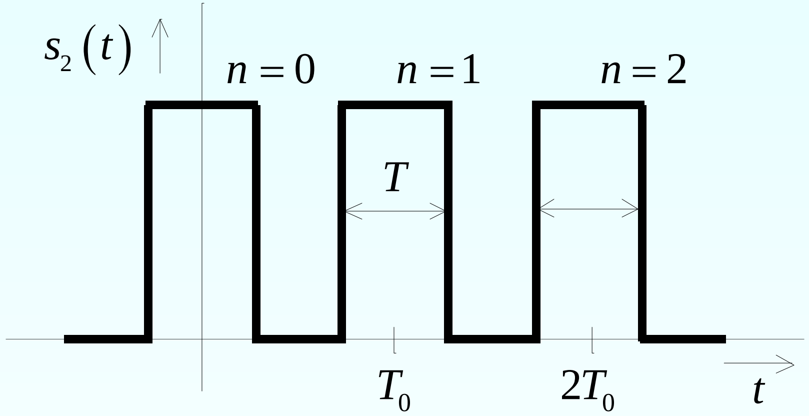
2.2.7 Periodic Signals

Example:

$$s_1(t) = rect\left(\frac{t}{T}\right)$$

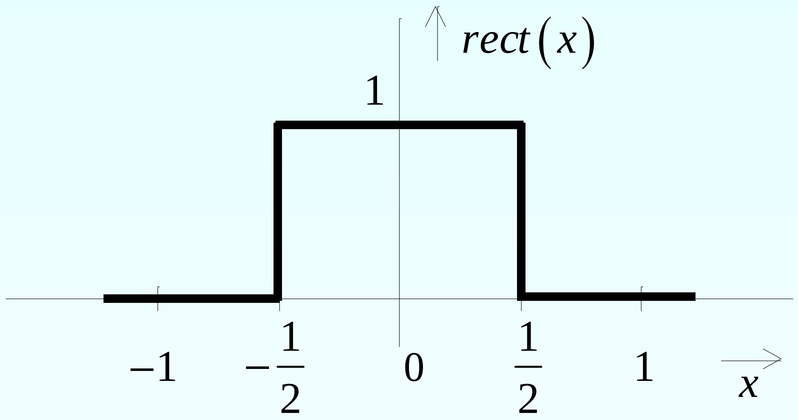
$$\Rightarrow s_2(t) = \sum_{n=-\infty}^{+\infty} rect\left(\frac{t-nT_0}{T}\right)$$

$$= \sum_{n=-\infty}^{+\infty} rect\left(\frac{t}{T} - n\frac{T_0}{T}\right)$$



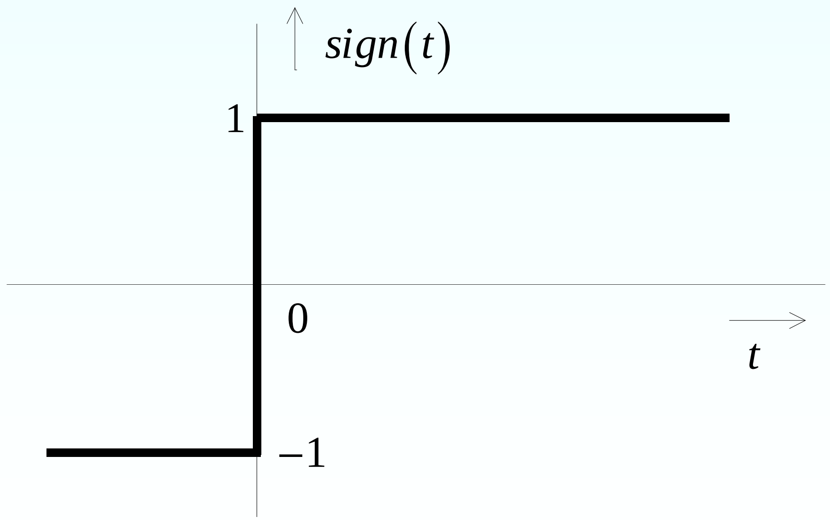
2.2.8 Impulse Type Signals

$$rect(x) = \begin{cases} 1 & \text{for} \quad |x| \leq \frac{1}{2} \\ 0 & \text{for} \quad |x| > \frac{1}{2} \end{cases}$$



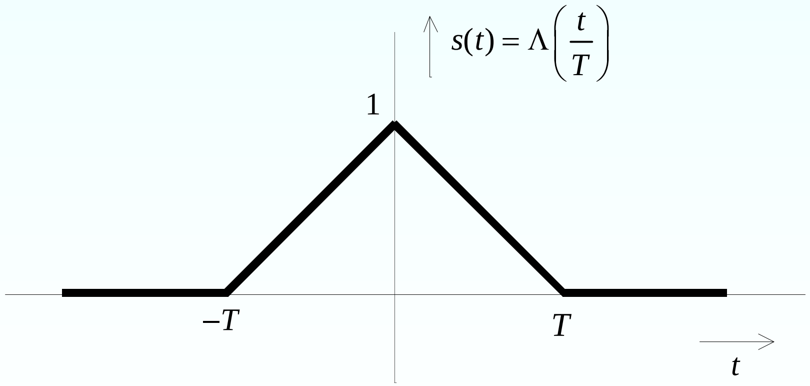
2.2.8 Impulse Type Signals

$$s(t) = sign(t) = \begin{cases} 1 & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t = 0 \\ -1 & \text{for} \quad t < 0 \end{cases}$$



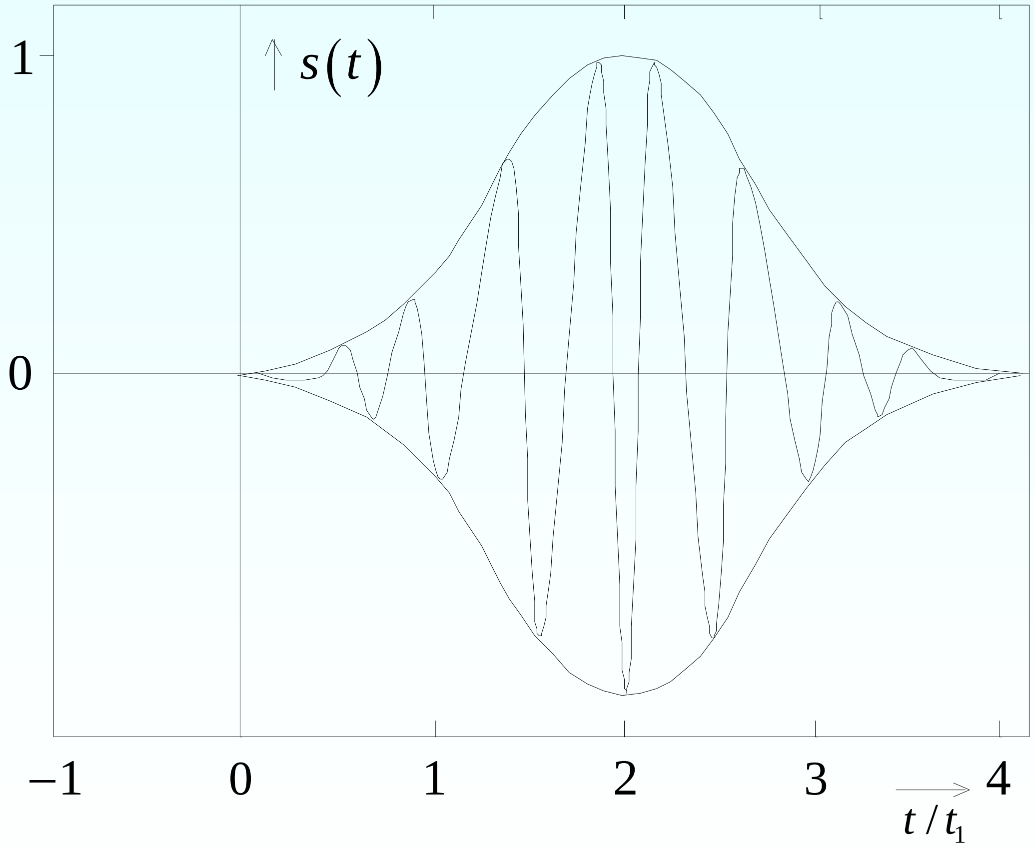
2.2.8 Impulse Type Signals

$$s(t) = \Lambda\left(\frac{t}{T}\right) = \begin{cases} 1 - \left|\frac{t}{T}\right| & \text{for} \quad |t| \le T \\ 0 & \text{otherwise} \end{cases}$$



2.2.8 Impulse Type Signals

$$s(t) = e^{-\left(\frac{t-t_1}{t_0}\right)^2} \cdot \cos(\omega_0 (t - t_1))$$



The figure shows $s(t)$ for $t_1 = t_0$

2.2.9 Adjustment of Time and Frequency Functions

Case1: Change of amplitude, compression & expansion with regard to time axis

$$s_2(t) = a \cdot s_1\left(\frac{t}{b}\right)$$

Example:

$$s_2(t) = u_0 \cdot rect\left(\frac{t}{2T}\right)$$

Case2: Shift (Time delay or advance)

$$s_2(t) = s_1(t - T_v)$$

2.2.9 Adjustment of Time and Frequency Functions

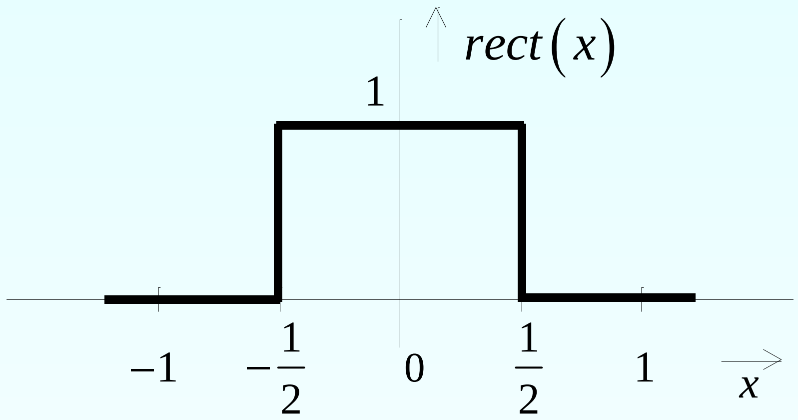
Example for expansion:

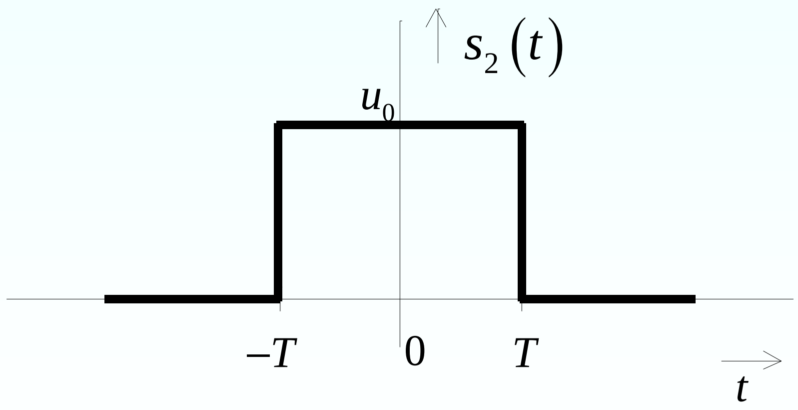
$$s_1(t) = rect\left(\frac{t}{T}\right)$$

$$s_2(t) = u_0 \cdot s_1\left(\frac{t}{2}\right)$$

$$= u_0 \cdot rect\left(\frac{t/2}{T}\right)$$

$$= u_0 \cdot rect\left(\frac{t}{2T}\right)$$





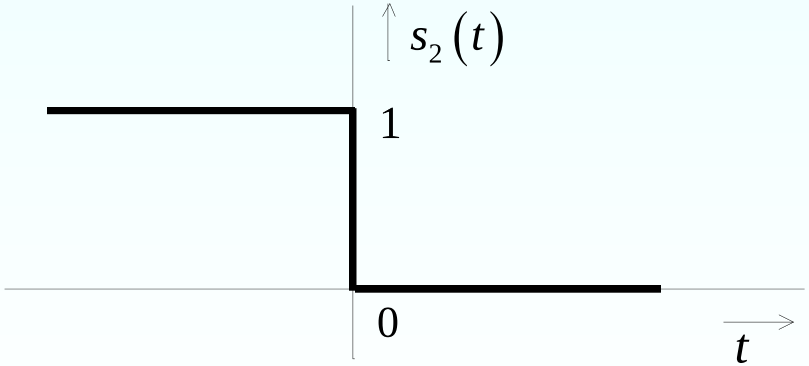
2.2.9 Adjustment of Time and Frequency Functions

Case3: Mirroring *(b = -1)*

$$s_2(t) = s_1(-t)$$

Example:

$$s_1(t) = \varepsilon(t)$$
$$s_2(t) = s_1(-t) = \varepsilon(-t)$$

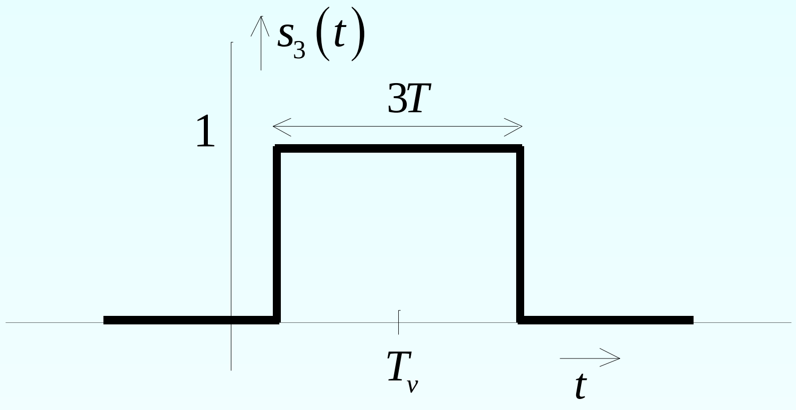


2.2.9 Adjustment of Time and Frequency Functions

Combination of expansion & shift:

$$s_1(t) = rect\left(\frac{t}{T}\right) \qquad s_2(t) = rect\left(\frac{t}{3T}\right)$$

$$s_3(t) = s_2(t - T_\nu) = rect\left(\frac{t - T_\nu}{3T}\right)$$



Combination of shift & expansion:

$$s_2(t) = s_1(t - T_\nu)$$

$$s_3(t) = as_2\left(\frac{t}{b}\right)$$ Replace in $s_2(t)$ argument t only by $\frac{t}{b}$

$$= as_1\left(\frac{t}{b} - T_\nu\right)$$ and do the same in $s_1(t)$

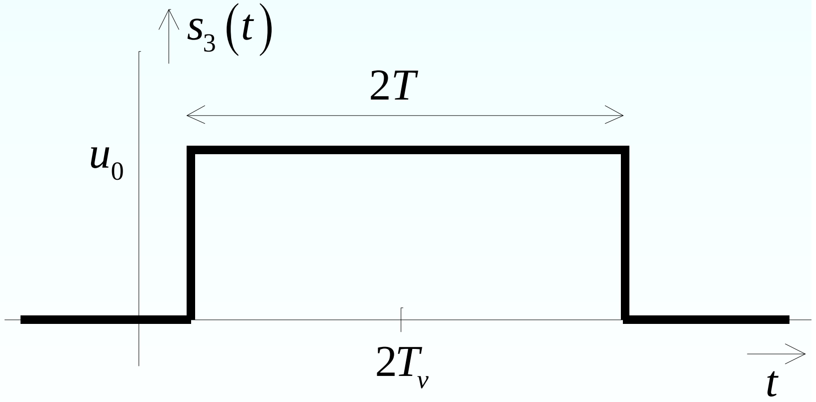
2.2.9 Adjustment of Time and Frequency Functions

