Fundamentals of Electrical Engineering 3

Professor Dr.-Ing. Ingolf Willms

and

Professor Dr. Dr.-Ing. Adalbert Beyer

based on the script of

Professor Dr. Dr.-Ing. Ingo Wolff

Fundamentals of Electrical Engineering 3
Contents

Literature

- Literature for the lecture:

R. Paul	Elektrotechnik 2, Grundlagenbuch Netzwerke Springer-Verlag, Heidelberg 1994
I. Wolff	Grundlagen der Elektrotechnik 4 Vorlesungs-Script

- Alternative Literature :

W. Ameling	Grundlagen der Elektrotechnik II
G. Bosse	Grundlagen der Elektrotechnik IV B.I. Wissenschaftverlag Mannheim, Wien Zürich 1996
R. Unbehauen	Grundlagen der Elektrotechnik I Springer-Verlag, Heidelberg 1994

1 Introduction

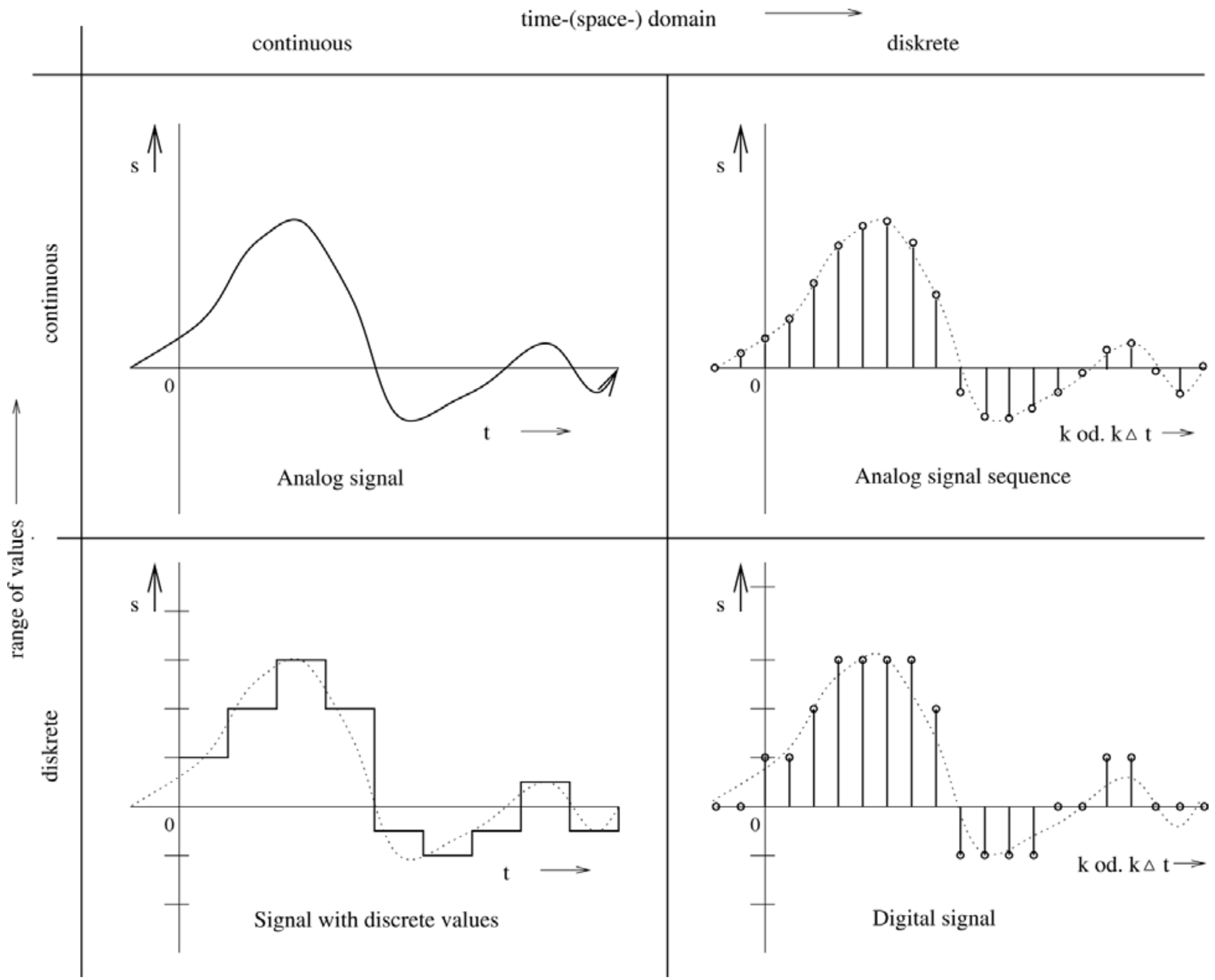
- GET3 contains predominantly theoretical bases to questins of information technology
- Information technology: Developed from computer science and communications
- IT: Efficient data processing, storage and transport
- IT contains 4 groups:
  - Bases and technologies (G1)
  - Structures, procedure, programs (G2)
  - Devices, mechanisms, plants (G3)
  - Applications (G4)

