Chapter 4:

# Discrete Systems


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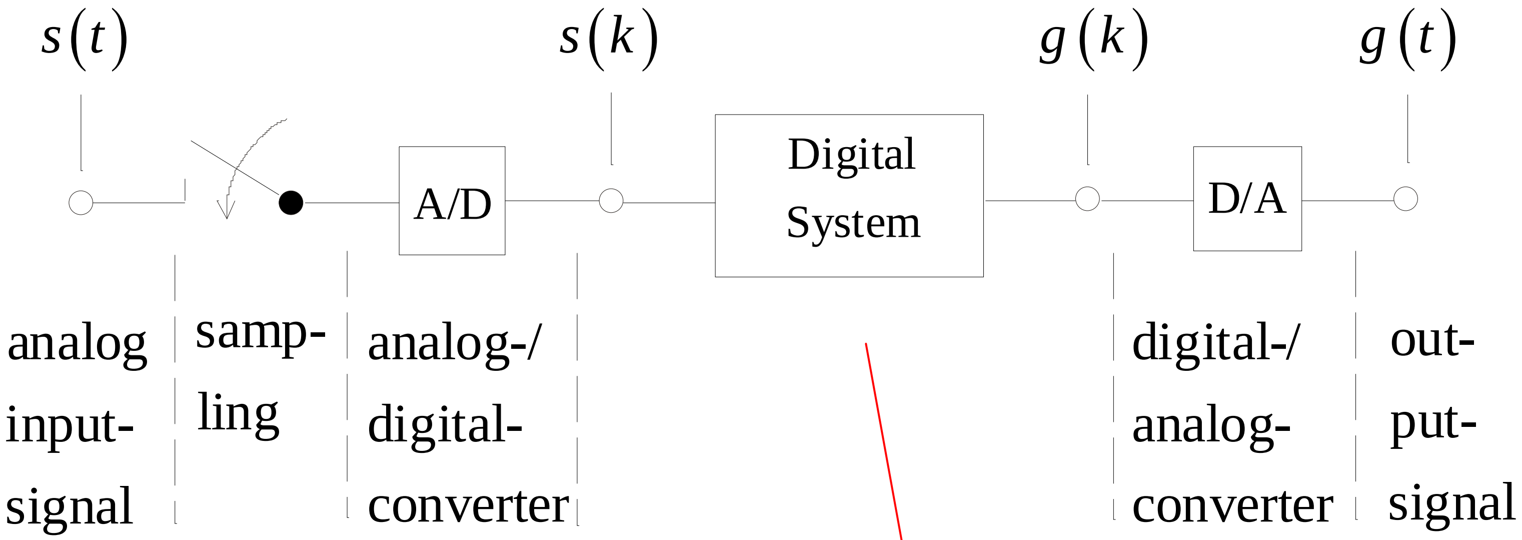
# 4.1 Introduction

General scheme of such a digital signal processing:

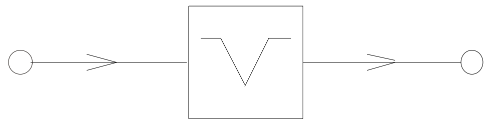



$$\{s(k)\} \rightarrow \{g(k)\} = T\left[\{s(k)\}\right] \quad \text{where} \quad k = -\infty \ldots +\infty$$

# 4.1 Introduction

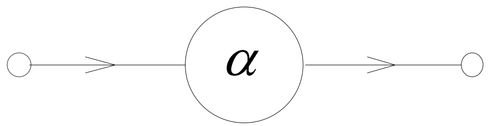
In practics, such a system is represented by an arithmetical network, consisting of the following elements:

**Delay:**



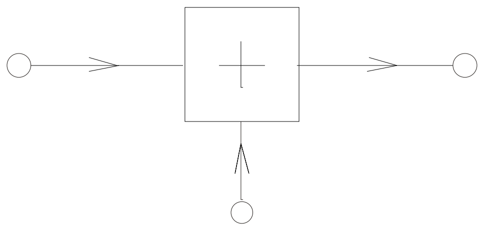
$s(k-1) \quad or \quad s(k-1) = V[s(k)]$

**Multiplier:**




$\alpha \cdot s(k)$

**Summer:** $s_1(k)$



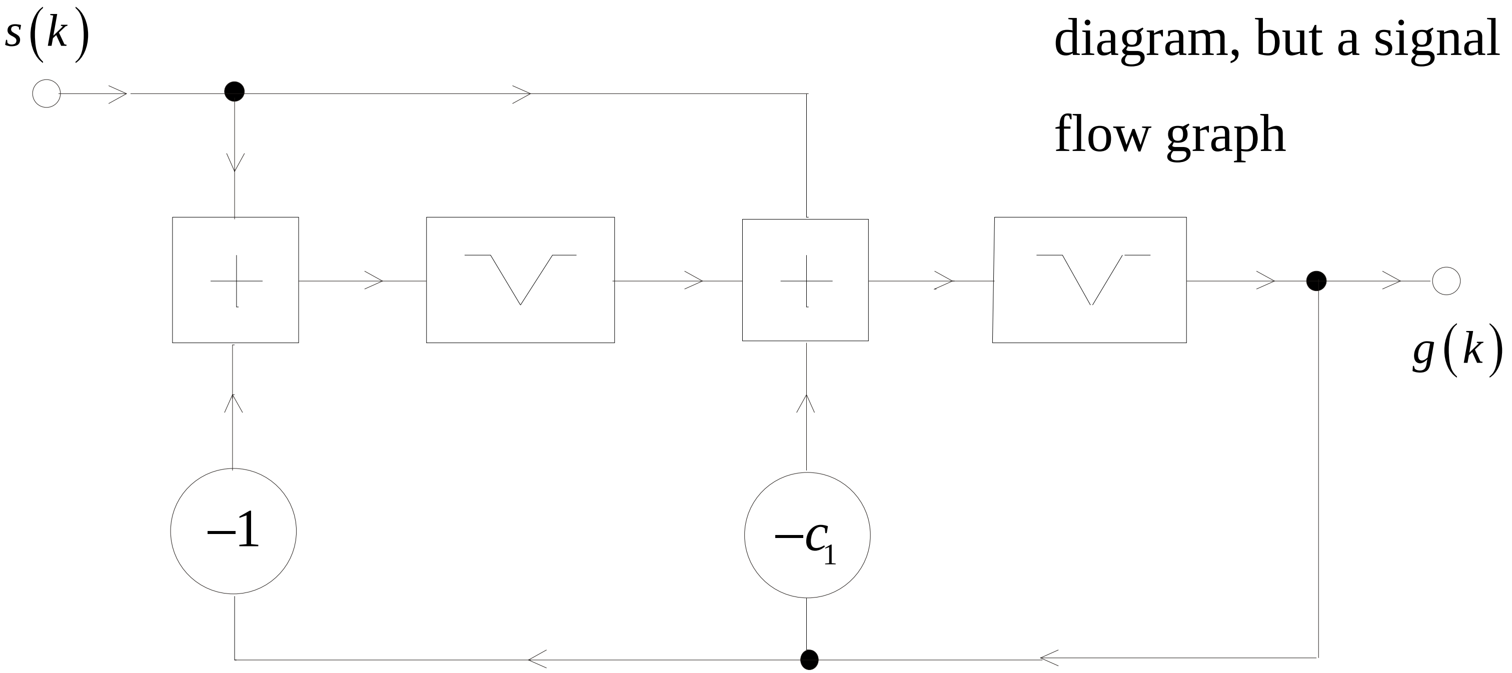
$s_1(k) + s_2(k)$

$s_2(k)$

# 4.1 Introduction

Example:

Note: This is not a circuit diagram, but a signal flow graph




**Signal Flow Graph of a Digital Filter**

# 4.2 Linear, Time-Invariant Systems

# 4.2.1 Difference Equations

The most important class of discrete LTI-systems is the one, describable by a n-th order equation of the following :

$$g(k)=\sum_{v=0}^{N}a_v s(k-\alpha)-\sum_{v=1}^{N}b_v g(k-v)$$

# 4.2.1 Difference Equations

Definition: A discrete-time LTI-system is called recursive if the calculation of each output value $g(k)$ from the preceding output values $g(k - v)$ with $v > 0$, which were determined before is possible.

Definition: A causal digital LTI-system is called non-recursive, if the calculation of each output value $g(k)$ is possible without the use of the before calculated output signals $g(k - v)$ with $v > 0$