Example:

$$s_1(t) = rect\left(\frac{t}{T}\right) \; ; \quad a = u_0 \; ; \quad b = 2$$

$$s_2(t) = rect\left(\frac{t - T_v}{T}\right) = rect\left(\frac{t}{T} - \frac{T_v}{T}\right)$$

$$s_3(t) = a\,rect\left(\frac{t}{bT} - \frac{T_v}{T}\right) = u_0 rect\left(\frac{t}{2T} - \frac{T_v}{T}\right)$$



2.2.9 Adjustment of Time and Frequency Functions

Mirroring & shifting:

$$s_2(t) = s_1(-t)$$

$$s_3(t) = s_2(t - T_\nu) = s_1(-(t - T_\nu)) = s_1(T_\nu - t)$$

New sequence: Shifting & mirroring:

$$s_4(t) = s_1(t - T_\nu)$$

$$s_5(t) = s_4(-t) = s_1(-t - T_\nu) \neq s_3(t)$$

2.2.9 Adjustment of Time and Frequency Functions

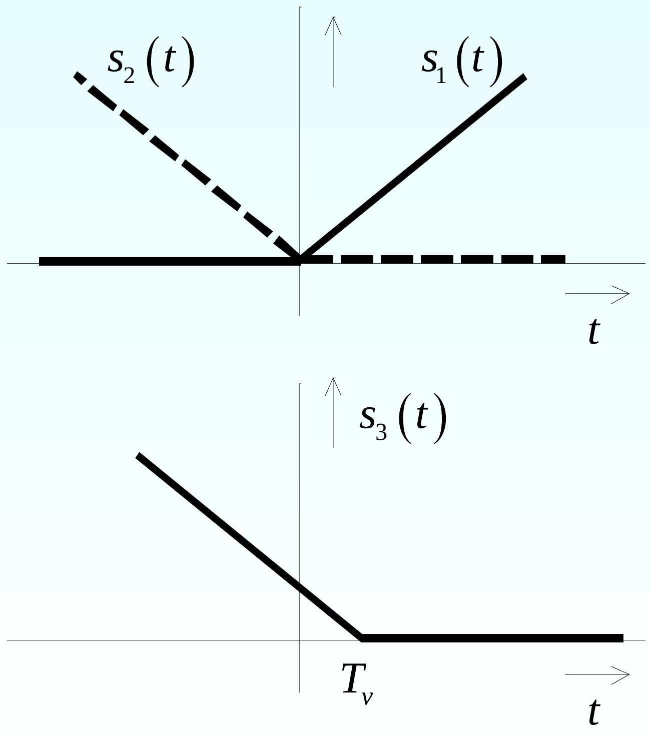
Example with a ramp function *r(t)*:

$$r\left(\frac{t}{T}\right) = \frac{t}{T}\cdot\varepsilon(t)$$

$$s_1(t) = r\left(\frac{t}{T}\right)$$

$$s_2(t) = s_1(-t) = r\left(\frac{-t}{T}\right)$$

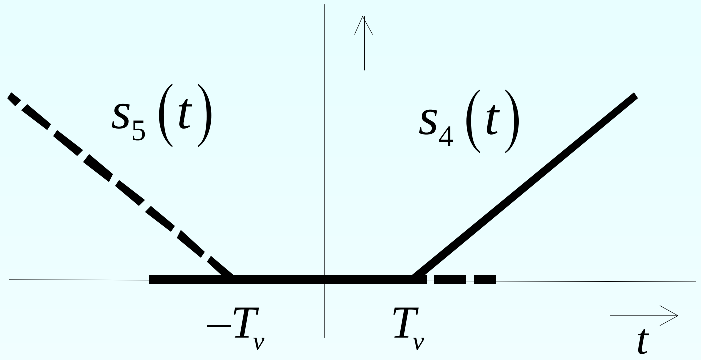
$$s_3(t) = s_2(t - T_v) = r\left(\frac{T_v - t}{T}\right)$$



2.2.9 Adjustment of Time and Frequency Functions

$$s_4(t) = s_1(t - T_v) = r\left(\frac{t - T_v}{T}\right)$$

$$s_5(t) = s_4(-t) = r\left(\frac{-t - T_v}{T}\right)$$



There are 4 cases: $\pm t \pm T_v$

2.2.9 Adjustment of Time and Frequency Functions

All methods described above can be extended to frequency functions.

Example:

$$f_1(\omega) = rect\left(\frac{\omega - \omega_0}{\omega_1}\right)$$

In general one function can be used as the argument of another function.

$$f_1(x) = f_2(y) \quad \text{where} \quad y = f_3(x)$$

$$\Rightarrow f_1(x) = f_2(f_3(x))$$

2.2.10 Energy and Power of Signals

Electrical Energy: $E_{el} = \frac{1}{R}\int_{-\infty}^{\infty} u^2(t)dt$

Signal Energy: $E = \int_{-\infty}^{\infty} s^2(t)dt$

Condition for energy signals: $0 < E < \infty$

Signal Power: $P = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{+T} s^2(t)dt$

Condition for power signals: $0 < P < \infty$ or $E \to \infty$

2.2.10 Energy and Power of Signals

Conditions for discrete signals:

$$E = \lim_{k\to\infty} \sum_{k=-K}^{k=+K} s^2(k) < \infty$$

$$P = \lim_{k\to\infty} \frac{1}{2K} \sum_{k=-K}^{k=+K} s^2(k) < \infty$$

2.3.1 Periodic Signals and the Fourier Series

Properties of periodic signals:

$$s(t) = s(t + kT) \quad k \text{ integer}, \ -\infty < k < \infty, \ \mathrm{T} = \text{Period}$$

Fourier series onset with 3 essential components:

$$s(t) = s_0 + \hat{s}_1 \cos(2\pi f_0 t + \varphi_1) + \hat{s}_2 \cos(2 \cdot 2\pi f_0 t + \varphi_2) + \ldots \quad \text{where } \omega_0 = 2\pi f_0$$

$$= s_0 + \sum_{n=1}^{\infty} \hat{s}_n \cos(n\omega_0 t + \varphi_n) \qquad f_n = n\frac{1}{T_0}\ , n > 1 \quad f_0 = \frac{1}{T_0}$$

and due to $\cos(x + y) = \cos x \cos y - \sin x \sin y$:

$$= s_0 + \sum_{n=1}^{\infty} \left[\hat{s}_n \cos(n\omega_0 t) \cos\varphi_n - \hat{s}_n \sin(n\omega_0 t) \sin\varphi_n \right]$$

2.3.1 Periodic Signals and the Fourier Series

Setting $a_n = \hat{s}_n \cos\varphi_n \; , \; -b_n = \hat{s}_n \sin\varphi_n \; , \; s_0 = \frac{a_0}{2}$, one obtains:

$$s(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos(n\omega_0 t) + b_n \sin(n\omega_0 t) \right] \quad \text{Trigonometric form}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[\sqrt{a_n^{\;2} + b_n^{\;2}} \cos(n\omega_0 t + \varphi_n) \right] \quad \text{Polar form}$$

where $\varphi_n = -\arctan \frac{b_n}{a_n}$ and $\hat{s}_n = \sqrt{a_n^{\;2} + b_n^{\;2}}$

Please observe limited range of values for the *arctan* function!

2.3.1 Periodic Signals and the Fourier Series

Determination of Fourier coefficients:

$$s_0 = \frac{a_0}{2} = \frac{1}{T_0} \int_{t_0}^{t_0+T} s(t)dt \quad \text{(this is the time averaged value of s(t))}$$

$$a_n = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} s(t)\cos(n\omega_0 t)dt$$

$$b_n = \frac{2}{T_0} \int_{t_0}^{t_0+T_0} s(t)\sin(n\omega_0 t)dt$$

2.3.1 Periodic Signals and the Fourier Series
The exponential form

Definition:

$$c_n = \frac{a_n - jb_n}{2} \quad \text{for } n \geq 0 \quad \text{with } b_0 = 0$$

$$c_{-n} = c_n^{\,*} = \frac{a_n + jb_n}{2}$$

Relation to trigonometric coefficients:

$$a_n = 2\,\mathrm{Re}\{c_n\} \quad \text{The amplitude of the } \cos(n\omega_0 t) \text{ for } n \geq 0$$

$$b_n = -2\,\mathrm{Im}\{c_n\} \quad \text{The amplitude of the } \sin(n\omega_0 t) \text{ for } n \geq 0$$

$$= +2\,\mathrm{Im}\{c_n\} \quad \text{for } n < 0$$

2.3.1 Periodic Signals and the Fourier Series
The exponential form

Periodic signals thus are represented by:

$$s(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t} = c_0 + \sum_{n=1}^{+\infty} 2\left|c_n\right| \cos(n\omega_0 t + \varphi_n)$$

$$\text{where } \left|c_n\right| = \frac{1}{2}\sqrt{a_n{}^2 + b_n{}^2} \quad \text{and} \quad \varphi_n = -\arctan\frac{b_n}{a_n} = \angle c_n$$

Please observe:

Complex coefficients represent pointers which are rotated by exponential function (clockwise rotating for positive n)

2.3.1 Periodic Signals and the Fourier Series
The exponential form

$$c_n = \frac{1}{2} a_n - j \frac{1}{2} b_n \quad \text{for} \quad n \geq 0$$

$$= \frac{1}{2} \frac{2}{T_0} \int_{t_0}^{t_0+T_0} s(t) \cos(n\omega_0 t) dt - \frac{j}{2} \frac{2}{T_0} \int_{t_0}^{t_0+T_0} s(t) \sin(n\omega_0 t) dt$$

$$= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} s(t) \left(\cos(n\omega_0 t) - j \sin(n\omega_0 t) dt \right) dt$$

$$= \frac{1}{T_0} \int_{t_0}^{t_0+T_0} s(t) e^{-jn\omega_0 t} dt$$

Additional Fourier series properties:

Linearity	$k \cdot s(t)$ leads to $\{k \cdot c_n\}$
Time delay	$s(t - t_v)$ leads to $\{c_n \cdot e^{jn\omega_0 t_v}\}$
Reversal	$s(-t)$ leads to $\{c_n^*\}$

2.3.1 Periodic Signals and the Fourier Series
The convergence of the exponential form

1) The Fourier series converges in the mean square average:

$$\lim_{\nu \to \infty} \int_0^T \left[s(t) - \sum_{n=-\nu}^{+\nu} c_n e^{jn\omega_0 t} \right]^2 dt = 0$$

2) At finite numbers of jumps in the period *T* the Fourier series approaches the jump, it is equal to *s(t)* before and after the jump and crosses the jump at its center

Interpretation of coefficients:

Complex coefficients as a pair represent one signal component with a certain frequency of *n* times the fundamental frequency ω_0.

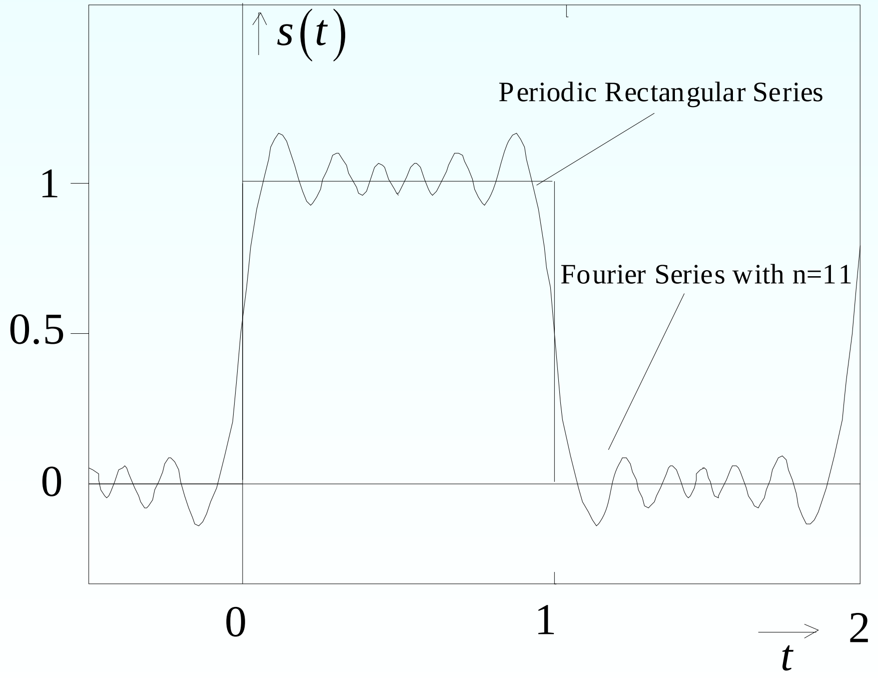
Magnitude and phase of the complex coefficient correspond to amplitude and phase (delay/advance) of that signal component.

2.3.1 Periodic Signals and the Fourier Series
The Gibb‘s Phenomenon

At jumps the Fouries series introduces overshoots into the signal

These overshoots can be observed for low-pass signals e.g.

This effect is always given, even for a perfect Fourier series with infinetely much components!