2 Signal theory of determined signals and applications

Chapter overview:

- 2.1 Prefaces
  - Model for the information transfer
  - Signal classes
  - Description and modification of signals
- 2.2 Description of periodic signals (Fourier Series)
  - Approximation of functions with the Fourier series
  - Applications to networks
- 2.3 Description of aperiodic signals (Fourier Transform)
  - The Fourier integral in different forms
  - Examples
  - Properties of the Fourier transform
  - Application to systems (convolution and the Fourier transform)-

2.1 Prefaces


time-(space-) domain
continuous
diskrete
range of values
continuous
diskrete
s
0
t
Analog signal
k od. k∆ t
Analog signal sequence
Signal with discrete values
Digital signal

2.1.1 The Exponential Signal

$$s(t) = e^{j\omega t} = \cos\omega t + j\sin\omega t$$

For voltages it holds:

$$u(t) = \hat{u}\cdot\cos(\omega t + \varphi_u) = \mathrm{Re}\left\{\hat{u}\cdot e^{j(\omega t+\varphi_u)}\right\} = \mathrm{Re}\left\{\underline{u}\cdot e^{j\omega t}\right\} \quad \text{where} \quad \underline{u} = \hat{u}\cdot e^{j\varphi_u}$$

For increasing/decreasing signals:

$$e^{(\sigma+j\omega)t} = e^{\sigma t}\cdot e^{j\omega t} = e^{pt}$$

2.1.2 The Dirac Function

Definition:

$$\Phi(t_0) = \int_{-\infty}^{+\infty} \delta(t - t_0) \cdot \Phi(t) dt$$

Properties:

$$\delta(at) = \frac{1}{|a|} \cdot \delta(t)$$

If $a = -1$ then: $\delta(-t) = \delta(t)$

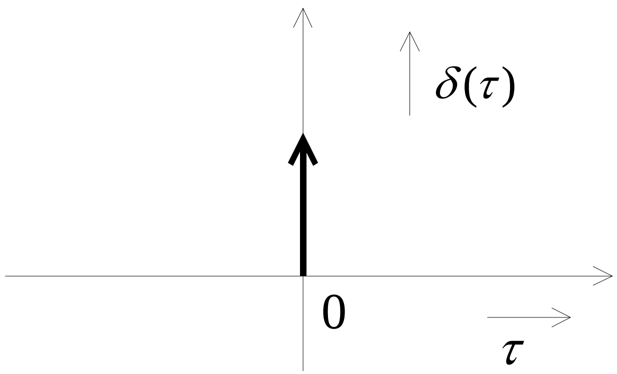
$$s(t) = \int_{-\infty}^{+\infty} \delta(\tau - t) \cdot s(\tau) d\tau$$

See formula above.

2.1.2 The Dirac Function

$$\int_{-\infty}^{+\infty} \delta(\tau)\,d\tau = s(0) = 1 \quad \text{where} \quad \delta(\tau) \equiv 0 \quad \text{for} \quad \tau \neq 0$$

$$\delta(t) = \lim_{T \to 0} \frac{1}{T}\, rect\left(\frac{t}{T}\right)$$

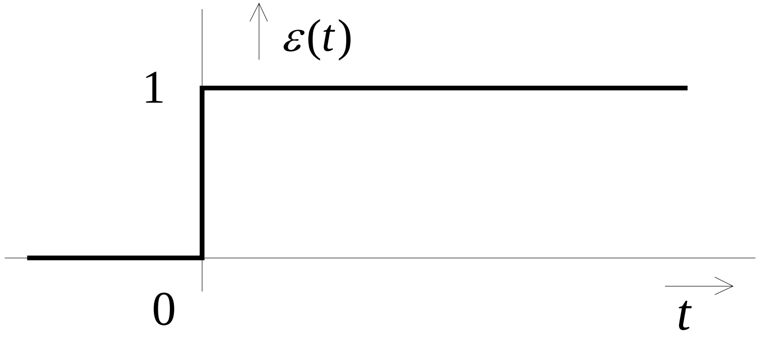


2.1.3 The Step Function

$$\varepsilon(t) = \begin{cases} 0 & \text{for} \quad t < 0 \\ 1 & \text{for} \quad t \geq 0 \end{cases}$$

$$\varepsilon(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$

$$\delta^{(-1)}(\tau) = \varepsilon(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$$