# 4.2.1 Difference Equations

**Example:** Given is a 2nd order discrete LTI-system with:

$$g(k) = a_0 s(k) + a_1 s(k-1) + a_2 s(k-2) - b_1 g(k-1) - b_2 g(k-2)$$

# 4.2.1 Difference Equations

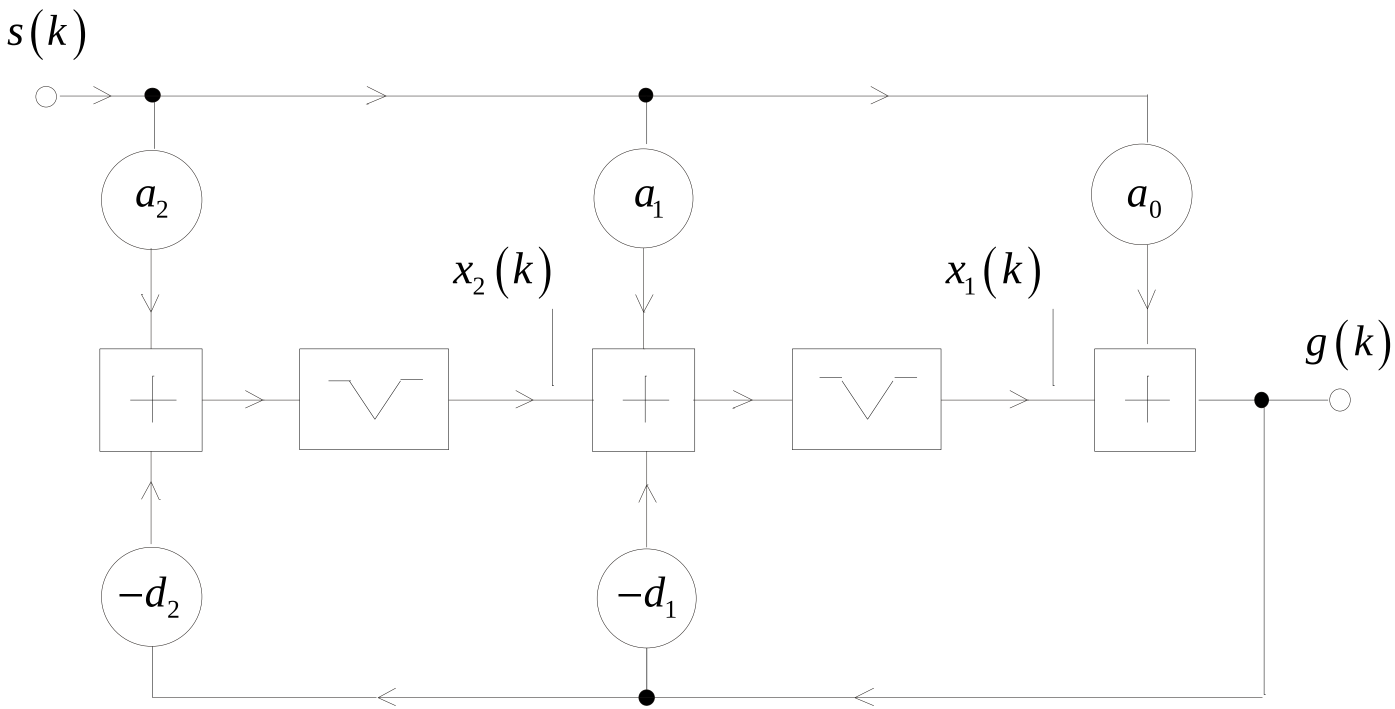



Figure 4.4: Structure of a 2nd Order (IIR-)Filter

# 4.2.2 The Discrete Impulse Response

Definition:

The impulse response is the response $g(k)$ of the system to $s(k)=\gamma_0(k)$

Denoted by: $h(k)$

The answer of the system to any causal excitation s(k) with s(k) $\equiv$ 0 for k < 0

$$g(k)=\sum_{\nu=0}^{+\infty} h(\nu)\cdot s(k-\nu)=h(k)*s(k)$$ → discrete convolution

# 4.2.3 The Discrete Transfer Function $H_z(z)$

* One can derive:

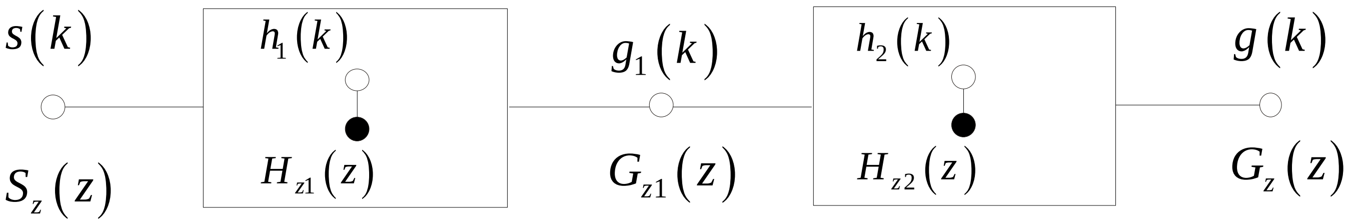
$$g(k) = \sum_{\nu=0}^{+\infty} h(\nu) \cdot s(k-\nu) = h(k) * s(k) \quad \circ\!\!-\!\!\bullet \quad G_z(z) = H_z(z) \cdot S_z(z)$$

$$\Leftrightarrow \quad H_z(z) = \frac{G_z(z)}{S_z(z)}$$

* The discrete transfer function is the z-transform of the impulse response {h(k)}

$$H_z(z) = \sum_{\nu=0}^{+\infty} h(\nu) \cdot z^{-\nu}$$

* With an effectless chain of two discrete LTI-systems results:



# 4.2.3 The Discrete Transfer Function $H_z(z)$

With the equations:

$$g_1(k) = h_1(k) * s(k) \quad \text{and} \quad g(k) = g_1(k) * h_2(k)$$

follows: $$g(k) = g_1(k) * h_2(k) = \left[h_1(k) * s(k)\right] * h_2(k)$$

or $$g(k) = h_1(k) * h_2(k) * s(k) \quad ○\!\!-\!\!● \quad G_z(z) = \underbrace{H_{z1}(z) \cdot H_{z2}(z)}_{H_z(z)} \cdot S_z(z)$$

$$G_z(z) = H_z(z) \cdot S_z(z) \longleftarrow H_z(z) = H_{z1}(z) H_{z2}(z)$$

# 4.2.3 The Discrete Transfer Function $H_z(z)$

The causality requires: $h(k) \equiv 0 \quad \text{for} \quad k < 0,$ thus

$$s(k) = g(k) \equiv 0 \quad \text{for} \quad k < 0$$

and the z-transform of both sides of the difference equation is then:

$$\sum_{\alpha=0}^{N} a_\alpha \cdot S_z(z) \cdot z^{-\alpha} = G_z(z) + \sum_{\beta=1}^{N} b_\beta \cdot G_z(z) \cdot z^{-\beta}$$

Rewriting this formula gives:

$$H_z(z) = \frac{G_z(z)}{S_z(z)} = \frac{\sum_{\alpha=0}^{N} a_\alpha \cdot z^{-\alpha}}{1 + \sum_{\beta=1}^{N} b_\beta \cdot z^{-\beta}}$$

# 4.2.3 The Discrete Transfer Function $H_z(z)$

The second form of the discrete transfer function *Hz(z)* is a rational function of the variable z:

$$H_z(z) = \frac{\text{Numerator polynom in z}}{\text{Denominator ploynom in z}} = \frac{P(z)}{Q(z)}$$

A third form of the discrete transfer function is based on the zeroes of the numerator and the denominator poly nom:

$$H_z(z) = K \cdot \frac{\prod_{\mu=1}^{m} (z - z_{0\mu})}{\prod_{\nu=1}^{n} (z - z_{\infty\mu})}$$

Roots of the of the numerator or the zeros

Roots of the denominator or the poles

# 4.2.4 General Properties of $H_z(z)$

1. Properties of system coefficients and of poles and zeros:

➡ The coefficients respectively are real constants. The zeros and poles are either real or conjugate complex:

$$z_{0_{\mu_1}} = \left|z_{0_{\mu_1}}\right| e^{j\psi_{0_{\mu_1}}} \quad \text{where } z_{0_{\mu_2}} = z^*_{0_{\mu_1}} = \left|z_{0_{\mu_1}}\right| e^{-j\psi_{0_{\mu_1}}}$$

$$z_{\infty_{\nu_1}} = \left|z_{\infty_{\nu_1}}\right| e^{j\psi_{\infty_{\nu_1}}} \quad \text{where } z_{\infty_{\nu_2}} = z^*_{\infty_{\nu_1}} = \left|z_{\infty_{\nu_1}}\right| e^{-j\psi_{\infty_{\nu_1}}}$$

2. Stability: (BIBO: bounded input bounded output criterion)

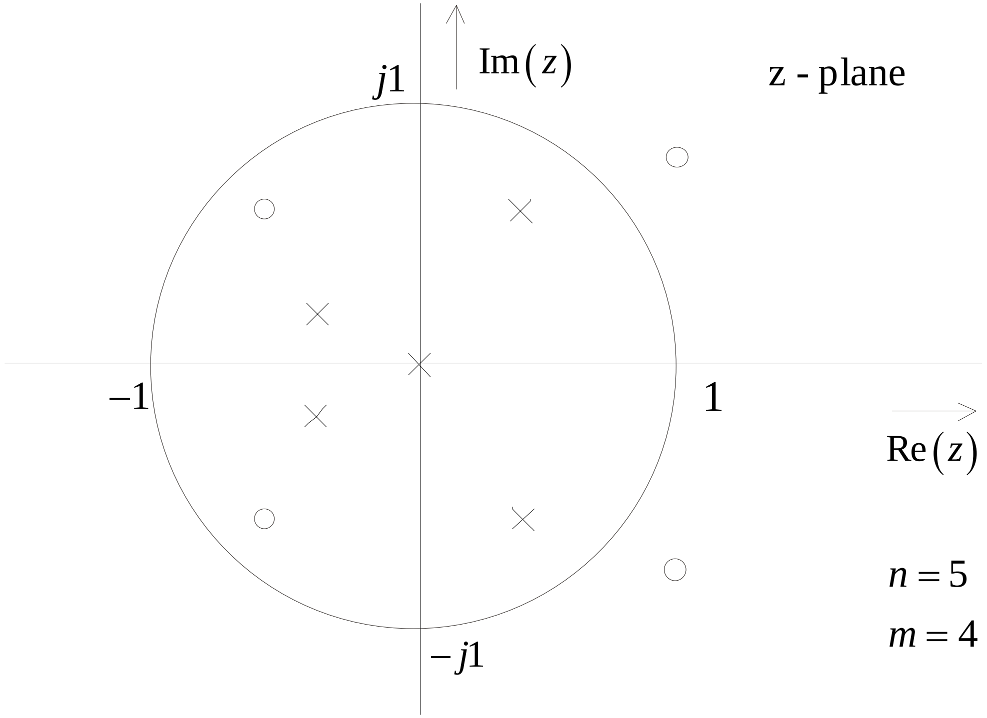
A discrete system obviously is stable, if

any bounded input signal $\left|s(k)\right| < M_1 < \infty \quad \forall k$ causes

a bounded output signal $\left|g(k)\right| < M_2 < \infty$

# 4.2.4 General Properties of $H_z(z)$

Or the stability is given, if the equation: $\left| z_{\infty_\nu} \right| < 1 \quad \forall \nu$