2.3.1 Periodic Signals and the Fourier Series
The distortion factor

Distortion factor is a measure for amount of higher harmonics in the signal

$$K = \frac{\text{rms-value of the signal harmonics}}{\text{rms-value of all harmonics}}$$

$$= \frac{\sqrt{s_{2,eff}^2 + s_{3,eff}^2 + s_{4,eff}^2 + \ldots}}{\sqrt{s_{1,eff}^2 + s_{2,eff}^2 + s_{3,eff}^2 + s_{4,eff}^2 + \ldots}}$$

Note:

DC component is no harmonic

where $s_{n,eff}^2 = |c_n|^2 + |c_{-n}|^2 = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} s_n(t)^2 \, dt = P_n$ yields in

$$K = \frac{\sqrt{\sum_{n=2}^{\infty} \left(|c_n|^2 + |c_{-n}|^2 \right)}}{\sqrt{\sum_{n=1}^{\infty} \left(|c_n|^2 + |c_{-n}|^2 \right)}}$$

2.3.2 The Fourier Transform

2.3.2 The Fourier Transform - Definition

Absolutely integrable signals are denoted by: $\int_{-\infty}^{+\infty} |s(t)|\, dt < \infty$

For such signals fufilling some additional conditions it holds:

$$s(t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} S(\omega) \cdot e^{j\omega t} d\omega \quad \circ\!\!\overset{\mathcal{F}}{-\!\!-}\!\!\bullet \quad S(\omega) = \int_{-\infty}^{+\infty} s(t) \cdot e^{-j\omega t} dt$$

Reasoning:

A periodic signal can be turned into a non-periodic one by extending the period to infinite. For periodic signals holds:

$$s(t) = \sum_{n=-\infty}^{+\infty} c_n e^{jn\omega_0 t}$$

In a narrow interval $\Delta\omega$ m summation terms (or lines) according to Fourier series exist: $m = \frac{\Delta\omega}{\omega_0} = \frac{T_0}{2\pi} \Delta\omega$

2.3.2 The Fourier Transform - Definition

The *m* (nearly not differently large) lines represent one part of the signal:

$$\Delta s_i(t) \approx m \cdot c_n e^{jn\omega_0 t} = \frac{T_0}{2\pi} \Delta\omega \cdot c_n e^{jn\omega_0 t}$$

The whole signal then is given by summing up all signal parts:

$$s(t) = \sum_i \Delta s_i(t)$$

In the limit (period growing over all limits) the summation turns to the integral:

$$s(t) = \int ds \qquad ds = c_n e^{jn\omega_0 t} \cdot \frac{T_0}{2\pi} d\omega$$

Now some rewriting is introduced:

$$S_F(\omega) = T_0 \cdot c_n \qquad \omega = n\omega_0$$

Finally the inverse Fourier Transform results:

$$s(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_F(\omega) e^{j\omega t} d\omega = F^{-1}\{S_F(\omega)\}$$

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2.3.2 The Fourier Transform - Interpretation

The Fourier transform is a measure of amplitudes and phases of the harmonics when evaluated at a specific frequency:

$$\left|S_F(\omega)\right| \text{ and } \sphericalangle S_F(\omega)$$

The Fourier Transform is determined by:

$$S_F(\omega) = \int_{-\infty}^{+\infty} s(t) e^{-j\omega t} dt = F\left\{s(t)\right\}$$

This function is also called spectrum or amplitude density spectrum!

The sub *F* only is used if it is not clear which transform is meant.

Special properties:

All frequencies in a certain interval are present

This transform relates the time-domain and the frequency domain

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2.3.2 The Fourier Transform – Convergence properties

Convergence properties have to be considered in special cases such as:

- Signals with jumps
- Signals with curves of infinite lenght (no limited variation)
- Signals including Dirac impulses

For absolutely integrable signals with limited variation in suitable intervals it holds:

$$\lim_{\alpha \to \infty} \frac{1}{2\pi} \int_{-\alpha}^{\alpha} S(\omega) e^{j\omega t} d\omega = \frac{s(t+0)+s(t-0)}{2}$$

(Reasonable point-for-point convergence)

Limited variation in a finite interval (a,b), which is partitioned means:

$$\sum_{\nu=0}^{n} \left| s(t_\nu) - s(t_{\nu-1}) \right| < \infty \qquad a = t_0 < t_1 < \ldots < t_n = b$$

Example: Dirac impulse has no limited variation.

2.3.2 The Fourier Transform - Convergence properties

Fourier transform and inverse transform show up similar relations.

Thus the convergence properties described before can be applied to the frequency domain:

$$\lim_{\alpha\to\infty}\int_{-\alpha}^{\alpha} s(t)\cdot e^{-j\omega t}dt=\frac{S(\omega+0)+S(\omega-0)}{2}$$

Additional remarks:

For voltage signals with the dimension of [V] the Fourier transform exhibits the dimension of [V · s] = [V / Hz] !

Verify the dimension of the expressions in: $S(\omega)=\int_{-\infty}^{+\infty} s(t)\cdot e^{-j\omega t}dt$

2.3.2 The Fourier Transform – Another interpretation of the transform values

A signal component gained by means of an ideal band pass is considered:

$$\Delta s(t) = \frac{1}{2\pi}\int_{-\omega_0-\Delta\omega/2}^{-\omega_0+\Delta\omega/2} S(\omega)e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{\omega_0-\Delta\omega/2}^{\omega_0+\Delta\omega/2} S(\omega)e^{j\omega t}d\omega$$

$$\approx \frac{\Delta\omega}{2\pi}\left(\underbrace{S(-\omega_0)}_{S^*(\omega_0)}e^{-j\omega_0 t} + S(\omega_0)e^{j\omega_0 t}\right)$$

Smooth form of the spectrum at ω_0 is assumed!

$$= \frac{\Delta\omega}{2\pi}\left(2\,\mathrm{Re}\left\{S(\omega_0)e^{j\omega_0 t}\right\}\right) = \frac{\Delta\omega}{2\pi}\left(2\,\mathrm{Re}\left\{S(\omega_0)(\cos\omega_0 t + j\sin\omega_0 t)\right\}\right)$$

$$= \frac{\Delta\omega}{2\pi}\left(2\,\mathrm{Re}\left\{S(\omega_0)\right\}\cos\omega_0 t - 2\,\mathrm{Im}\left\{S(\omega_0)\right\}\sin\omega_0 t\right)$$

$$= \frac{\Delta\omega}{\pi}\left|S(\omega_0)\right|\cos(\omega_0 t + \sphericalangle S(\omega_0))$$

The transform is a measure of amplitude & phase of the signal component!

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2.3.2 The Fourier Transform – Important properties

For <u>complex signals</u> it holds:

$$s(t) = s_1(t) + js_2(t) \quad \circ\!\!\overset{\mathcal{F}}{-\!\!-\!\!-}\!\!\bullet \quad S(\omega) = R(\omega) + jX(\omega)$$

$$S(\omega) = \int_{-\infty}^{+\infty} s(t) e^{-j\omega t} dt = \int_{-\infty}^{+\infty} \left(s_1(t) + js_2(t)\right)\left(\cos\omega t - j\sin\omega t\right) dt$$

Thus it results:

$$R(\omega) = \int_{-\infty}^{+\infty} \left(s_1(t)\cos\omega t + s_2(t)\sin\omega t\right) dt$$

$$X(\omega) = -\int_{-\infty}^{+\infty} \left(s_1(t)\sin\omega t - s_2(t)\cos\omega t\right) dt$$

2.3.2 The Fourier Transform – Important properties

For <u>real signals</u> some further simplifications can be used:

$$R(\omega) = \int_{-\infty}^{+\infty} s(t)\cos\omega t dt \quad \text{due to } s_2(t) = 0 \text{ and } s(t) = s_1(t)$$

$$X(\omega) = -\int_{-\infty}^{+\infty} s(t)\sin\omega t dt$$

These integrals show <u>very important</u> properties:

$$R(-\omega) = R(\omega) \qquad X(-\omega) = -X(\omega)$$ or in other words:

$$S(-\omega) = R(-\omega) + jX(-\omega) = R(\omega) - jX(\omega) = S^*(\omega)$$

Summary: Real part of the transform is even, imaginary is odd!
Magnitude of the transform is even, phase is odd!

Left part of spectrum is conjugated complex compared to right part!

2.3.2 The Fourier Transform – Important properties

Note the general mathematical properties of functions:

Any function can be separated

into even and odd parts:

$$s(t) = s_g(t) + s_u(t) \quad \text{with} \quad s_g(t) = \frac{s(t) + s(-t)}{2}, \quad s_u(t) = \frac{s(t) - s(-t)}{2}$$

For these parts it holds:

$$s_g(-t) = s_g(t)$$

$$s_u(-t) = -s_u(t)$$

2.3.2 The Fourier Transform – Important poperties

Summary:

If only even or only odd parts of a signal are regarded the Fourier transform formulas simplify a bit:

$$s_g(t) \;\circ\!\!\overset{\mathcal{F}}{-\!\!-}\!\!\bullet\; R(\omega)$$

$$s_u(t) \;\circ\!\!\overset{\mathcal{F}}{-\!\!-}\!\!\bullet\; jX(\omega)$$

$$R(\omega) = 2 \cdot \int_0^{+\infty} s_g(t) \cdot \cos(\omega t) dt \quad ; \quad X(\omega) = -2 \cdot \int_0^{+\infty} s_u(t) \cdot \sin(\omega t) dt$$

$$s_g(t) = \frac{1}{\pi} \int_0^{+\infty} R(\omega) \cos(\omega t) dt \quad ; \quad s_u(t) = -\frac{1}{\pi} \int_0^{+\infty} X(\omega) \sin(\omega t) dt$$

2.3.2 The Fourier Transform of special signals

Special signals: $e^{-j\omega t}$, $\sin\omega t$, $\cos\omega t$, $\delta(t)$, $\varepsilon(t)$

$$s(t) = a\cdot\delta(t) \quad \circ\!\!\overset{\mathcal{F}}{-\!\!-}\!\!\bullet \quad S(\omega)=\int_{-\infty}^{+\infty} \mathrm{a}\cdot\delta(t)\cdot e^{-j\omega t}dt = a\int_{-\infty}^{+\infty}\delta(t)\cdot e^{0}dt = a$$

Application of shifting theorem in the time domain:

$$s(t) = a\cdot\delta(t-t_0) \quad \circ\!\!\overset{\mathcal{F}}{-\!\!-}\!\!\bullet \quad S(\omega) = a\cdot e^{-j\omega t_0}$$

Application of symmetry theorem: $S(-t) \circ\!\!\overset{\mathcal{F}}{-\!\!-}\!\!\bullet\, 2\pi\cdot s(\omega)$ with $t_0 \rightarrow \omega_0$

$$a\cdot e^{j\omega_0 t} \quad \circ\!\!\overset{\mathcal{F}}{-\!\!-}\!\!\bullet \quad 2\pi\cdot a\cdot\delta(\omega-\omega_0)$$

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2.3.2 The Fourier Transform of special signals

Now a cosine is written by 2 exponential functions. Also this last result is used:

$$a \cdot e^{j\omega_0 t} \;\circ\!\!-\!\!\!\stackrel{\mathcal{F}}{-}\!\!\bullet\; 2\pi \cdot a \cdot \delta(\omega - \omega_0)$$

$$s(t) = a \cdot \cos(\omega_0 t) = \frac{a}{2} \cdot (e^{j\omega_0 t} + e^{-j\omega_0 t})$$

Thus we obtain:

$$\circ\!\!-\!\!\!\stackrel{\mathcal{F}}{-}\!\!\bullet$$

$$S(\omega) = \pi \cdot a \cdot \left[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right]$$

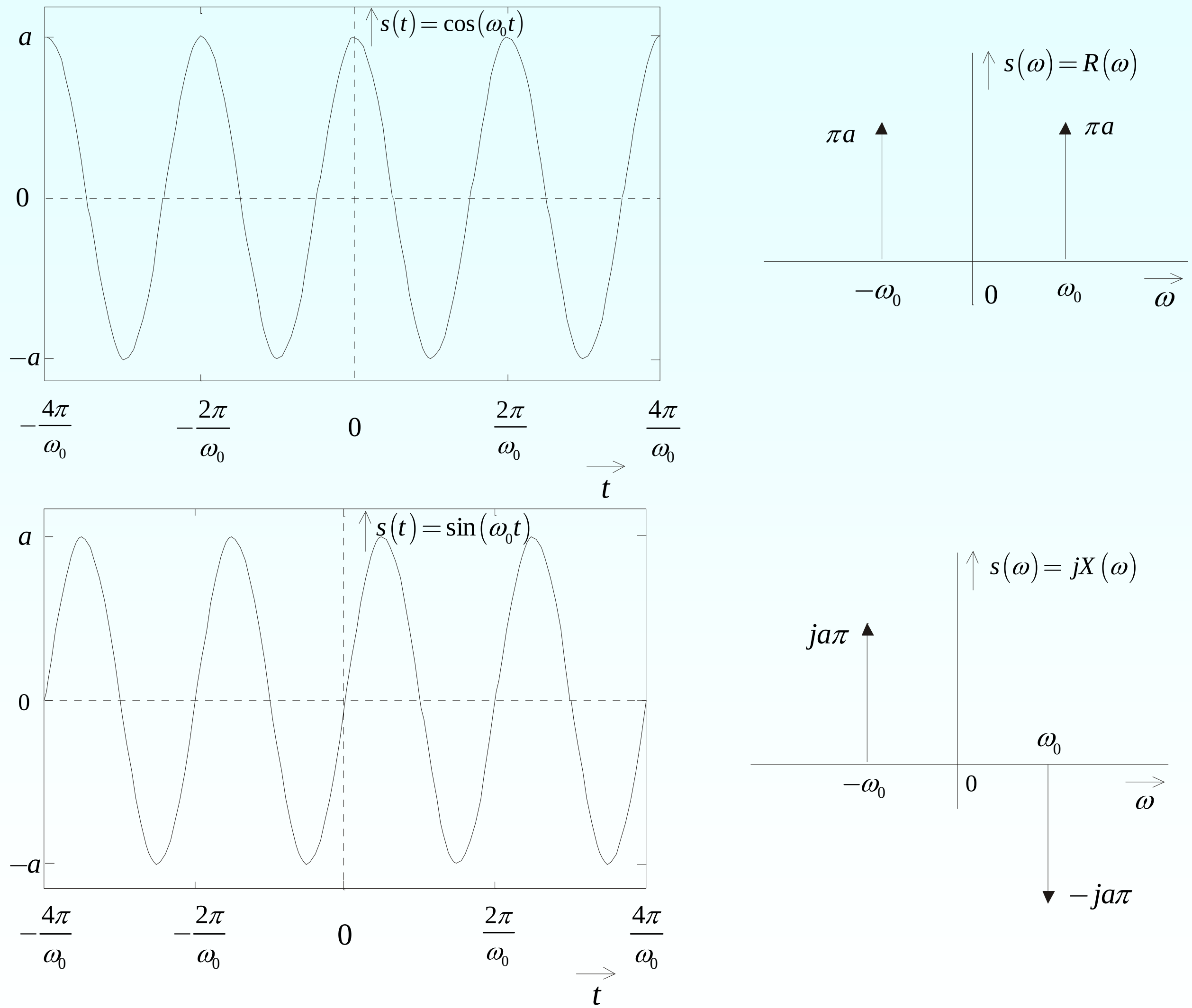
Same procedure is applied for a sine function:

$$s(t) = a \cdot \sin(\omega_0 t) = \frac{a}{2j} \cdot (e^{j\omega_0 t} - e^{-j\omega_0 t})$$

$$\circ\!\!-\!\!\!\stackrel{\mathcal{F}}{-}\!\!\bullet$$

$$S(\omega) = j\pi \cdot a\left[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right]$$