2.1.4 Periodic Signals

General formula:

$$s(t) = s(t + nT) \quad \text{where} \quad n = -\infty, \ldots, -1, +1, \ldots, +\infty$$

Examples:

$$s_2(t) = \sum_{n=-\infty}^{+\infty} s_1(t - nT_0)$$

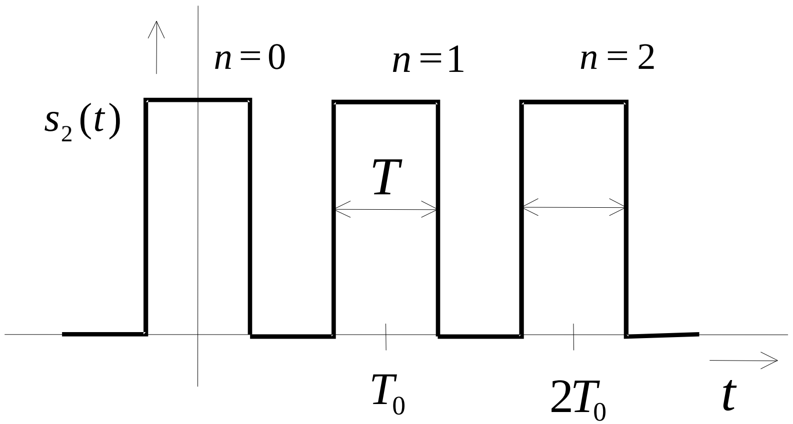
$$s_2(t) = \sum_{n=-\infty}^{+\infty} c_n s_1(t - nT_0)$$

2.1.4 Periodic Signals

$$s_1(t) = rect\left(\frac{t}{T}\right)$$

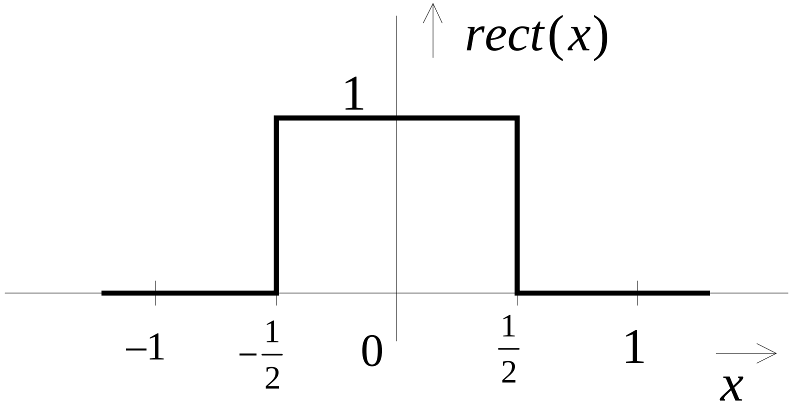
$$\Rightarrow s_2(t) = \sum_{n=-\infty}^{+\infty} rect\left(\frac{t - nT_0}{T}\right)$$