**Poles-Zeroes diagram of a real, causal and stable system**

# 4.2.5 Behaviour of $H_z(z)$ on the unit circle

If we observe the $H_Z(z)$ for any point z on the unit circle with $|z| = 1$ and $z=e^{j\omega T}$

we obtain the so-called frequency response.

$$H_z(e^{j\omega T}) = \frac{\sum_{\alpha=0}^{N} a_\alpha e^{-j\mu\omega T}}{1+\sum_{\beta=1}^{N} b_\beta e^{-j\mu\omega T}} = K \frac{\prod_{\mu=1}^{m}(e^{j\omega T} - z_{0_\mu})}{\prod_{\nu=1}^{n}(e^{j\omega T} - z_{\infty_\nu})}$$

→ periodic function with $\omega T = 2\pi$ or $\omega = \dfrac{2\pi}{T}$

Frequency normalised function:

$$H_z(e^{j\omega T}) = H_a(\omega) \quad \bullet\!\!-\!\!\circ \quad h(t)\sum_{i=-\infty}^{+\infty} \delta(t - iT)$$

# 4.2.5 Behaviour of $H_z(z)$ on the unit circle

In general :

$$G_z(z) = H_z(z)S_z(z) \quad \Rightarrow \quad G_z(e^{j\omega T}) = H_z(e^{j\omega T})S_z(e^{j\omega T})$$

A normalized representation using $\omega T = \Omega \;\; or \;\; f = \Omega \cdot 2\pi$, leads to:

$$H_z(e^{j\omega T}) = H_z(e^{j\Omega}) = H_N(\Omega) = K \frac{\prod_{\mu=1}^{m}(e^{j\Omega} - z_{0_\mu})}{\prod_{\nu=1}^{n}(e^{j\Omega} - z_{\infty_\nu})}$$

Magnitude $|H_N(\Omega)|$, phase $\varphi_N(\Omega)$ and group delay $\tau(\Omega)$

can be determined as:

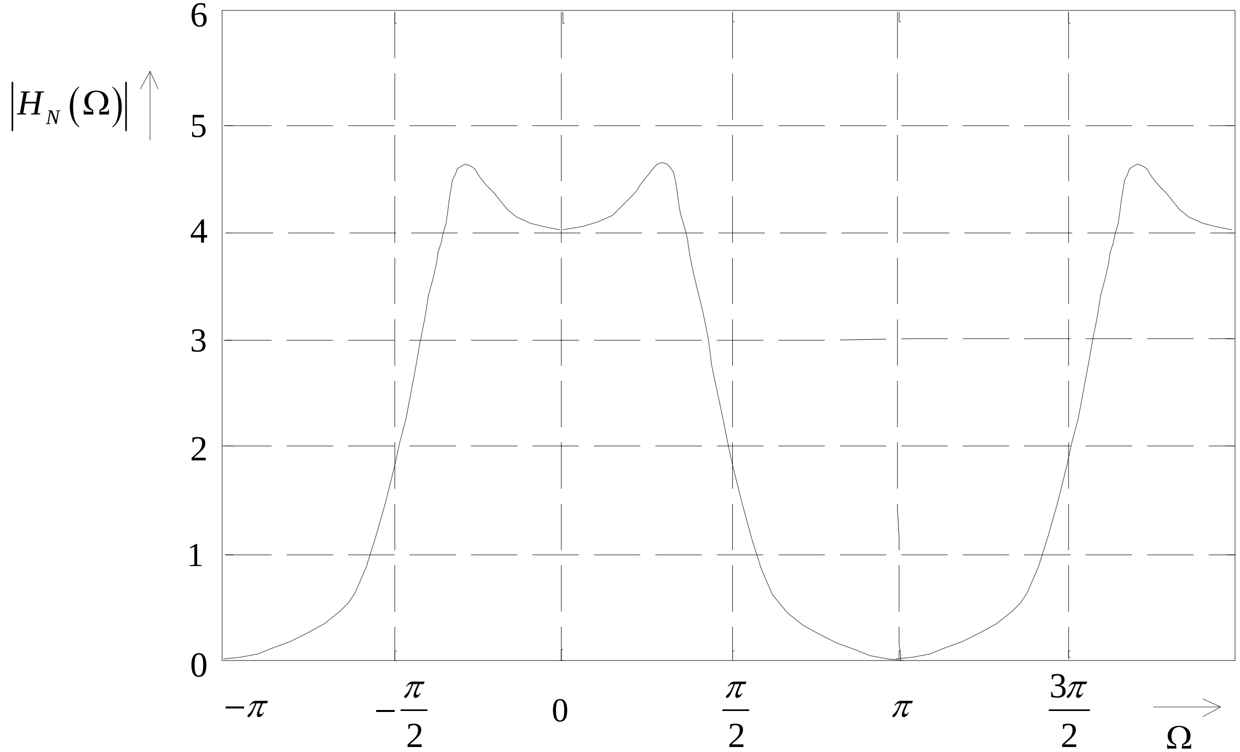
$$H_N(\Omega) = |H_N(\Omega)| e^{j\varphi_N(\Omega)}$$

$$\tau(\Omega) = -\frac{d\varphi_N(\Omega)}{d\Omega}$$

# 4.2.5 Behaviour of $H_z(z)$ on the unit circle

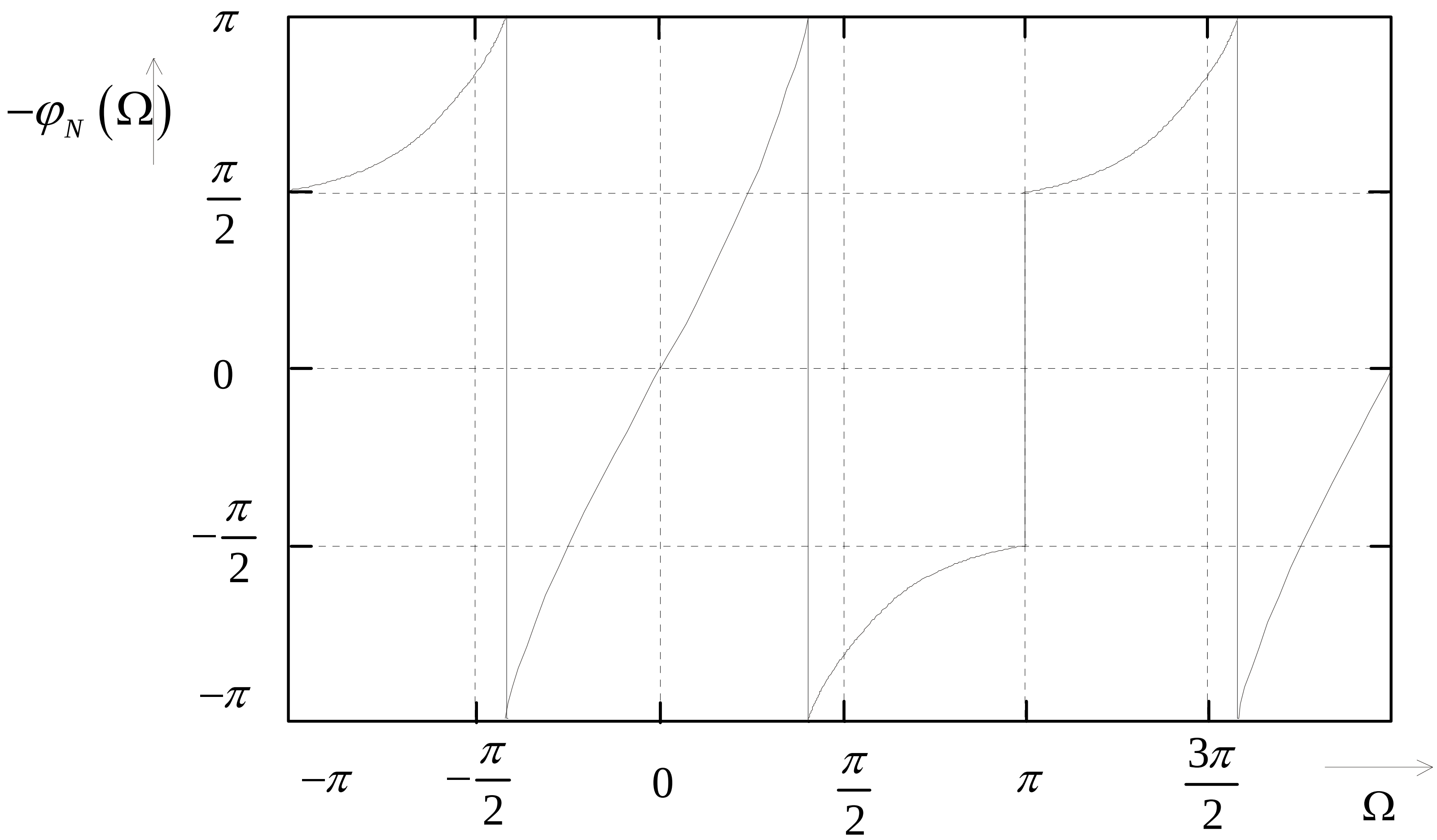
**Example:** 3rd order system with the discrete transfer function