2.3.2 The Fourier Transform of special signals



2.3.4 Laplace Transform of Signals

For causal signals (see following property) the Laplace transform exists.

$$s(t) \equiv 0 \quad \text{for} \quad t < 0$$

$$s(t) = \lim_{\omega \to \infty} \frac{1}{2\pi j} \cdot \int_{\sigma - j\omega}^{\sigma + j\omega} S_L(p) \cdot e^{pt} dp \qquad p = \sigma + j\omega$$

$$S_L(p) = \int_0^{\infty} s(t) \cdot e^{-pt} dt = \int_0^{\infty} s(t) \cdot e^{-\sigma t} \cdot e^{-j\omega t} dt$$

Interpretation:

Laplace transform is a Fourier transform of the damped causal signal:

$$s(t) \cdot \varepsilon(t) \cdot e^{-\sigma t} \quad \text{where} \quad \sigma > 0 \text{ and real}$$

Abbreviation similar to Fourier transform:

$$s(t) \quad \circ\!\!\overset{\mathscr{L}}{-\!\!-\!\!-}\!\!\bullet \quad S_L(p) = \int_0^{\infty} s(t) \cdot e^{-pt} dt$$

2.3.4 Laplace Transform of Signals

Convergence of the Laplace integral

It converges for all *s(t)* growing slower with *t* than $e^{\sigma t}$

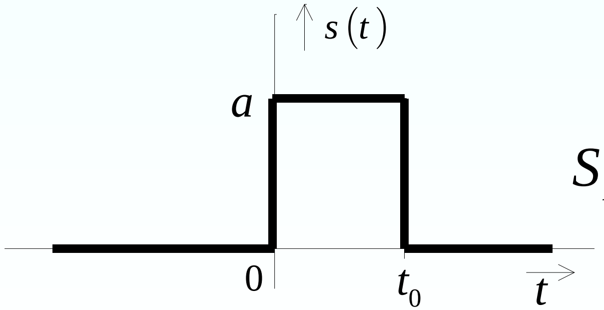
If there is convergence in one point p_0 , then there is also convergence in all points p with higher real part of p.

The area of convergence is always a half p plane!

Areas with no convergence are of high interest because location of poles is important in many aspects!

Example 1:

$$s(t) = a \cdot rect\left(\frac{t - t_0/2}{t_0}\right)$$



$$S_L(p) = \frac{a}{p} \cdot (1 - e^{-pt_0})$$

2.3.4 Laplace Transform of Signals

Example 2:

$$s(t) = a \cdot \varepsilon(t) \sin \omega_0 t$$

$$S_L(p) = \frac{a \cdot \omega_0}{p^2 + \omega_0^2} = \frac{a \cdot \omega_0}{(p + j\omega_0) \cdot (p - j\omega_0)}$$

<u>Some first properties of the Laplace transform</u>

The Laplace transform develops to the Fourier transform on the vertical axis if some conditions are met:

$$S_L(j\omega) = S_F(\omega)$$

For real *p* the Laplace transform is also real, if other conditions are met

2.3.5 Z-Transform of Discrete-Time Sequences

For discrete signals in most cases instead of the Laplace the z-transform is used:

$$\left\{s(k)\right\} \quad \overset{\mathcal{Z}}{\circ\!-\!\bullet} \quad S_z(z) = \sum_{k=0}^{\infty} s(k) \cdot z^{-k} = Z\left\{s(k)\right\}$$

with

$$s(k) = \begin{cases} 0 & \text{for } k < 0 \\ s(k) & \text{for k} \geq 0 \end{cases}$$

This transform results from the Laplace transform for the case of discrete signals with constant clock period.

Here an ideally sampled continous-time (analog) signal $s_a(t)$ is assumed.

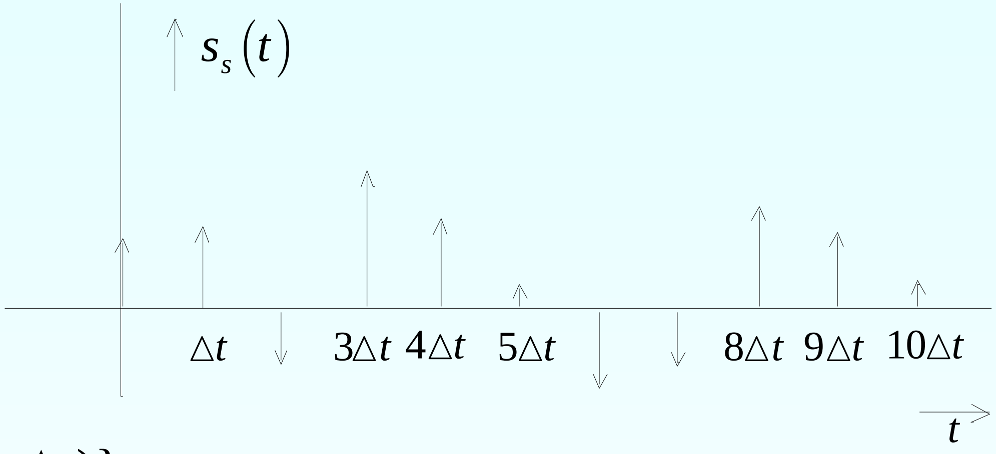
$$s_s(t) = \sum_{k=0}^{\infty} s_a(k \cdot \Delta t) \cdot \delta(t - k \cdot \Delta t)$$

All samples of the continous-time signal can also be written in short:

$$s_a(k \cdot \Delta t) = s(k)$$

2.3.5 Z-Transform of Discrete-Time Sequences

$$s_s(t) = \sum_{k=0}^{\infty} s_a(k \cdot \Delta t) \cdot \delta(t - k \cdot \Delta t)$$



$$L\{s_s(t)\} = \sum_{k=0}^{\infty} s_a(k \cdot \Delta t) \cdot L\{\delta(t - k \cdot \Delta t)\}$$

$$= \sum_{k=0}^{\infty} s_a(k \cdot \Delta t) \cdot 1 \cdot e^{-pk\Delta t}$$

$$= \sum_{k=0}^{\infty} s_a(k \cdot \Delta t) \cdot e^{-k \cdot \Delta t p}$$

2.3.5 Z-Transform of Discrete-Time Sequences

The exponential expression can also be written in short.

$$z = e^{\Delta t \cdot p} \qquad s(k) = s_a(k \cdot \Delta t)$$

Thus we obtain an expression which is no more directly depending on *p*:

$$L\{s(k)\} = S_z(z) = \sum_{k=0}^{\infty} s(k) z^{-k}$$

This is the z-transform. The inverse transform looks as follows:

$$s(k) = \frac{1}{2\pi j} \oint_c S_Z(z) z^{k-1} dz \qquad k = 0, 1, 2, \ldots$$

2.3.5 Z-Transform of Discrete-Time Sequences

Rules and properties of the z-transform

1 Shifting

$$\{s(k-1)\} \circ\!\!-\!\!\bullet\; z^{-1}S_Z(z)$$

$$\{s(k+1)\} \circ\!\!-\!\!\bullet\; z\left[S_Z(z)-s(0)\right]$$

2 Modulation

$$\{e^{akT}s(k)\} \circ\!\!-\!\!\bullet\; S_Z(e^{-aT}z)$$

3 Damping

$$\{\alpha^{-k}s(k)\} \circ\!\!-\!\!\bullet\; S_Z\{\alpha z\}$$

2.3.5 Z-Transform of Discrete-Time Sequences

4 Differentiation of the z-transform

$$\{k s(k)\} \circ\!\!-\!\!\bullet \; -z \frac{dS_Z(z)}{dz}$$

5 Convolution

$$\{s(k)\} \circ\!\!-\!\!\bullet \; S_z(z)$$

$$\{g(k)\} \circ\!\!-\!\!\bullet \; G_z(z)$$

$$\{s(k)\} * \{g(k)\} = \sum_{\nu=0}^{k} s(\nu) g(k-\nu) \circ\!\!-\!\!\bullet \; S_z(z) G_z(z)$$

6 Linearity

2.3.5 Z-Transform of Discrete-Time Sequences

Example (2nd order processing of an input sequence)

$$y(k-2)+c_1 y(k-1)+y(k)=s(k-1)+s(k) \quad \text{with} \quad c_1=-2.5$$

For this situation the output sequence in terms of the input sequence is required (zero state and causal input sequence is assumed).

$$\{y(k)\} \circ\!\!\overset{\mathcal{Z}}{-\!\!-}\!\!\bullet\; Y_Z(Z) \qquad \{s(k)\} \circ\!\!\overset{\mathcal{Z}}{-\!\!-}\!\!\bullet\; S_Z(Z)$$

$$y(k)=0 \quad \forall k<0 \qquad s(k)=0 \quad \forall k<0$$

2.3.5 Z-Transform of Discrete-Time Sequences

$$y(k-2)+c_1 y(k-1)+y(k) \;=\; s(k-1)+s(k)$$

$$\mathcal{Z}\;\big\downarrow$$

$$z^{-2}Y_Z(z)+c_1 z^{-1}Y_Z(z)+Y_Z(z) \;=\; z^{-1}S_Z(z)+S_Z(z)$$

$$\Rightarrow\quad Y_Z(z)\left[z^{-2}+c_1 z^{-1}+1\right] \;=\; S_Z(z)\left[z^{-1}+1\right]$$

$$\Rightarrow\quad Y_Z(z) \;=\; S_Z(z)\underbrace{\frac{z^{-1}+1}{z^{-2}+c_1 z^{-1}+1}}_{H_Z(z)}$$

In short the result reads as follows:

2.3.5 Z-Transform of Discrete-Time Sequences

$$H_Z(z) = \frac{z^{-1}+1}{z^{-2}+c_1 z^{-1}+1} = \frac{z^2+z}{z^2-2.5z+1} = \frac{z(z+1)}{(z-0.5)(z-2)}$$

$$= z\cdot\left(\frac{z}{(z-0.5)(z-2)}+\frac{1}{(z-0.5)(z-2)}\right)$$

$$H_Z(z)\cdot z^{-1} = \frac{z}{(z-0.5)(z-2)}+\frac{1}{(z-0.5)(z-2)}$$

$$\mathcal{Z}\;\bullet\!\!-\!\!\circ$$

$$h(k-1) = -\frac{2}{3}(0.5^k-2^k)-\frac{2}{3}(0.5^{k-1}-2^{k-1}) \quad \Bigg|_{\substack{k-1\to k \text{ or} \\ k\to k+1}}$$

$$h(k) = \frac{2}{3}(-0.5^{k+1}+2^{k+1}-0.5^k+2^k)$$

2.3.5 Z-Transform of Discrete-Time Sequences

Also for the z-transform the frequency response of a system is of high importance. As before the sampling of an analog signal is considered, but now the Fourier transform is applied:

$$s_a(t) \;\overset{\mathcal{F}}{\circ\!\!-\!\!\bullet}\; S_a(\omega)$$

$$s_s(t) = \sum_{k=0}^{\infty} s_a(k\Delta t)\,\delta(t - k\Delta t)$$

$$\mathcal{F}\;\circ\!\!-\!\!\bullet$$

$$S_s(\omega) = \sum_{k=0}^{\infty} s_a(k\Delta t)\,e^{-j\omega k\Delta t} = \sum_{k=0}^{\infty} s(k)\,e^{-j\omega k\Delta t}$$

due to $s_a(k\Delta t) = s(k)$

Comparison to $S_Z(z) = \sum_{k=0}^{\infty} s(k)\,z^{-k}$ gives the relation:

$$S_s(\omega) = S_Z(e^{j\omega\Delta t})$$

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2.3.5 Z-Transform of Discrete-Time Sequences

Conclusion:

To obtain the properties of the discrete signal in the frequency domain the z-transform has to be evaluted only at the following points:

$$z = e^{j\omega\Delta t} \qquad \Rightarrow |z| = 1 \text{ for all } \omega\Delta t$$

Thus we evaluate the z-transform on the unit-circle:

$$S_Z(e^{j\omega\Delta t}) = \sum_{k=0}^{\infty} s(k) \cdot e^{-j\omega\Delta t \cdot k}$$

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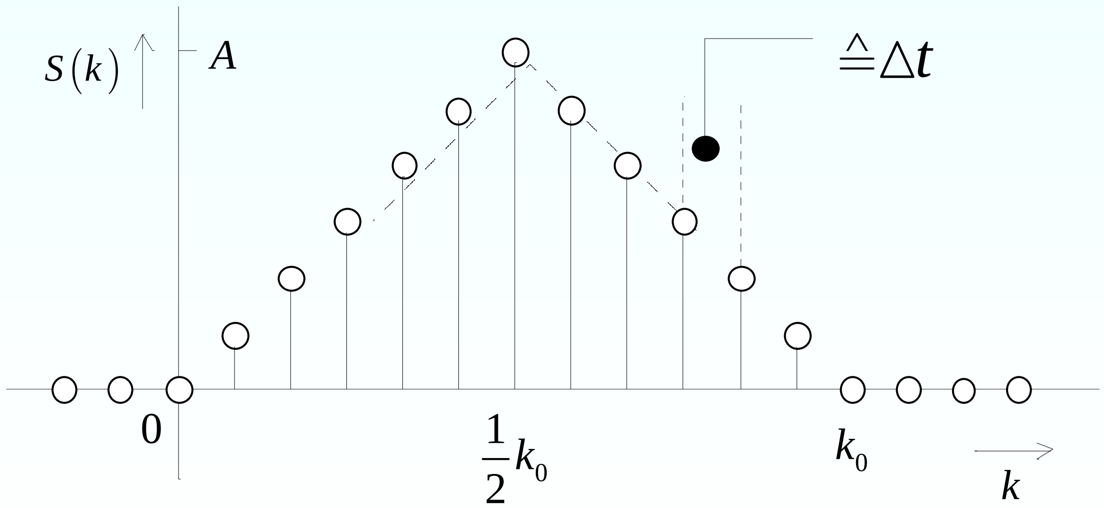
2.3.5 Z-Transform of Discrete-Time Sequences

Example (triangular sequence)

$$s_a(t) = A \cdot \Lambda\left(\frac{t - t_0}{0.5 \cdot k_0 \cdot \Delta t}\right) \circ\!\!-\!\!\bullet \; A \cdot \frac{k_0}{2} \cdot \Delta t \cdot \operatorname{sinc}^2\left(\omega \cdot \frac{k_0 \cdot \Delta t}{4}\right) \cdot e^{-j\omega t_0}$$