$$= \sum_{n=-\infty}^{+\infty} rect\left(\frac{t}{T} - n\frac{T_0}{T}\right)$$



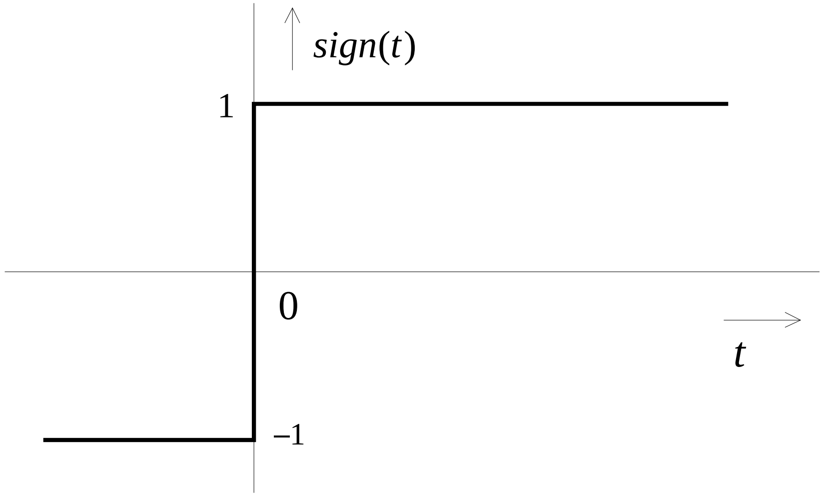
2.1.5 Impulse Type Signals

$$rect(x) = \begin{cases} 1 & \text{for} \quad |x| \le \frac{1}{2} \\ 0 & \text{for} \quad |x| > \frac{1}{2} \end{cases}$$



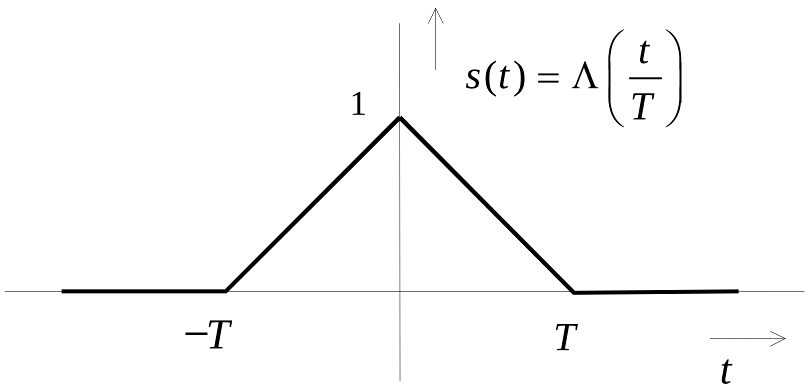
2.1.5 Impulse Type Signals

$$s(t) = sign(t) = \begin{cases} 1 & \text{for} \quad t > 0 \\ 0 & \text{for} \quad t = 0 \\ -1 & \text{for} \quad t < 0 \end{cases}$$



2.1.5 Impulse Type Signals

$$s(t) = \Lambda\left(\frac{t}{T}\right) = \begin{cases} 1 - \left|\frac{t}{T}\right| & \text{for} \quad |t| \le T \\ 0 & \text{otherwise} \end{cases}$$



2.1.6 Adjustment of Time and Frequency Functions

Case1: Compression & expansion

$$s_2(t) = a \cdot s_1\left(\frac{t}{b}\right)$$

Example:

$$s_2(t) = u_0 \cdot rect\left(\frac{t}{2T}\right)$$

Case2: Shift

$$s_2(t) = s_1(t - T_v)$$

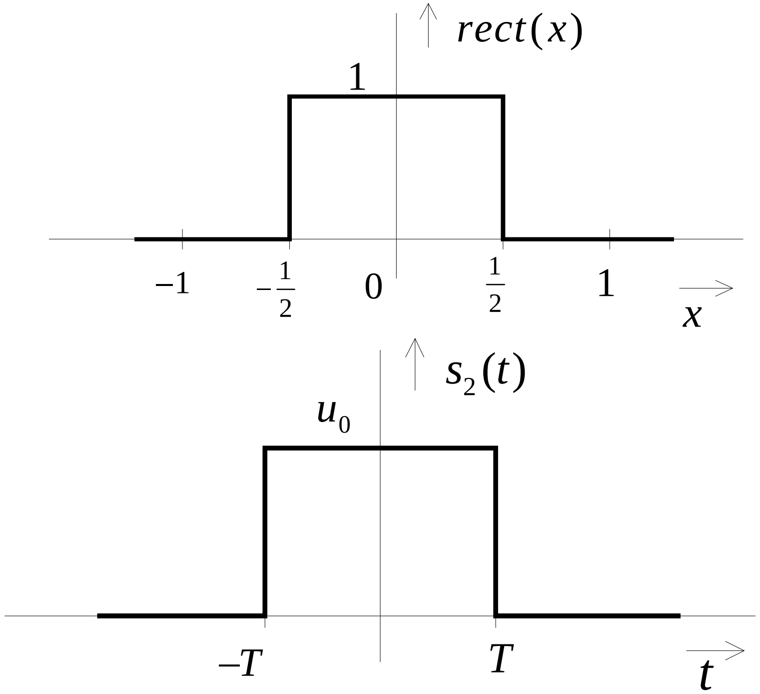
2.1.6 Adjustment of Time and Frequency Functions