$$H(z) = \frac{(z+1)(z+0.2)}{(z-0.3)(z-0.3-j0.6)(z-0.3+j0.6)}$$



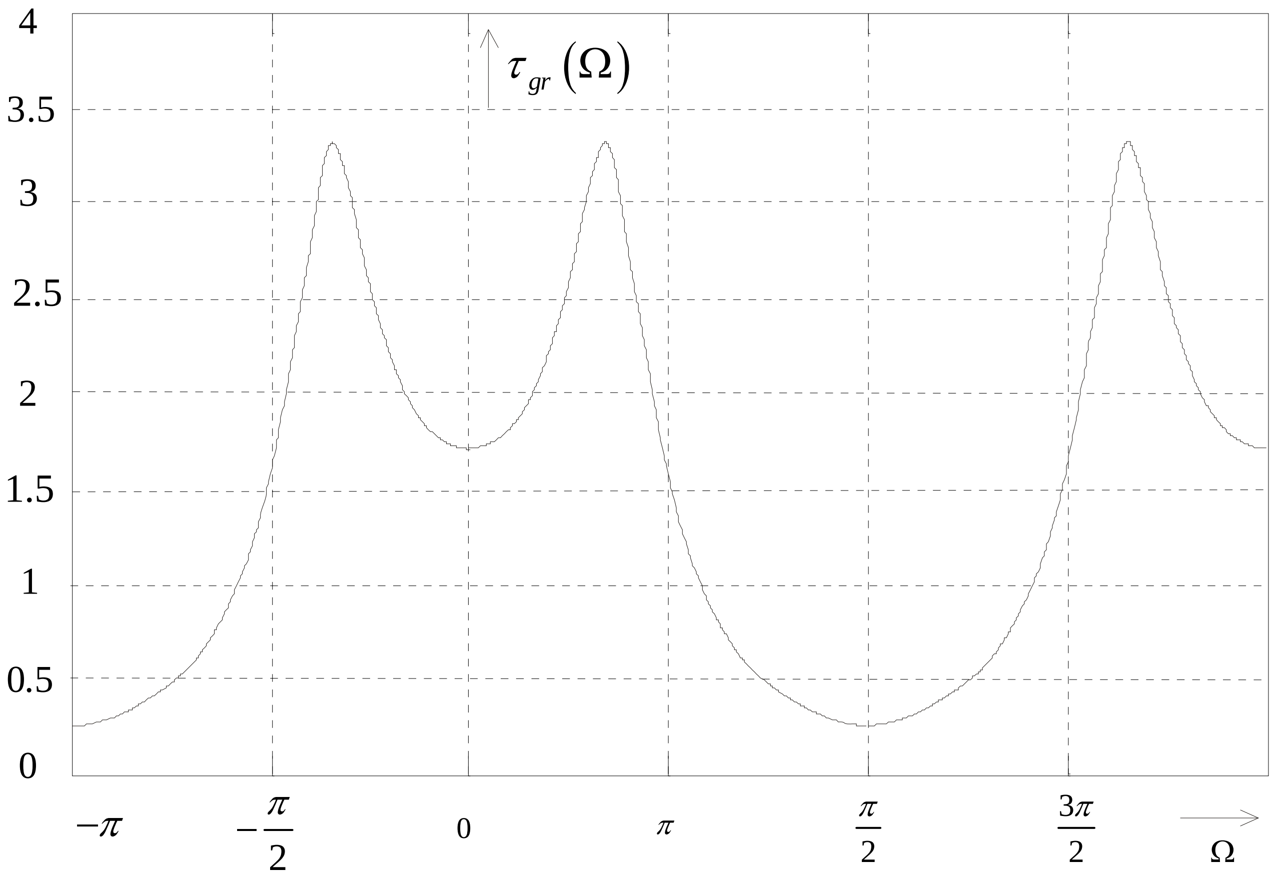

Magnitude of discrete transfer function of 3rd order system

# 4.2.5 Behaviour of $H_z(z)$ on the unit circle



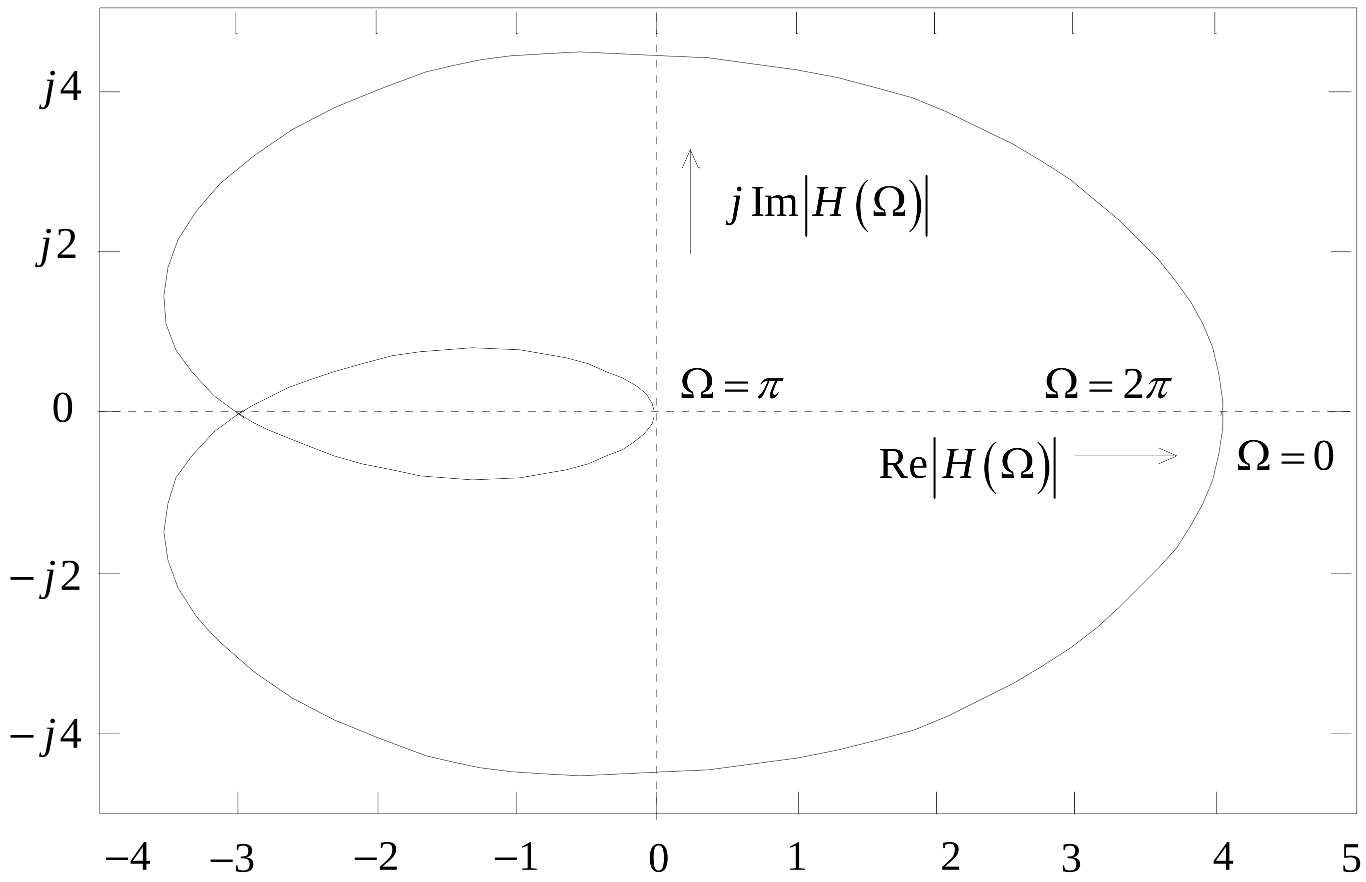

Phase of discrete transfer function of 3rd order system

# 4.2.5 Behaviour of $H_z(z)$ on the unit circle




Envelope delay of a 3rd order system

# 4.2.5 Behaviour of $H_z(z)$ on the unit circle




Locus of the discrete transfer function of 3rd order system

# 4.2.5 Behaviour of $H_z(z)$ on the unit circle

If the regarded system is causal and stable then:

1. $H_L(p)$ is analytically regular for $\operatorname{Re} p > 0$ and $H_Z(z)$ is analytically regular for $|z| < 1$

2. $H_L(j\omega) = H(\omega)$ shows the frequency behaviour of analog filter

   $\mathrm{H}_Z(e^{j\omega t}) = H_Z(e^{j\Omega}) = H_a(\omega)$ shows the frequency behaviour of digital filter.

with
$$H_z(e^{j\Omega}) = \operatorname{Re}\left\{H_z(e^{j\Omega})\right\} + j\operatorname{Im}\left\{H_z(e^{j\Omega})\right\}$$
or

$$H_{Na}(\Omega) = \operatorname{Re}\left\{H_{Na}(\Omega)\right\} + j\operatorname{Im}\left\{H_{Na}(\Omega)\right\}$$