2.4 Important General Signal Representations

If all physical signals in the time domain are real, it follows:

$$R(\omega) = R(-\omega)$$

$$X(\omega) = -X(-\omega)$$

$$|S(\omega)| = |S(-\omega)|$$

$$\varphi(-\omega) = -\varphi(\omega)$$

or

$$S(-\omega) = S^*(\omega)$$

$$s(t) \circ\!\!\overset{\mathcal{F}}{-\!\!-}\!\!\bullet\; S(\omega) = R(\omega) + j \cdot X(\omega) = |S(\omega)| \cdot e^{j\varphi(\omega)} \quad \text{with} \quad \varphi(\omega) = \angle S(\omega)$$

2.4.1 Low-Pass Signals

„Low-pass signal“ are signals s(t) with a spectrum S(ω) that vanishes completely or negligible for

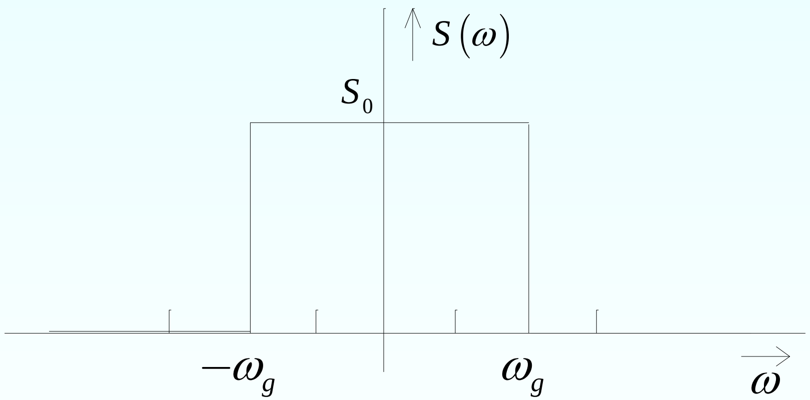
$$|S(\omega)| = 0 \quad \text{or} \quad |S(\omega)| \approx 0 \quad \text{for } |\omega| > \omega_g = 2\pi f_g$$

The spectrum of a low-pass signal exhibits at ω = 0 always non-zero values

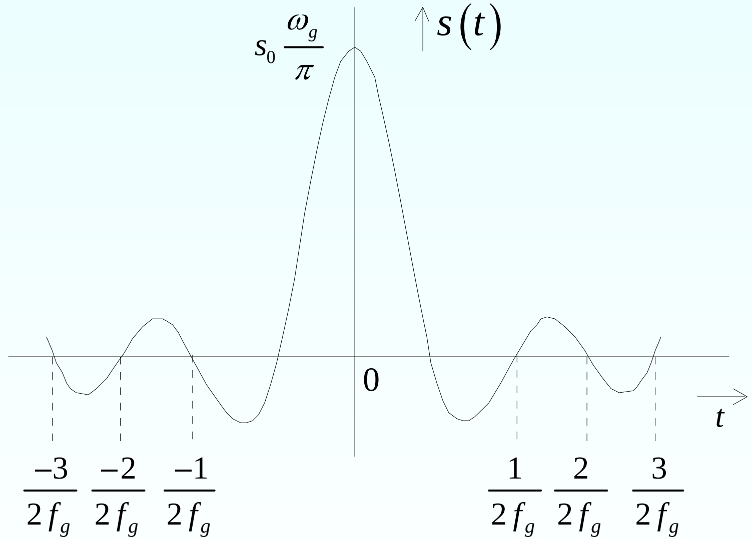
2.4.1 Low-Pass Signals

Example 1: Ideal low-pass signal

$$S(\omega) = S_0 \cdot rect\left(\frac{\omega}{2\omega_g}\right) \quad \circ\!\!-\!\!\bullet \quad s(t) = S_0 \cdot \frac{\omega_g}{\pi} \cdot si(\omega_g t)$$



Spectrum S(w) of an Ideal low-pass signal

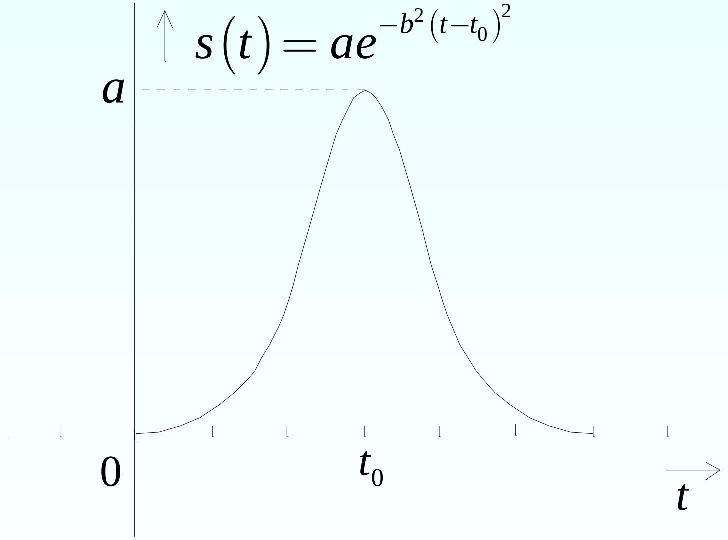


Ideal low-pass signal s(t)

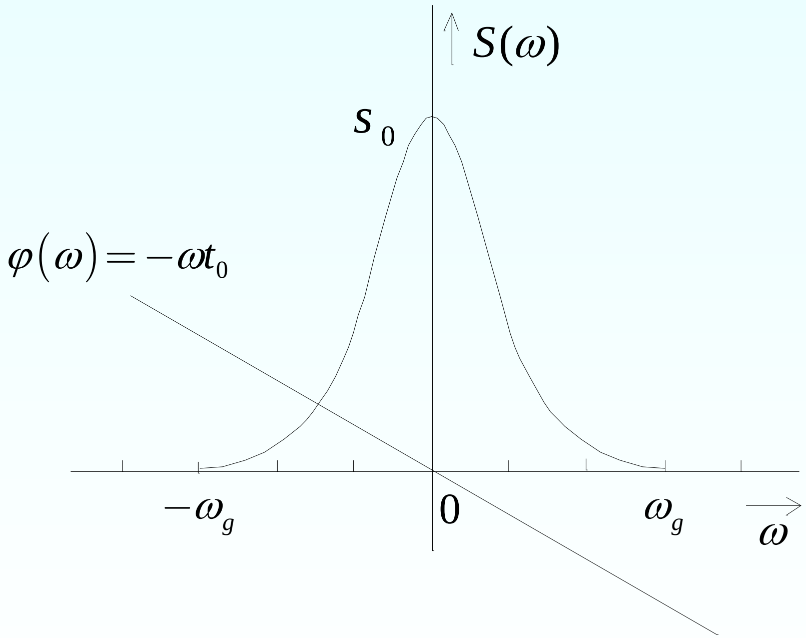
2.4.1 Low-Pass Signals

Example 2: the Gaussian impulse

$$s(t) = a \cdot e^{-b^2 \cdot (t-t_0)^2} = S_0 \cdot \frac{b}{\sqrt{\pi}} \cdot e^{-b^2 \cdot (t-t_0)^2} \;\circ\!\!-\!\!\bullet\; S(\omega) = a \cdot \frac{\sqrt{\pi}}{b} \cdot e^{-\frac{\omega^2}{(2b)^2}} \cdot e^{-j\omega t_0} = S_0 \cdot e^{-\frac{\omega^2}{(2b)^2}}$$



Time function of the Gaussian Impulse



Spectrum S(w) (mag. and phase)

2.4.2 The Hilbert Transform and the Analytic Signal

For signals: $\int\limits_{-\infty}^{+\infty} s^2(t)dt < \infty$

the Hilbert transform of the signal s(t) is given as:

$$\hat{s}(t) = H\{s(t)\} = \frac{1}{\pi}.V.P \int\limits_{-\infty}^{+\infty} \frac{s(\tau)}{t-\tau} d\tau$$

where $V.P \int\limits_{-\infty}^{+\infty} \frac{s(\tau)}{t-\tau} d\tau = \lim\limits_{\varepsilon \to 0} \left[\int\limits_{-\infty}^{t-\varepsilon} ...d\tau + \int\limits_{t+\varepsilon}^{+\infty} ...d\tau \right]$

Accordingly, the inverse Hilbert transform is given by:

$$s(t) = -\frac{1}{\pi} V.P. \int\limits_{-\infty}^{+\infty} \frac{\hat{s}(\tau)}{t-\tau} d\tau = -H\{\hat{s}(t)\}$$

2.4.2 The Hilbert Transform and the Analytic Signal

Hilbert transform can be interpreted by means of convolution integrals in case the intergrals converge:

$$\hat{s}(t) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{s(\tau)}{t-\tau}\,d\tau = s(t) * \frac{1}{\pi t} \quad \text{and} \quad s(t) = -\frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\hat{s}(\tau)}{t-\tau}\,d\tau = -\hat{s}(t) * \frac{1}{\pi t}$$

$\mathcal{F}$ ○—● Thus, the Fourier transforms can be derived directly: $\mathcal{F}$ ○—●

$$F\{\hat{s}(t)\} = -j \cdot sign(\omega) \cdot S(\omega) = \hat{S}(\omega) \qquad F\{s(t)\} = j \cdot sign(\omega) \cdot \hat{S}(\omega) = S(\omega)$$

2.4.2 The Hilbert Transform and the Analytic Signal

If s(t) is real with $S(\omega) = R(\omega) + j \cdot X(\omega)$, it gives result:

$$s(t) = \frac{1}{\pi}\int_0^{\infty} R(\omega)\cdot\cos(\omega t)\,d\omega - \frac{1}{\pi}\int_0^{\infty} X(\omega)\cdot\sin(\omega t)\,d\omega$$

$$\hat{s}(t) = \frac{1}{\pi}\int_0^{\infty} X(\omega)\cdot\cos(\omega t)\,d\omega + \frac{1}{\pi}\int_0^{\infty} R(\omega)\cdot\sin(\omega t)\,d\omega$$

$s(t)$ and $\hat{s}(t)$ are called <u>conjugated functions</u>.

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2.4.2 The Hilbert Transform and the Analytic Signal

With: $$s(t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} S(\omega) \cdot e^{j\omega t} d\omega$$

$$= \frac{1}{2\pi} \cdot \int_{-\infty}^{0} S(\omega) \cdot e^{j\omega t} d\omega + \frac{1}{2\pi} \cdot \int_{0}^{+\infty} S(\omega) \cdot e^{j\omega t} d\omega$$

$$\mathrm{S}(\omega) = S^{-}(\omega) + S^{+}(\omega)$$

the <u>analytic signal</u> is defined as following:

$$s^{\circ}(t) = s(t) + j \cdot \hat{s}(t)$$

Real part: the signal itself

Imaginary part: Hilbert transform of s(t)

$$\hat{s}(t) = -\frac{1}{\pi} \cdot \int_{-\infty}^{+\infty} \frac{s(\tau)}{\tau - t} d\tau$$

2.4.2 The Hilbert Transform and the Analytic Signal

The properties of the analytic signal:

1. If s(t) $\circ\!\!\overset{\mathcal{F}}{\longrightarrow}\!\!\bullet$ S(ω), then

$$\hat{s}(t) \circ\!\!\overset{\mathcal{F}}{\longrightarrow}\!\!\bullet \hat{S}(\omega) = \begin{cases} -j \cdot S(\omega) & \text{for} \quad \omega > 0 \\ 0 & \text{for} \quad \omega = 0 \\ +j \cdot S(\omega) & \text{for} \quad \omega < 0 \end{cases}$$

$$\hat{s}(t) \circ\!\!\overset{\mathcal{F}}{\longrightarrow}\!\!\bullet \hat{S}(\omega) = -j \cdot S(\omega) \cdot sign(\omega)$$

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2.4.2 The Hilbert Transform and the Analytic Signal

2. With s(t) $\circ\!\!\overset{\mathcal{F}}{\longrightarrow}\!\!\bullet$ S(ω), result:

$$s^{\circ}(t) \;\circ\!\!\overset{\mathcal{F}}{\longrightarrow}\!\!\bullet\; S^{\circ}(\omega) = \begin{cases} 0 & \text{for} \quad \omega < 0 \\ S(\omega) & \text{for} \quad \omega = 0 \\ 2S(\omega) & \text{for} \quad \omega > 0 \end{cases}$$

or $$s^{\circ}(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S^{\circ}(\omega)\cdot e^{j\omega t}\,d\omega = \frac{1}{\pi}\int_{0}^{\infty} S(\omega)\cdot e^{j\omega t}\,d\omega$$

Proof:

$$s^{\circ}(t) = s(t) + j\hat{s}(t) \;\circ\!\!\overset{\mathcal{F}}{\longrightarrow}\!\!\bullet\; S^{\circ}(\omega) = S(\omega) + j\cdot\hat{S}(\omega) = S(\omega) + j(-jsign(\omega)S(\omega))$$

$$= S(\omega)\cdot\left[1 + sign(\omega)\right] = \begin{cases} 0 & \text{for} \quad \omega < 0 \\ S(\omega) & \text{for} \quad \omega = 0 \\ 2\cdot S(\omega) & \text{for} \quad \omega > 0 \end{cases}$$

2.4.2 The Hilbert Transform and the Analytic Signal

3. Real part $s(t)$ and imaginary part $\hat{s}(t)$ are orthogonal:

$$\int_{-\infty}^{+\infty} s(t) \cdot \hat{s}(t) dt = 0$$

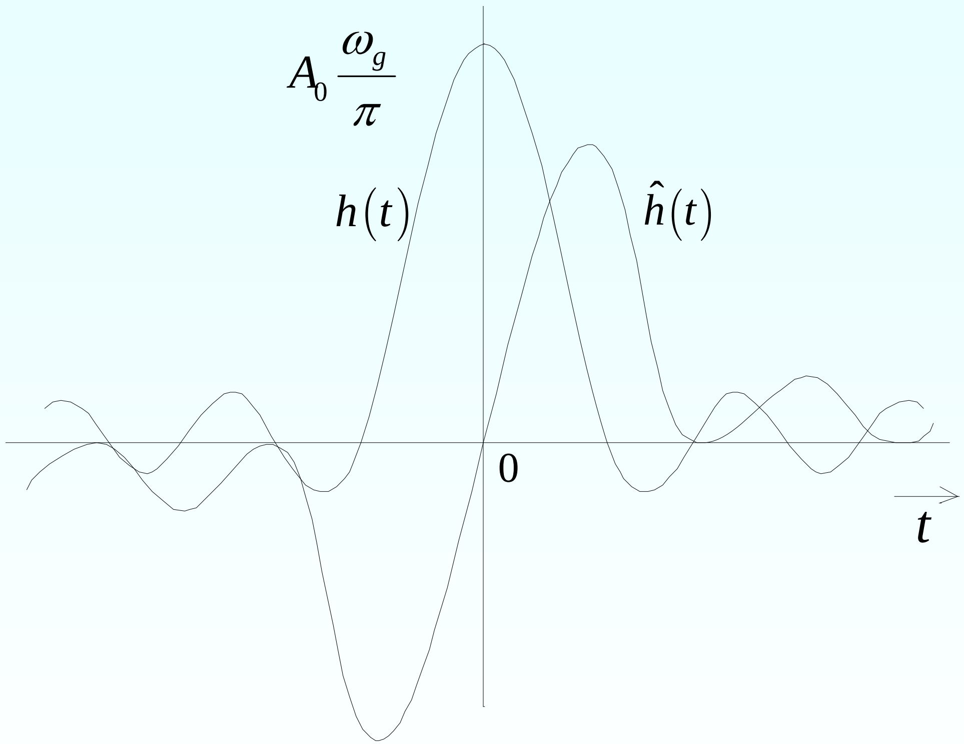
Example:

$$h(t) = \frac{A_0 \cdot \omega_g}{\pi} \cdot si(\omega_g \cdot t)$$

$$h^{\circ}(t) = h(t) + j \cdot \hat{h}(t) = \frac{A_0 \cdot \omega_g}{\pi} \cdot \left[\frac{\sin \omega_g t}{\omega_g t} + j \cdot \frac{1 - \cos \omega_g t}{\omega_g t} \right]$$