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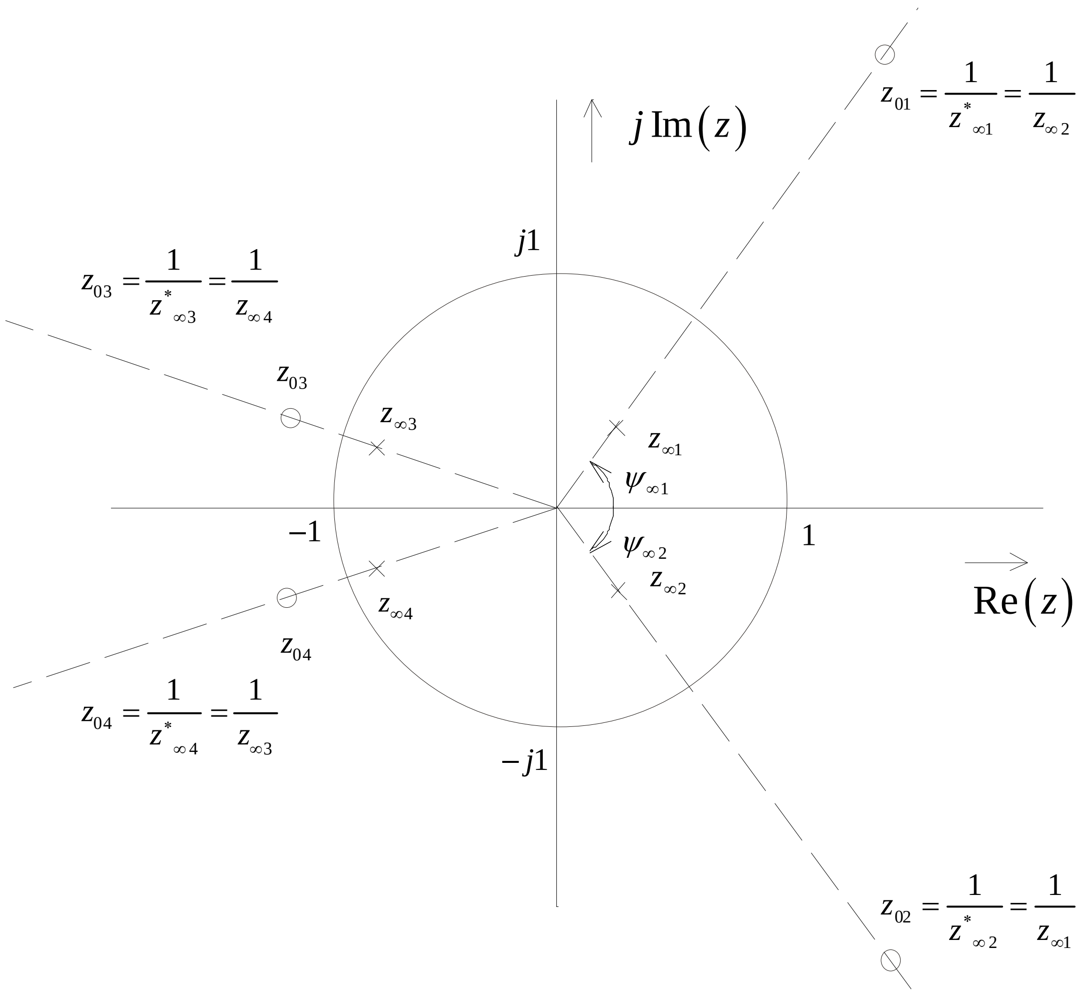
# 4.2.6 All-Pass Filters




**Pole-zero diagram of an all-pass in the z-plane**

# 4.2.6 All-Pass Filters

One gets in case of: $|z_{\infty\nu}| = \dfrac{1}{|z_{0\nu}|}$ and $\psi_{\infty\nu} = \psi_{0\nu}$

with $z_{\infty\nu} = |z_{\infty\nu}| e^{\psi_{\infty\nu}}$ and $z_{0\nu} = |z_{0\nu}| e^{\psi_{0\nu}}$

**the phase :**

$$\varphi_{Na}(\Omega) = \sum_{\nu=1}^{n} \arctan \frac{\left(1-|z_{\infty\nu}|^2\right)\sin(\Omega - \Psi_{\infty\nu})}{(1+|z_{\infty\nu}|^2)\cos(\Omega - \Psi_{\infty\nu}) - 2|z_{\infty\nu}|}$$

**and the group delay:**

$$\tau_{Nga}(\Omega) = \sum_{\nu=1}^{n} \frac{1-|z_{\infty\nu}|^2}{1-2|z_{\infty\nu}|\cos(\Omega - \Psi_{\infty\nu}) + |z_{\infty\nu}|^2} \qquad \text{where} \quad |z_{\infty\nu}| < 1 \quad \text{for all } \nu$$

# 4.2.7 Minimum-phase Systems

A discrete system can be devided into an all-pass and a minimum-phase system

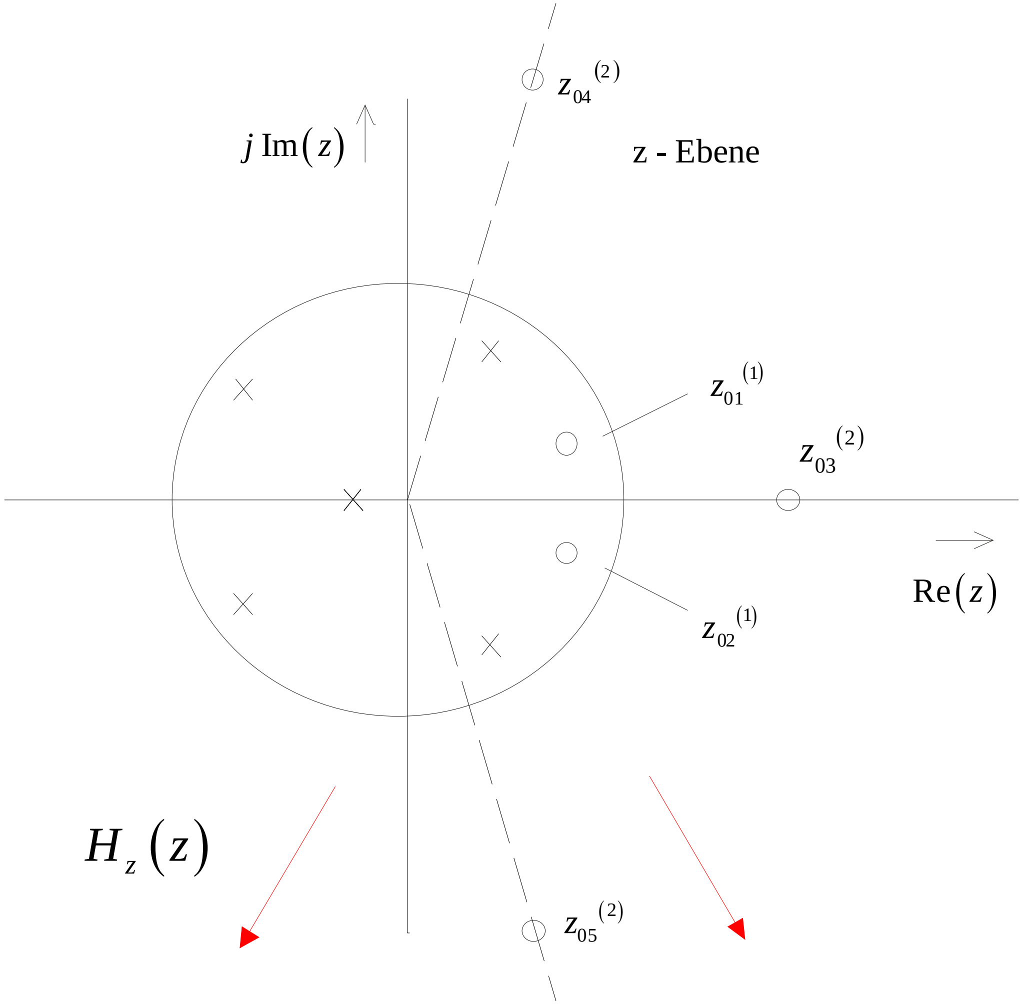
Let‘s assume an stable LTI discrete system with the transfer funsction:

$$H_z(z) = K \frac{\prod_{\mu=1}^{m}(z - z_{0\mu})}{\prod_{\nu=1}^{n}(z - z_{\infty\nu})} = K \frac{\prod_{\mu=1}^{m_1}(z - z^{(1)}{}_{0\mu})}{\prod_{\nu=1}^{n}(z - z_{\infty\nu})} \prod_{\mu=m_1+1}^{m}(z - z^{(2)}{}_{0\mu})$$

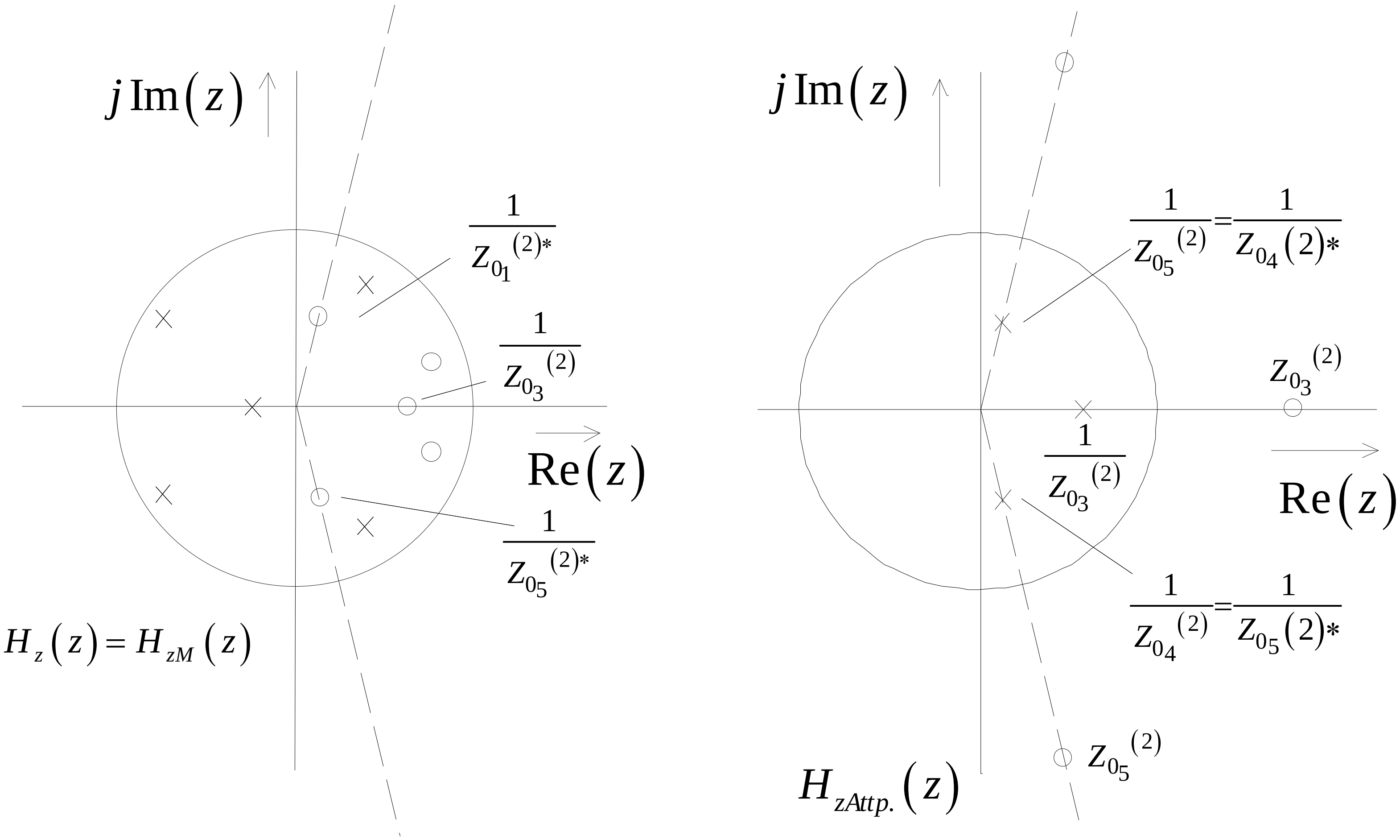
with $\left|z^{(1)}{}_{0\mu}\right| \leq 1$ for $\mu = 1, \ldots, m_1$ the first $m_1$ zeros in the unit-circle

$\left|z^{(2)}{}_{0\mu}\right| > 1$ for $\mu = m_1 + 1, \ldots, m$ other zeros outside the unit-circle

# 4.2.7 Minimum-phase Systems



# 4.2.7 Minimum-phase Systems




**Minimum phase system** **All pass system**

# 4.2.9 Non-Recursive Systems (FIR-Filters)

Non-recursive systems have been defined by:

$$g(k) = \sum_{\alpha=0}^{N} a_\alpha s(k-\alpha) \quad ○\!\!-\!\!● \quad G(z) = \sum_{\alpha=0}^{N} a_\alpha S(z) z^{-\alpha}$$

or

$$H_z(z) = \frac{G(z)}{S(z)} = \sum_{\alpha=0}^{N} a_\alpha z^{-\alpha}$$

**Properties:**

1. Because of $H_z(z) = \sum_{\nu=0}^{\infty} h(\nu) z^{-\nu}$, the impulse response:

$$h(k) = a_k \quad \text{for} \quad k = 0 \ldots N \quad \text{with} \quad h(k) = 0 \quad \text{for} \quad k < 0 \quad \text{and} \quad k > N$$

→ finite duration ⇒ also called FIR (Finite Impulse Response) - systems.

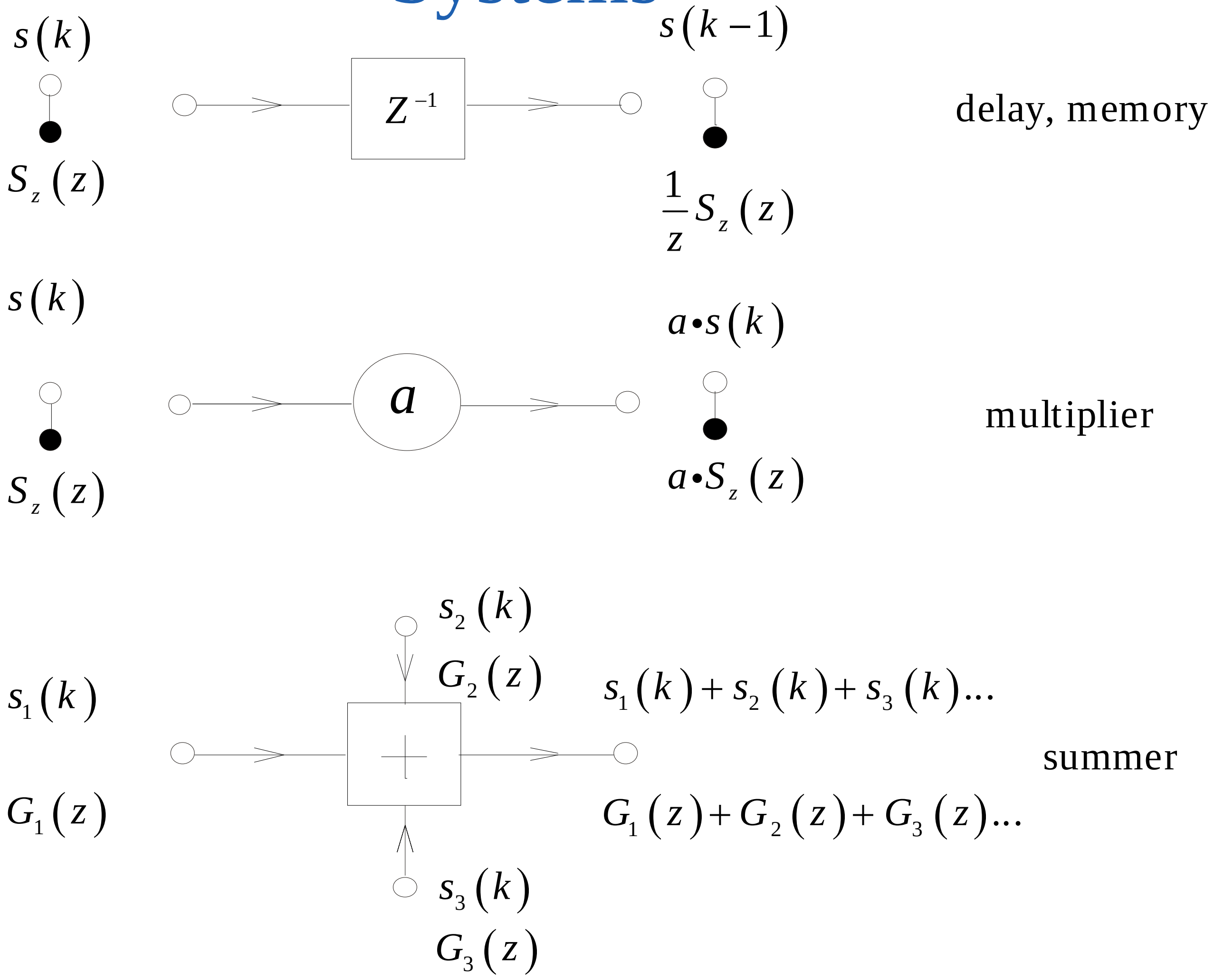
With the output signal: $g(k) = s(k) * h(k) = \sum_{\nu=0}^{n} h(\nu) s(k-\nu)$

# 4.2.9 Non-Recursive Systems (FIR-Filters)

2. From this results: $H_z(z) = \sum_{\alpha=0}^{N} a_\alpha z^{-\alpha} = \dfrac{\sum_{\alpha=0}^{N} a_\alpha z^{N-\alpha}}{z^N}$

Thus non-recursive systems have just an nth order pole at z = 0
→ They are always stable!

# 4.3 System Structures for Discrete LTI-Systems



# 4.3.1 The First Canonical Form of a Discrete System

**A canonical form** is a system structure with a minimized number of memories (delay elements).

From chapter 4.2.1it holds:

$$g(k) = \sum_{v=0}^{N} a_v s(k-v) - \sum_{v=1}^{N} b_v g(k-v)$$

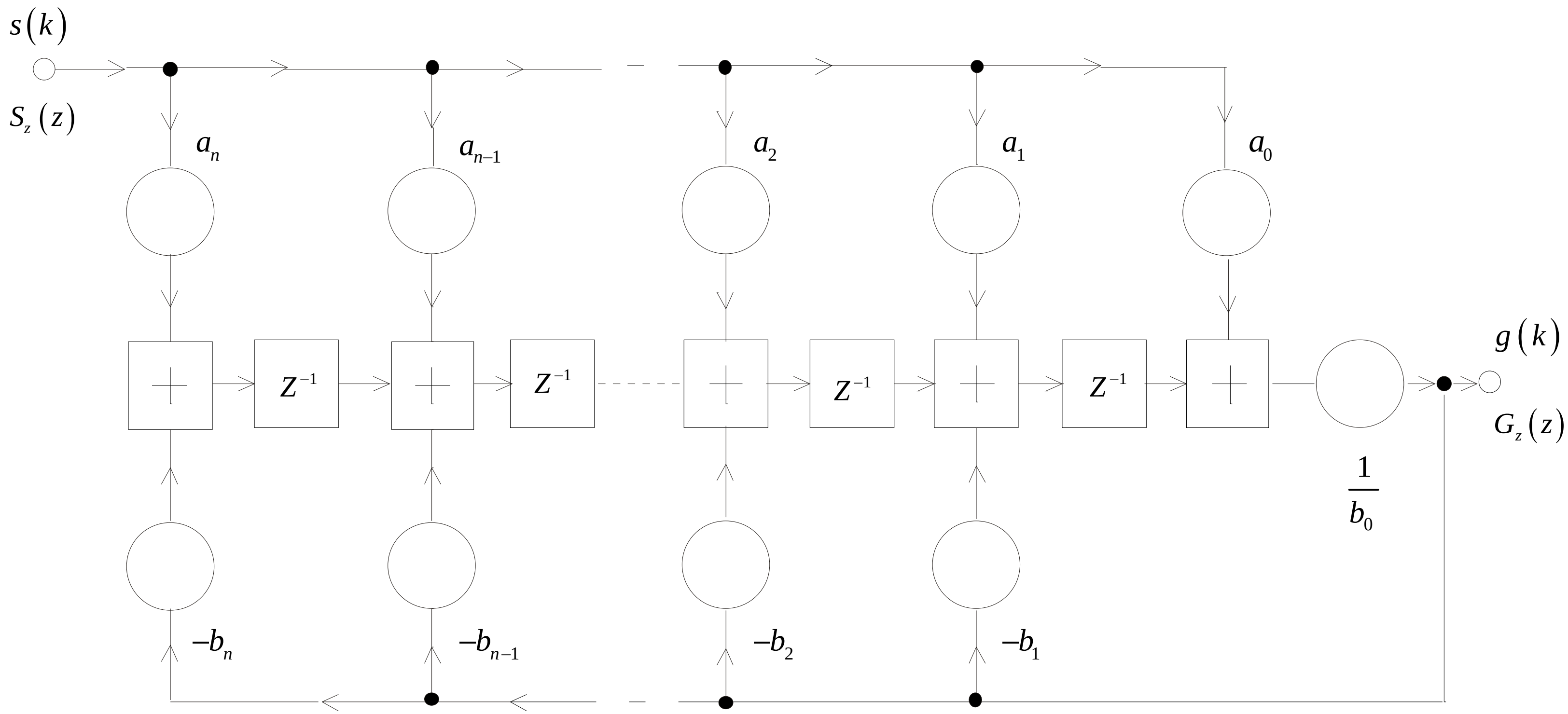
After a z-transform this gives:

$$\circ\!\!-\!\!\bullet$$

$$G_z(z) = a_0 S_z(z) + \sum_{v=1}^{N} \left\{ a_v S_z(z) z^{-v} - b_v G_z(z) z^{-v} \right\}$$

$$= \frac{\sum_{v=0}^{N} a_v S_z(z) z^{-v}}{1 + \sum_{v=1}^{N} b_v G_z(z) z^{-v}}$$