2.4.2 The Hilbert Transform and the Analytic Signal

Example:



Real and Imaginary part of the analytic Signal of an Ideal low-pass

2.4.3 Band-Pass Signals

Band-pass signals are signals s(t) with spectrum S(ω) limited to a certain interval on the frequency axis

This interval does not include the frequency w = 0

$$|S(\omega)| = 0 \text{ or } |S(\omega)| \approx 0 \quad \text{for all } \omega \text{ outside of } \Delta\omega$$

The two versions of band-pass signal which will be described following are:

- Symmetrical band-pass signal
- More generalized version

2.4.3 Band-Pass Signals

The symmetric band-pass signal:

$$S(\omega) = S_0\left[rect\left(\frac{\omega-\omega_0}{\Delta\omega}\right)+rect\left(\frac{\omega+\omega_0}{\Delta\omega}\right)\right]$$

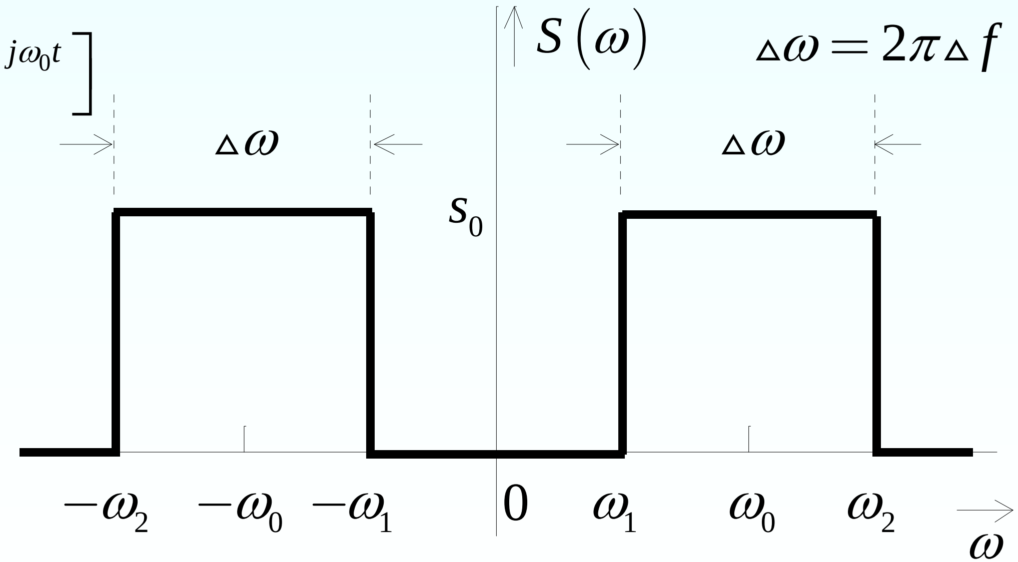
$$\mathcal{F}$$

$$s(t) = S_0\left[\frac{\Delta\omega}{2\pi}\cdot si\left(\frac{\Delta\omega}{2}t\right)\cdot e^{j\omega_0 t}+\frac{\Delta\omega}{2\pi}\cdot si\left(\frac{\Delta\omega}{2}t\right)\cdot e^{-j\omega_0 t}\right]$$

$$= S_0\cdot\frac{\Delta\omega}{2\pi}si\left(\frac{\Delta\omega}{2}t\right)\cdot\left[e^{j\omega_0 t}+e^{-j\omega_0 t}\right]$$

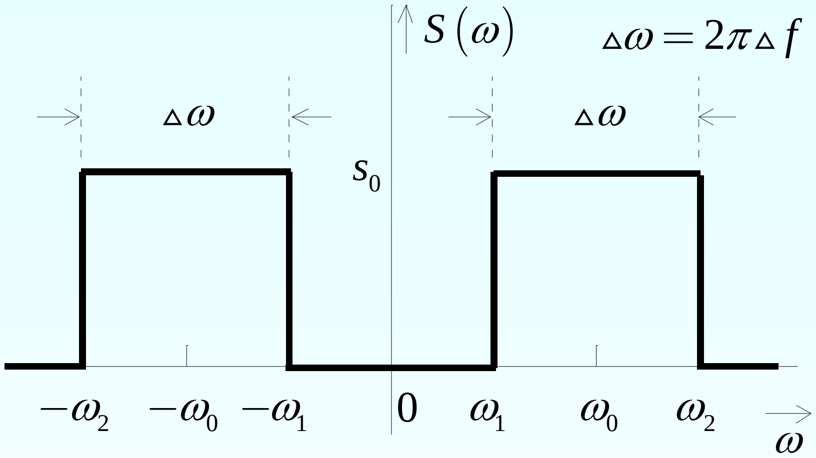
$$= S_0\cdot\frac{\Delta\omega}{\pi}\cdot si\left(\frac{\Delta\omega}{2}t\right)\cdot\cos\omega_0 t$$

$$= s_T(t)\cdot\cos\omega_0 t$$

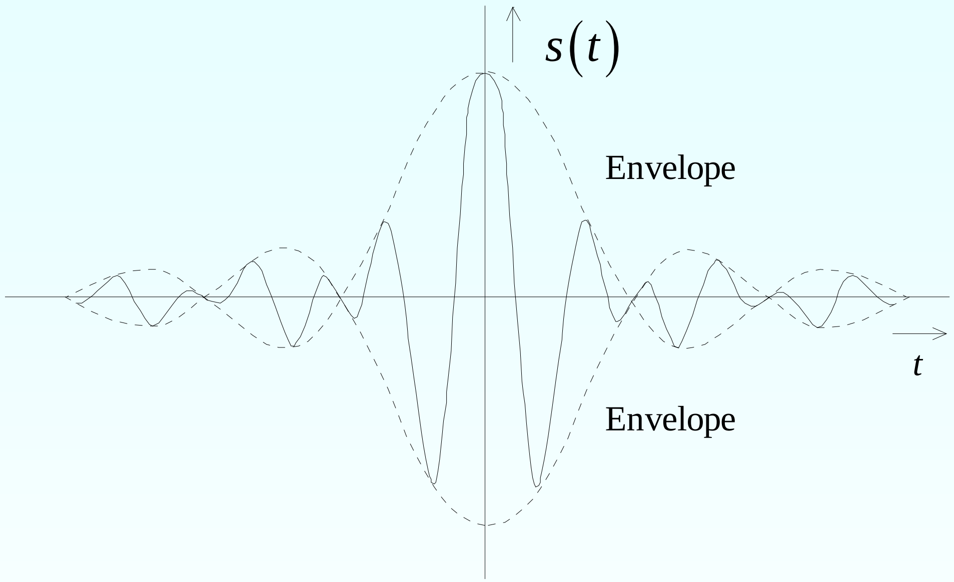


2.4.3 Band-Pass Signals

Example:



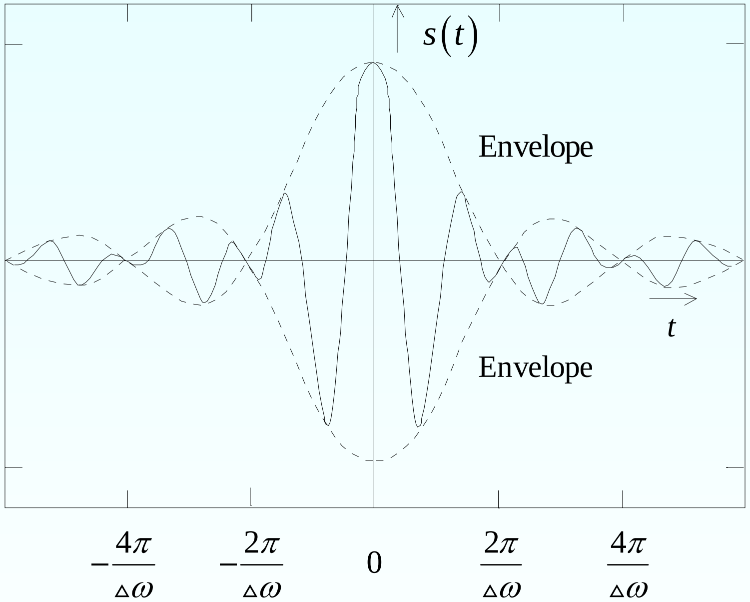
Spectrum of a symmetric (Ideal) Band-pass signal

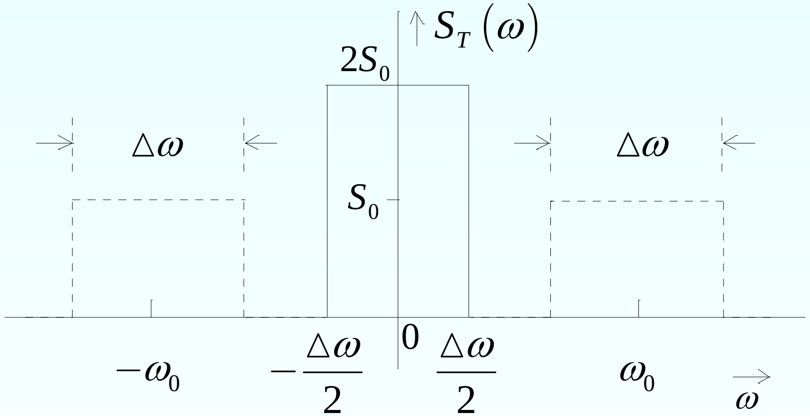


Symmetric (Ideal) Band-pass signal and its envelope

2.4.3 Band-Pass Signals

Example: Equivalent low-pass signal and its Spectrum





$$s_T(t) = 2S_0 \cdot \frac{\Delta\omega}{2\pi} \cdot si\left(\frac{\Delta\omega}{2} t\right) \quad \circ\!\!-\!\!\bullet \quad S_T(\omega) = 2S_0 \cdot rect\left(\frac{\omega}{\Delta\omega}\right)$$

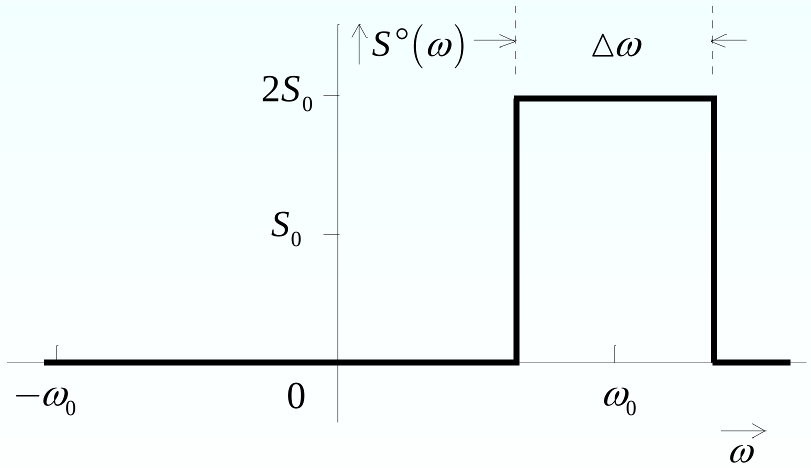
2.4.3 Band-Pass Signals

Presentation of symmetrical band-pass signals using equivalent low-pass signals:

$$s^{\circ}(t) = s(t) + j \cdot \hat{s}(t) \quad \circ\!\!-\!\!\bullet \quad S^{\circ}(\omega) = S(\omega) \cdot [1 + sign(\omega)] = 2 \cdot S(\omega) \cdot \varepsilon(\omega)$$

$$S^{\circ}(\omega) = \begin{cases} 0 & \text{for} \quad \omega < 0 \\ S(\omega) & \text{for} \quad \omega = 0 \\ 2S(\omega) & \text{for} \quad \omega > 0 \end{cases} = 2 \cdot S_0 \cdot rect\left(\frac{\omega - \omega_0}{\Delta\omega}\right)$$

Analytic signal of a Symmetric band-pass signal



2.4.3 Band-Pass Signals

By shifting on the frequency axis, the equivalent low-pass signal can be derived as:

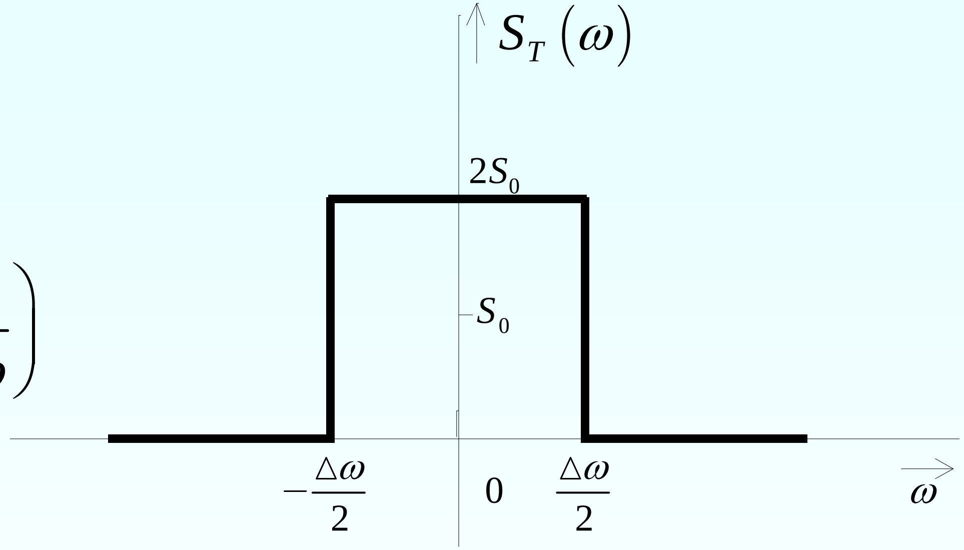
$$s_T(t) = s^\circ(t) \cdot e^{-j\omega_0 t}$$

$$\mathcal{F} \; \circ\!\!-\!\!\bullet$$

$$S_T(\omega) = S^\circ(\omega + \omega_0) = 2S_0 \cdot rect\left(\frac{\omega}{\Delta\omega}\right)$$

$$\mathcal{F} \; \bullet\!\!-\!\!\circ$$

$$s_T(t) = \frac{S_0 \Delta\omega}{\pi} \cdot si\left(\frac{\Delta\omega}{2} t\right)$$