# 4.3.1 The First Canonical Form of a Discrete System




**First Canonical form of a digital filter**

# 4.3.2 The Second Canonical Form of a Discrete Filter

A second canonical form results as follows:

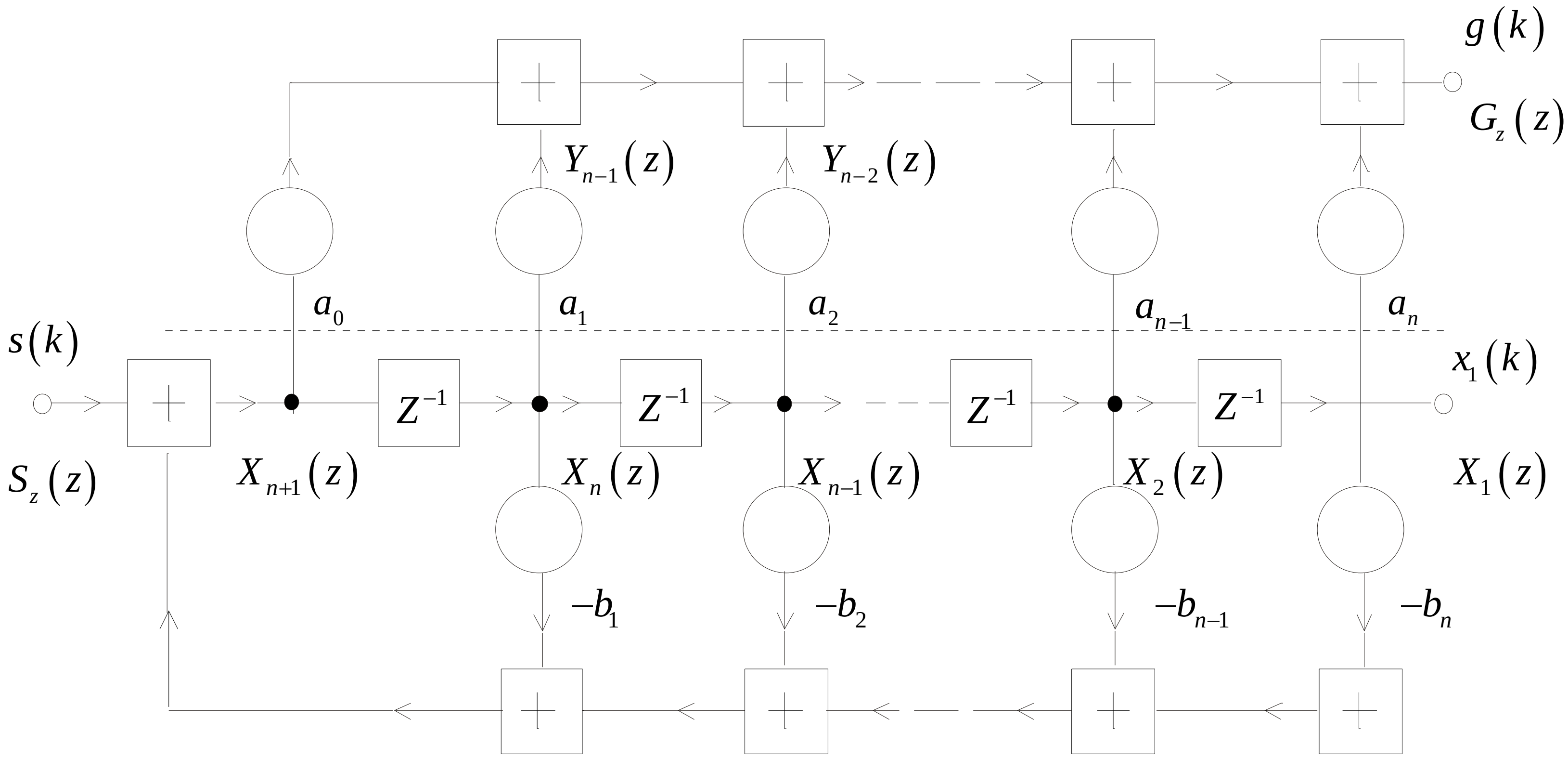
$$G_z(z) = \frac{\sum_{\mu=0}^{n} d_\mu z^\mu}{\sum_{\nu=0}^{n} c_\nu z^\nu} S_z(z)$$

The second form works equal to the first canonical form; this is proved in the following:

- The representation in following figure with

$$H_z(z) = \frac{G_z(z)}{S_z(z)} = \frac{\sum_{\mu=0}^{n} b_\mu z^\mu}{\sum_{\nu=0}^{n} c_\nu z^\nu}$$ equals the one in figure of the 1st Form

# 4.3.2 The Second Canonical Form of a Discrete Filter




**The Second Canonical Form of a Digital Filter**

# 4.3.2 The Second Canonical Form of a Discrete Filter

The following onset is used here:

$$H_z(z) = \frac{\sum_{\nu=0}^{N} a_\nu z^{-\nu}}{1+\sum_{\nu=1}^{N} b_\nu z^{-\nu}} = H_{z1}(z) \cdot H_{z2}(z) = \frac{G_z(z)}{S_z(z)} = \frac{G_z(z)}{X_z(z)} \cdot \frac{X_z(z)}{S_z(z)}$$

$$\mathrm{H}_{z1}(z) = \frac{1}{1+\sum_{\nu=1}^{N} b_\nu z^{-\nu}} = \frac{X_z(z)}{S_z(z)} \Rightarrow X_z(z) = S_z(z) - \sum_{\nu=1}^{N} b_\nu X_z(z)$$

The inverse z-transform gives then: $x\;(k) = s(k) - \sum_{\nu=1}^{N} b_\nu x(k-n)$

# 4.3.2 The Second Canonical Form of a Discrete Filter

$$\mathrm{H}_{z2}(z) = \sum_{\nu=0}^{N} a_\nu z^{-\nu} = \frac{G_z(z)}{X_z(z)} \Rightarrow G_z(z) = \sum_{\nu=1}^{N} a_\nu X_z(z)$$

Another inverse z-transform gives: $g(k) = \sum_{\nu=0}^{n} a_\nu x_\nu(k)$

Combining both differential equations for the two sub systems gives:

$$H_z(z) = \frac{G_z(z)}{S_z(z)} = \frac{\sum_{\nu=0}^{N} a_\nu z^{-\nu}}{1 + \sum_{\nu=1}^{N} b_\nu z^{-\nu}}$$

# 4.3.3 The Third Canonical Form of a Digital System

Example:

A 3rd canonical form is realised by a cascade of a 1st and 2nd order systems.

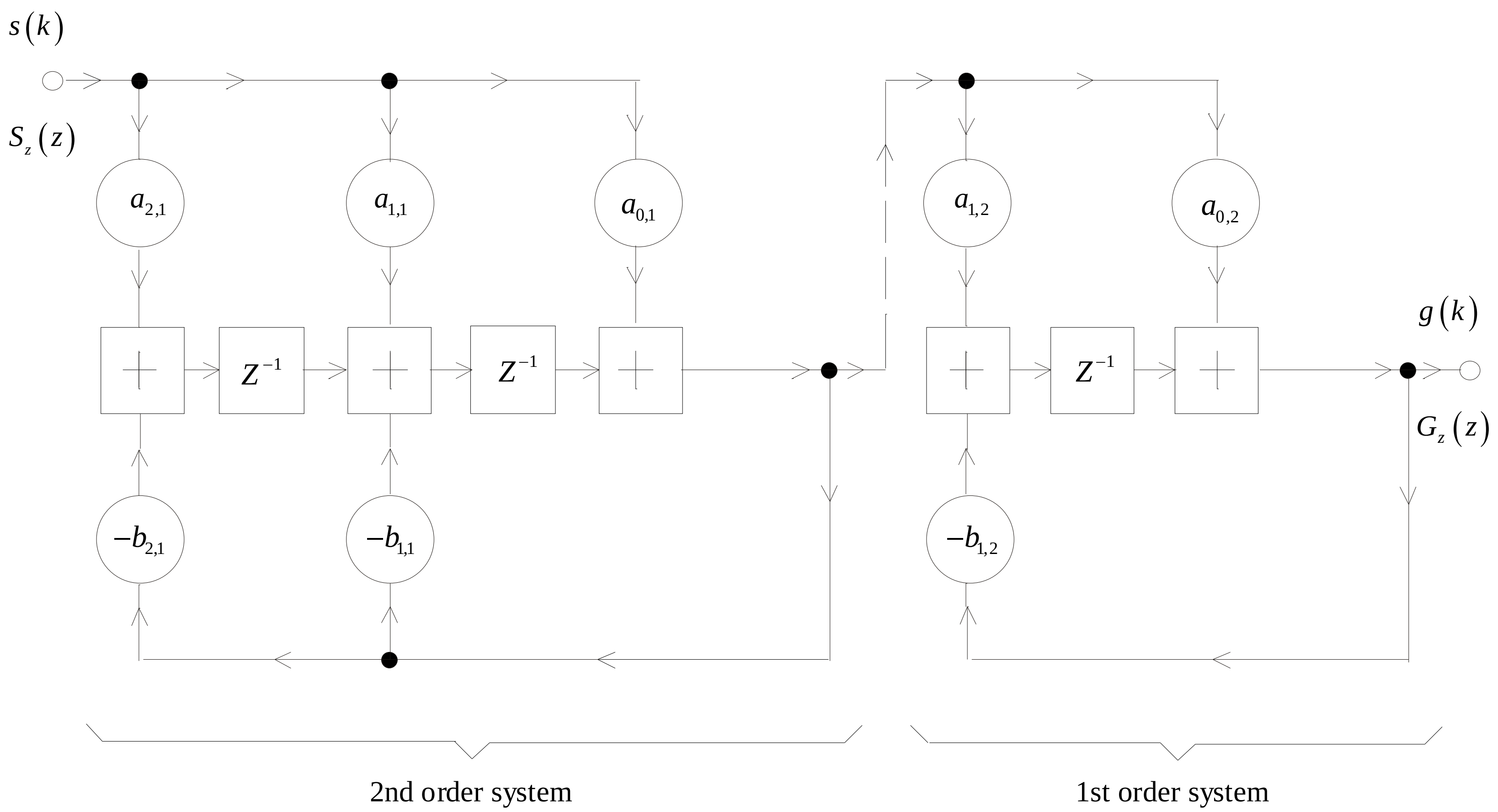
$$H_z(z) = \frac{\prod_{\nu=1}^{M}(z - z_{0\nu})}{\prod_{\nu=0}^{N}(z - z_{\infty\nu})} = \prod_{\gamma=1}^{L} H_{z\gamma}(z)$$

Each 1st order system realises one real-valued pole

Each 2nd order system realises a pair of conjugated complex poles like

$$H_{z\gamma}(z) = \frac{a_{0,\gamma} + a_{1,\gamma} z^{-1}}{1 - b_{1,\gamma} z^{-1}} \quad \text{and} \quad H_{z\gamma}(z) = \frac{a_{0,\gamma} + a_{1,\gamma} z^{-1} + a_{2,\gamma} z^{-2}}{1 - b_{1,\gamma} z^{-1} + b_{2,\gamma} z^{-2}}$$

# 4.3.4 The third Canonical form of a Digital Filter




**Cascade of a 1st and 2nd order system**

# 4.3.4 The Fourth Canonical form of a Digital Filter

Another canonical structure can be obtained by converting the transfer function into a partional fractions sum:

$$H_z(z) = K\frac{\sum_{\mu=0}^{M} d_\mu z^\mu}{\prod_{\nu=1}^{N}(z - z_{\infty\nu})} = R_\infty + \sum_{\nu=1}^{N}\frac{R_\nu}{(z - z_{\infty\nu})}$$

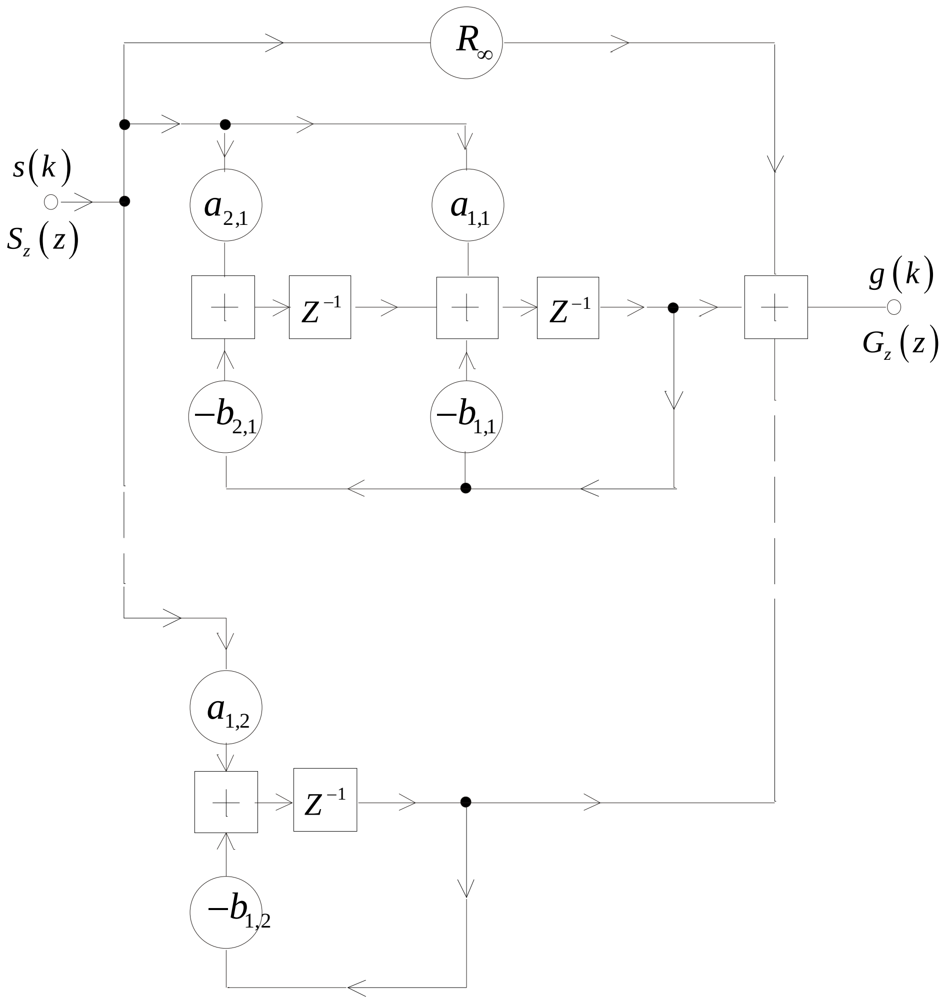
The residues in this simple case follow from:

$$R_\infty = \lim_{z\to\infty} H_z(z) = Kd_M \quad \text{for} \quad \mathrm{M} = N$$

$$R_\nu = \lim_{z\to z_{\infty\nu}}\left\{(z - z_{\infty\nu})H_z(z)\right\} \quad \text{for single poles (real or complex)}$$

➡ **This leads to a parallel connection of the subsystems!**

# 4.3.4 The Fourth Canonical form of a Digital Filter



# 4.3.4 The Fourth Canonical form of a Digital Filter

For any real-valued poles the subsystem is given by:

$$H_{z\nu}(z) = \frac{R_\nu}{z - z_\infty} = \frac{R_\nu z^{-1}}{1 - z_\infty z^{-1}}$$

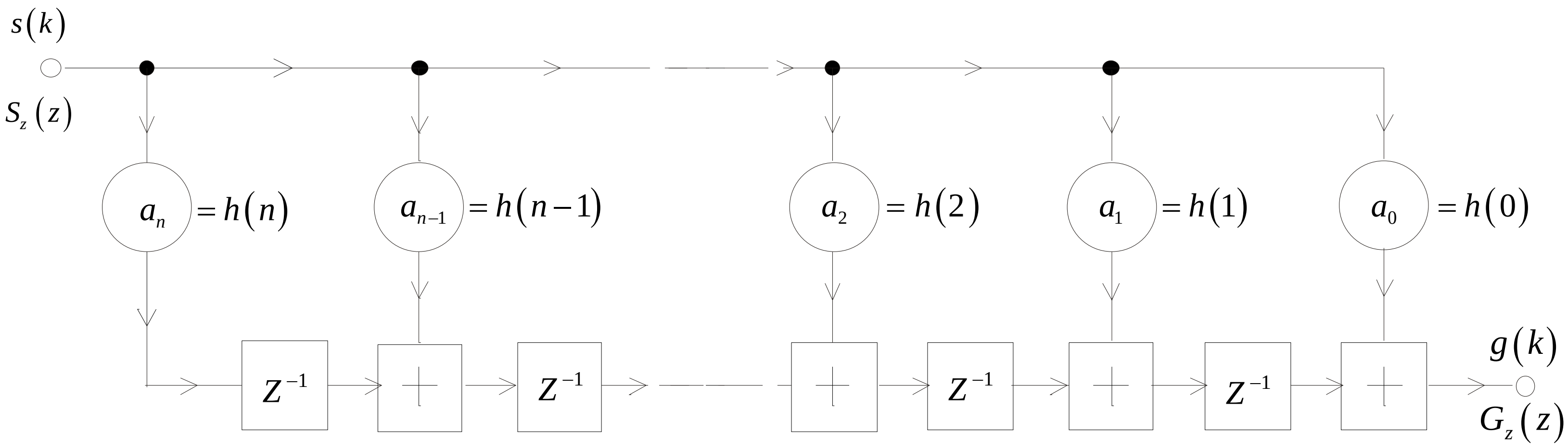
For any pair of conjugate complex poles, the two corresponding terms must be combined for having real-valued coefficients:

$$H_{z\nu}(z) = \frac{2\,\mathrm{Re}\{R_\nu\}z - 2\,\mathrm{Re}\{R_\nu z_{\infty\nu}^*\}}{z^2 - 2\,\mathrm{Re}\{z_{\infty\nu}\}\,z + |z_{\infty\nu}|^2} = \frac{2\,\mathrm{Re}\{R_\nu\}z^{-1} - 2\,\mathrm{Re}\{R_\nu z_{\infty\nu}^*\}z^{-2}}{1 - 2\,\mathrm{Re}\{z_{\infty\nu}\}\,z^{-1} + |z_{\infty\nu}|^2\, z^{-2}}$$

# 4.3.5 System Structures for Non-Recursive Systems (FIR-Filters)

For any of the first three canonical forms, an appropriate non-recursive system can be directly derived from equation

$$g(k) = \sum_{\alpha=0}^{N} a_\alpha s(k-\alpha)$$




**First Canonical Form of a FIR - Filter**