The equivalent low-pass signal of a Symmetric Band-pass signal

As $s_T(t)$ is real, the following relation holds:

$$s(t) = \mathrm{Re}\left\{s^\circ(t)\right\} = \mathrm{Re}\left\{s_T(t) \cdot e^{j\omega_0 t}\right\} = s_T(t)\,\mathrm{Re}\left\{e^{j\omega_0 t}\right\} = s_T(t) \cdot \cos\omega_0 t$$

2.4.3 Band-Pass Signals

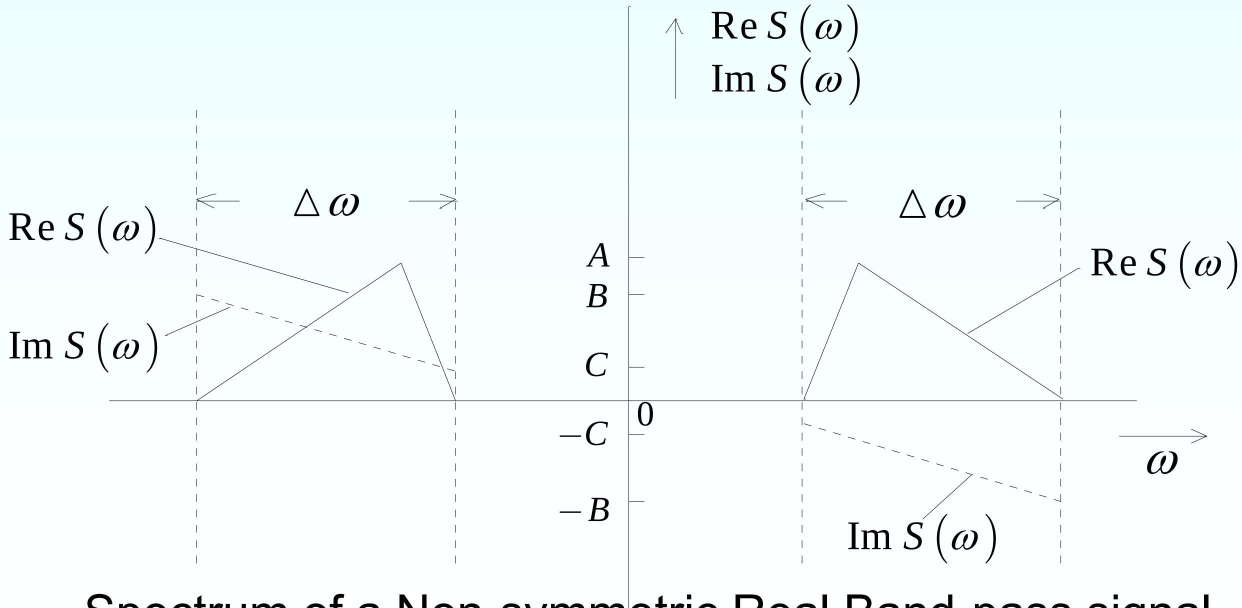
The general, real band-pass signal:

$$s_T(t) = u(t) + j \cdot v(t) = s_0(t) \cdot e^{j\phi(t)} \quad \circ\!\!\stackrel{\mathcal{F}}{-}\!\!\bullet \quad S_T(\omega)$$

Signal envelope

$u(t) = s_0(t) \cdot \cos(\phi(t))$: inphase component

$v(t) = s_0(t) \cdot \sin(\phi(t))$: quadrature component



Spectrum of a Non-symmetric Real Band-pass signal

2.4.3 Band-Pass Signals

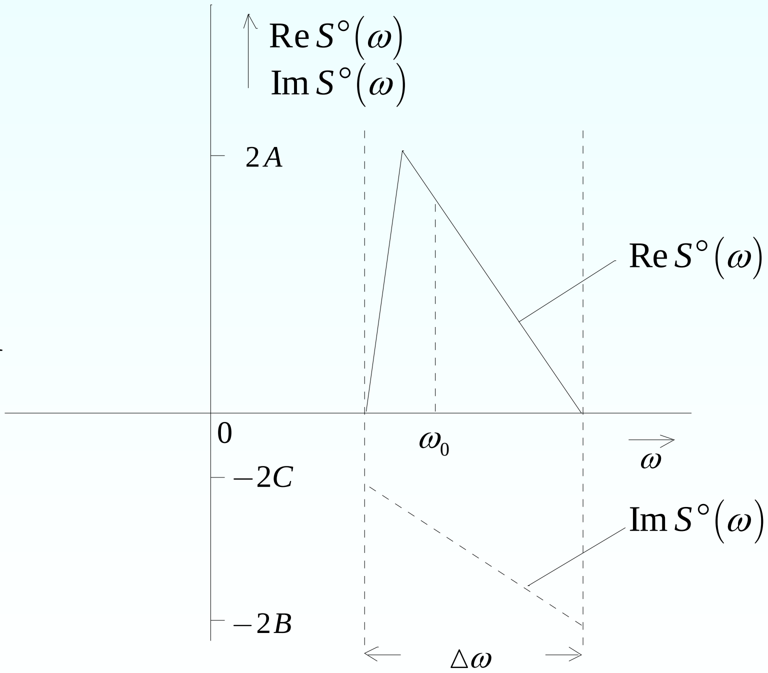
One can find the signal $s^0(t)$ developed from the equation:

$$s^\circ(t) = s_T(t) \cdot e^{j(\omega_0 t + \phi_0)}$$

$\mathcal{F}$

$$S^\circ(\omega) = e^{j\phi_0} \cdot S_T(\omega - \omega_0)$$

by choosing ω_0: "midband frequency"
as following description in the figure



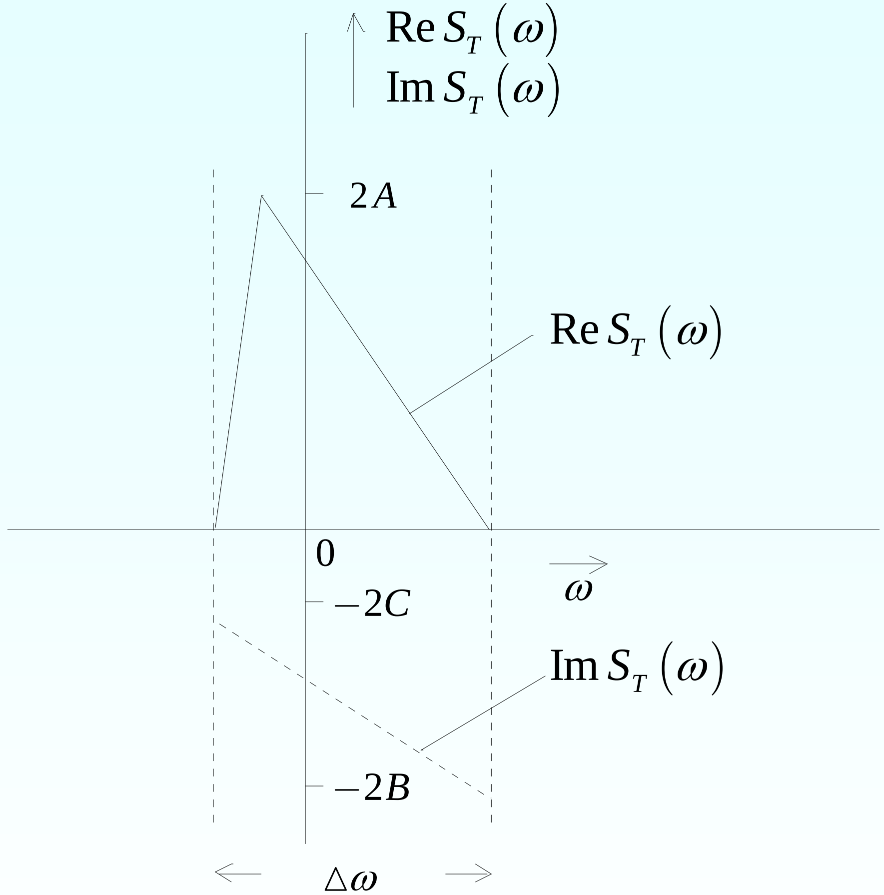
Spectrum $S^0(\omega)$ of the Analytic Signal of the Non-symmetric Real band-pass signal

In the follwing we set without loss of generality :

$$\phi_0 = 0 \Rightarrow e^{j\phi_0} = 1$$

2.4.3 Band-Pass Signals

Example:



Complex envelope (Real and Imaginary part) of the Non-symmetric real band-pass signal

2.4.3 Band-Pass Signals

In general some followings relations hold:

1. $S^{\circ}(\omega) = S(\omega) \cdot 2 \cdot \varepsilon(\omega)$

2. The spectrum's relationship between complex envelope and analytic signal is as follows:

$$S_T(\omega) = S^{\circ}(\omega + \omega_0)$$

3. The band-pass signal s(t) can be represented in the form:

$$s(t) = \mathrm{Re}\left\{s^{\circ}(t)\right\} = \mathrm{Re}\left\{s_T(t) \cdot e^{j(\omega_0 t)}\right\} = s_0(t) \cdot \cos(\omega_0 t + \phi(t))$$

or
$$s(t) = \mathrm{Re}\left\{s_T(t) \cdot (\cos(\omega_0 t) + j \cdot \sin(\omega_0 t))\right\}$$
$$= u(t) \cdot \cos(\omega_0 t) - v(t) \cdot \sin(\omega_0 t)$$

For $\phi_0 \neq 0$ it holds: $s(t) = u(t) \cdot \cos(\omega_0 t + \phi_0) - v(t) \cdot \sin(\omega_0 t + \phi_0)$

2.4.4 Causal Signal Functions

A causal signal function has the property:

$$s(t) \equiv 0 \quad \text{for} \quad t < 0 \longrightarrow$$ for analog signal

$$s(k) \equiv 0 \quad \text{for} \quad \mathrm{k} < 0 \longrightarrow$$ for discrete signal

Causal, analog signals s(t):

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2.4.4 Causal Signal Functions

Example:

$$s(t) = e^{-at}\varepsilon(t) \quad \circ\!\!-\!\!\!\overset{\mathcal{L}}{-}\!\!\!-\!\!\bullet \quad S_L(p) = \frac{1}{p+a}$$ is causal for $a > 0$

The unique relation between real part and imaginary part of causal signal spectra:

For $s(t) \quad \circ\!\!-\!\!\!\overset{\mathcal{F}}{-}\!\!\!-\!\!\bullet \quad S(\omega) = S_1(\omega) + jS_2(\omega)$, applies:

$S_2(\omega) = \hat{S}_1(\omega)$ and $S_1(\omega) = -\hat{S}_2(\omega)$

$$S(\omega) = \frac{1}{2\pi}\left[S(\omega) * \left(\frac{1}{j\omega} + \pi\delta(\omega)\right)\right]$$

$$= \frac{1}{2\pi} S(\omega) * \frac{1}{j\omega} + \frac{1}{2} S(\omega) * \delta(\omega)$$

$$= \frac{1}{2\pi} S(\omega) * \frac{1}{j\omega} + \frac{1}{2} S(\omega)$$

2.4.4 Causal Signal Functions

$$s(t) \overset{!}{=} s(t)\varepsilon(t)$$

$$\mathcal{F} \; \circ\!\!-\!\!\bullet$$

$$S(\omega) = \frac{1}{2\pi}\left[S(\omega) * \left(\frac{1}{j\omega} + \pi\delta(\omega) \right) \right]$$

$$= \frac{1}{2\pi} S(\omega) * \frac{1}{j\omega} + \frac{1}{2} S(\omega) * \delta(\omega)$$

$$= \frac{1}{2\pi} S(\omega) * \frac{1}{j\omega} + \frac{1}{2} S(\omega)$$

2.4.4 Causal Signal Functions

$$S(\omega) = S_1(\omega) + jS_2(\omega)$$

$$S(\omega) = \frac{1}{j\pi} S(\omega) * \frac{1}{\omega}$$

$$S_1(\omega) + jS_2(\omega) = \frac{1}{j\pi}\left[S_1(\omega) * \frac{1}{\omega} + jS_2(\omega) * \frac{1}{\omega} \right]$$

2.4.4 Causal Signal Functions

$$S_1(\omega) = \frac{1}{\pi} S_2(\omega) * \frac{1}{\omega} = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{S_2(\eta)}{\omega - \eta} d\eta = \hat{S}_1(\omega)$$

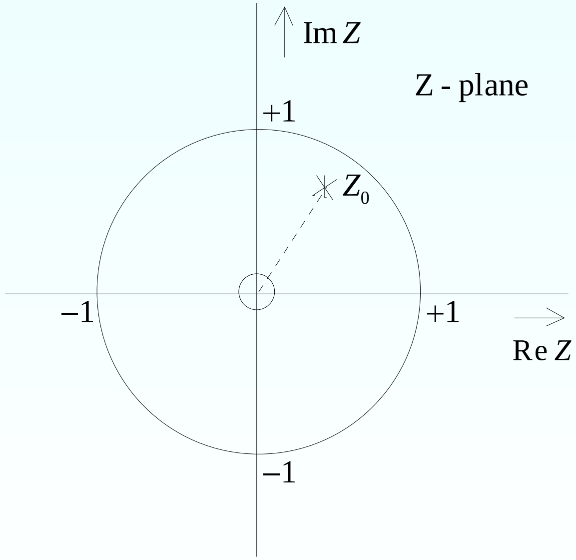
$$S_2(\omega) = -\frac{1}{\pi} S_1(\omega) * \frac{1}{\omega} = -\frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{S_1(\eta)}{\omega - \eta} d\eta = -\hat{S}_2(\omega)$$

$$\sum_{0}^{\infty} |s(k)| M^{-k} < \infty \quad \text{where } M > 1 \text{ and arbitrary real}$$

2.4.4 Causal Signal Functions

$$\{s(k)\} = \begin{cases} z_0^k & \text{for } k \geq 0 \\ 0 & \text{for } k < 0 \end{cases} \quad \text{where } z_0 = |z_0| e^{j\phi_0} \text{ and } |z_0| \leq 1$$

$$S_Z(z) = \frac{z}{z - z_0}$$



2.4.4 Causal Signal Functions

$$z = e^{j\Omega} \qquad |z| = 1$$

$$\{s(k)\} \circ\!\!-\!\!\!\overset{\mathcal{Z}}{-}\!\!\!-\!\!\bullet \; S_Z(z) = R_Z(z) + jX_Z(z) \quad \Rightarrow \quad S_Z(e^{j\Omega}) = R_Z(e^{j\Omega}) + jX_Z(e^{j\Omega})$$

$$R_Z(e^{j\Omega}) = s(0) - \frac{1}{\pi}\sum_{k=1}^{\infty}\left[\int_{-\pi}^{+\pi} X_Z(e^{j\eta})\sin(k\eta)d\eta\right]\cos(k\Omega)$$

$$X_Z(e^{j\Omega}) = -\frac{1}{\pi}\sum_{k=1}^{\infty}\left[\int_{-\pi}^{+\pi} R_Z(e^{j\eta})\cos(k\eta)d\eta\right]\sin(k\Omega)$$

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2.5.1 Correlation Functions and Energy Spec. of Deterministic Analog Energy Signals

$$E_s = \int_{-\infty}^{+\infty} |s(t)|^2 \, dt < \infty$$

$$s_n(t) = \frac{s(t)}{\sqrt{E_s}} \qquad\qquad g_n(t) = \frac{g(t)}{\sqrt{E_g}}$$

$$E_{s-g} = \int_{-\infty}^{+\infty} (s_n(t) - g_n(t))^2 \, dt = 2 - r_{sg}$$

$$r_{sg} = \frac{\int_{-\infty}^{+\infty} s(t) g(t) dt}{\sqrt{E_s E_g}}$$

2.5.1 Correlation Functions and Energy Spec. of Deterministic Analog Energy Signals

$$s(t) = s_{\mathrm{Re}}(t) + js_{\mathrm{Im}}(t)$$

$$E_s = \int_{-\infty}^{+\infty} s(t)s^*(t)dt = \int_{-\infty}^{+\infty} |s(t)|^2\, dt$$

$$r_{sg} = \frac{\int_{-\infty}^{+\infty} s(t)g^*(t)dt}{\sqrt{E_s E_g}} \qquad -1 \le r_{sg} \le +1$$

$$s(t) = kg(t) \qquad E_s = k^2 E_g \qquad r_{sg} = +1$$

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2.5.1 Correlation Functions and Energy Spec. of Deterministic Analog Energy Signals

$$s(t) = -kg(t)$$

$$E_s = k^2 E_g \qquad\qquad r_{sg} = -1$$

$$r_{sg} = 0$$

$$\rho_{sg}(\tau) = \int_{-\infty}^{+\infty} s(t) g^*(t+\tau) dt$$

$$\rho_{sg}(\tau) = \int_{-\infty}^{+\infty} s(t) g^*(t+\tau) dt = s(\tau) \otimes g(\tau)$$

2.5.1 Correlation Functions and Energy Spec. of Deterministic Analog Energy Signals

$$\rho_{sg}(\tau) = \int_{+\infty}^{-\infty} s(-\theta) g^*(-\theta+\tau) d(-\theta) = \int_{-\infty}^{+\infty} s(-\theta) g^*(\tau-\theta) d\theta = s(-\tau) * g(\tau)$$

$$\rho_{sg}(\tau) \circ\!\!-\!\!\bullet\ R_{sg}(\omega) = \int_{-\infty}^{+\infty} \rho_{sg}(\tau) e^{-j\omega\tau} d\tau = S(-\omega) G^*(-\omega)$$

$$\rho_{sg}(\tau) = s(-t) * g^*(t)$$

$$f(-t) \circ\!\!-\!\!\bullet\ F(-\omega)$$

$$\mathcal{F}: \rho_{sg}(\tau) \circ\!\!-\!\!\bullet\ R_{sg}(\omega)$$

$$R_{sg}(\omega) = S(-\omega) * G^*(-\omega)$$

$$f^*(t) \circ\!\!-\!\!\bullet\ F^*(-\omega)$$

2.5.1 Correlation Functions and Energy Spec. of Deterministic Analog Energy Signals

$$\rho_{sg}(-\tau) = \rho_{gs}^{*}(\tau) \quad \circ\!\!-\!\!\!\overset{\mathcal{F}}{-}\!\!\!-\!\!\bullet \quad R_{sg}(-\omega) = R_{gs}^{*}(-\omega)$$

$$\rho_{sg}(\tau) = \rho_{gs}^{*}(-\tau) \quad \circ\!\!-\!\!\!\overset{\mathcal{F}}{-}\!\!\!-\!\!\bullet \quad R_{sg}(\omega) = R_{gs}^{*}(\omega)$$

$$s(t) \otimes g(t) \neq g(t) \otimes s(t)$$

$$s(t) \otimes g(t) = g(-t) \otimes s(-t)$$

2.5.1 Correlation Functions and Energy Spec. of Deterministic Analog Energy Signals

$$\rho_{ss}(\tau) = \int_{-\infty}^{+\infty} s(t)s^*(t+\tau)dt = s(t) \otimes s(t)$$

$$\mathcal{F} \; \circ\!\!-\!\!\bullet$$

$$R_{ss}(\omega) = S(-\omega)S^*(\omega) = \left|S(-\omega)\right|^2$$

$$\rho_{ss}(\tau) = s(t) \otimes s(t) = s(-t) * s(t)$$

$$\rho_{ss}(t) = s(t) \otimes s(t) = s(-t) * s(t)$$

$$\mathcal{F} \; \circ\!\!-\!\!\bullet$$

$$R_{ss}(\omega) = S^*(\omega)S(\omega) = \left|S(\omega)\right|^2$$

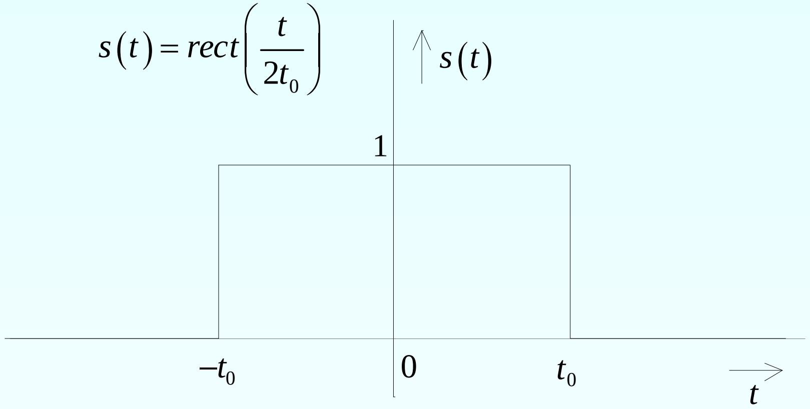
2.5.1 Correlation Functions and Energy Spec. of Deterministic Analog Energy Signals

$$\rho_{ss}(0) = \int_{-\infty}^{+\infty} s(t)s(t)dt = E_s \quad \text{(signal energy of s(t))}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} R_{ss}(\omega) e^{j\omega t} d\omega \bigg|_{t=0} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |S(\omega)|^2 d\omega = \int_{-\infty}^{+\infty} |S(2\pi f)|^2 df$$

2.5.1 Correlation Functions and Energy Spec. of Deterministic Analog Energy Signals

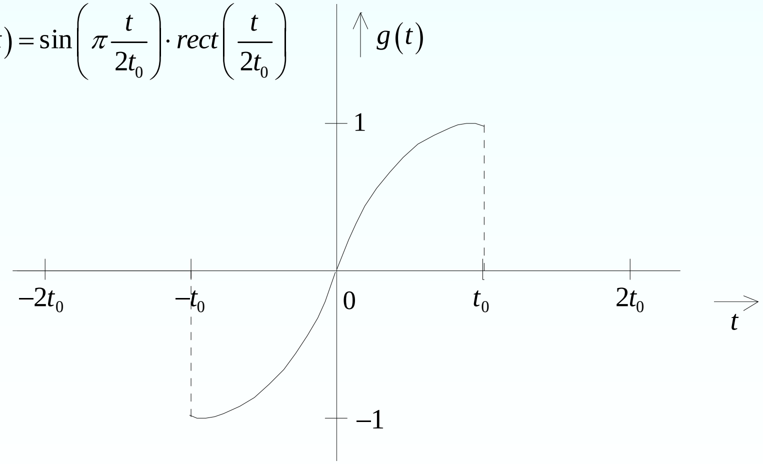
$s(t) = rect\left(\frac{t}{2t_0}\right)$



$$\rho_{sg}(\tau) = s(t) \otimes g(t) = \int_{-\infty}^{+\infty} s(t) g^*(t+\tau) dt$$

$$= 2\frac{t_0}{\pi} \sin\left(\frac{\pi\tau}{2t_0}\right) rect\left(\frac{\tau}{4t_0}\right)$$

$g(t) = \sin\left(\pi \frac{t}{2t_0}\right) \cdot rect\left(\frac{t}{2t_0}\right)$



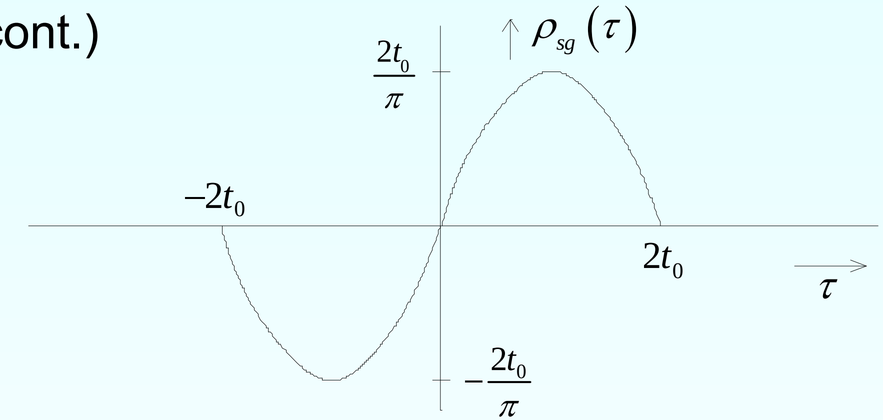
$$\rho_{gs}(\tau) = \rho_{sg}^*(-\tau) = \rho_{sg}(-\tau)$$

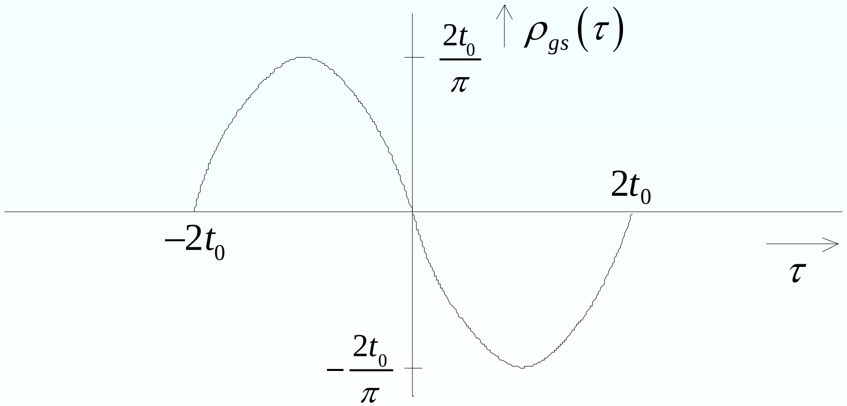
$$= -2\frac{t_0}{\pi} \sin\left(\frac{\pi\tau}{2t_0}\right) rect\left(\frac{\tau}{4t_0}\right)$$

Two deterministic energy signal s(t) and g(t)

2.5.1 Correlation Functions and Energy Spec. of Deterministic Analog Energy Signals

Example 1: (cont.)





The resulted Cross-correlation function for s(t) and g(t)

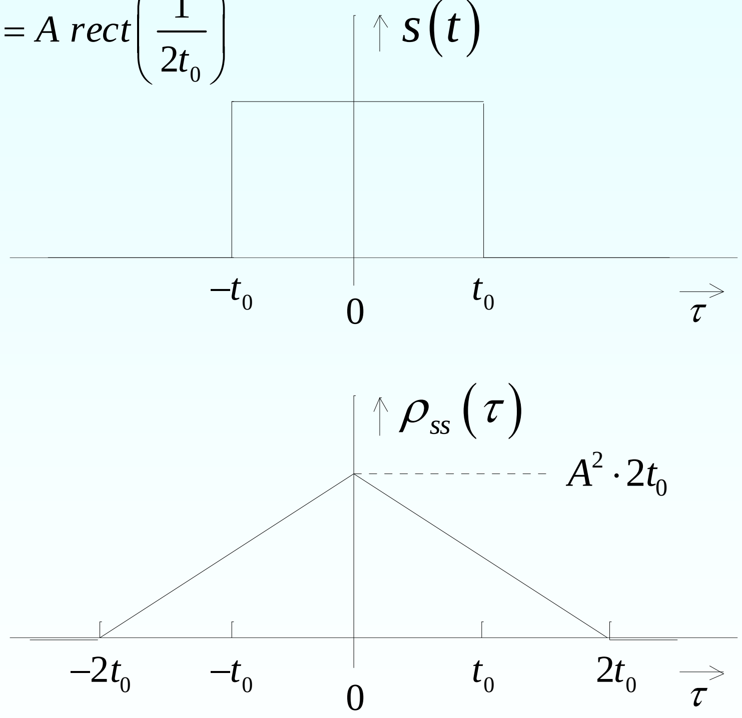
2.5.1 Correlation Functions and Energy Spec. of Deterministic Analog Energy Signals

$$\rho_{sg}(\tau) = s(t) \otimes g(t) = \int_{-\infty}^{+\infty} s(t) g^*(t+\tau) dt = 2\frac{t_0}{\pi} \sin\left(\frac{\pi\tau}{2t_0}\right) rect\left(\frac{\tau}{4t_0}\right)$$

$$\rho_{gs}(\tau) = \rho_{sg}^*(-\tau) = \rho_{sg}(-\tau) = -2\frac{t_0}{\pi} \sin\left(\frac{\pi\tau}{2t_0}\right) rect\left(\frac{\tau}{4t_0}\right)$$

2.5.1 Correlation Functions and Energy Spec. of Deterministic Analog Energy Signals

$$s(t) = A\, rect\left(\frac{1}{2t_0}\right)$$



2.5.1 Correlation Functions and Energy Spec. of Deterministic Analog Energy Signals

$$\rho_{ss}(\tau) = s(t) \otimes s(t) = s(-t) * s^*(t) = A.rect\left(-\frac{t}{2t_0}\right) * A.rect\left(\frac{t}{2t_0}\right)$$

$$= A^2\, 2t_0 \Lambda\left(\frac{\tau}{4t_0}\right)$$

$\mathcal{F}$ ⫯

$$R_{ss}(\omega) = 4A^2 t_0^2 si^2(\omega t_0)$$