Example:

$$s_1(t) = rect\left(\frac{t}{T}\right)$$

$$s_2(t) = u_0 \cdot s_1\left(\frac{t}{2}\right)$$

$$= u_0 \cdot rect\left(\frac{t/2}{T}\right)$$



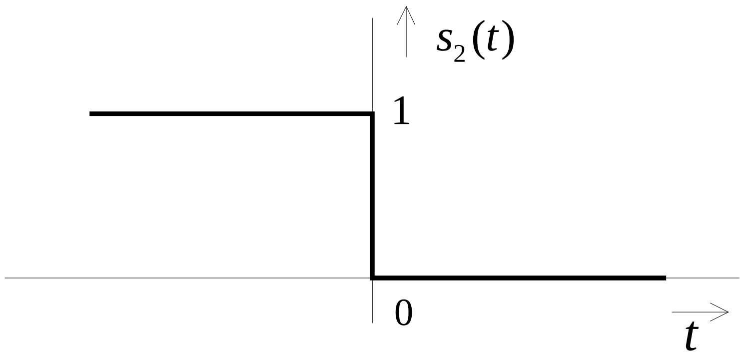
2.1.6 Adjustment of Time and Frequency Functions

Case3: Mirroring *(b = -1)*

$$s_2(t) = s_1(-t)$$

Example:

$$s_1(t) = \varepsilon(t)$$
$$s_2(t) = s_1(-t) = \varepsilon(-t)$$

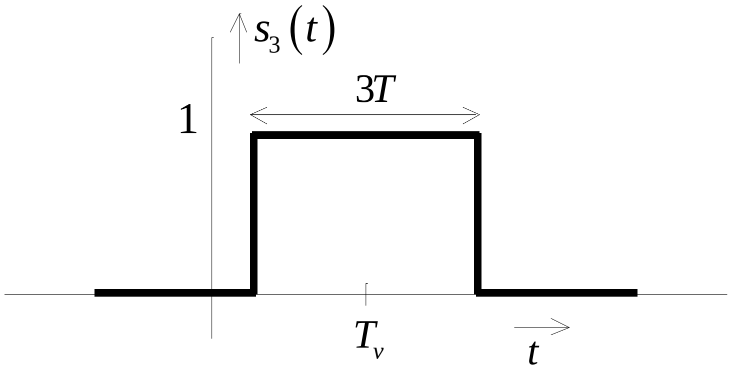


2.1.6 Adjustment of Time and Frequency Functions

Expansion & shift:

$$s_1(t) = rect\left(\frac{t}{T}\right) \qquad s_2(t) = rect\left(\frac{t}{3T}\right)$$

$$s_3(t) = s_2(t - T_\nu) = rect\left(\frac{t - T_\nu}{3T}\right)$$



Shift & expansion:

$$s_2(t) = s_1(t - T_\nu)$$

$$s_3(t) = a s_2\left(\frac{t}{b}\right) \qquad \text{Replace } t \text{ by } \frac{t}{b} \text{ in } s_1(t)$$

$$= a s_1\left(\frac{t}{b} - T_\nu\right)$$

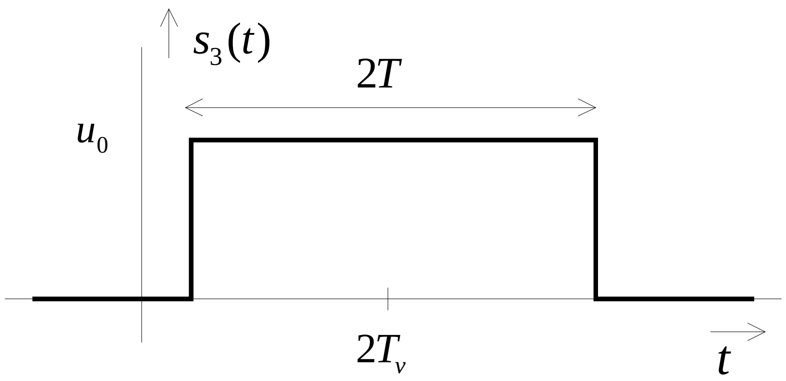
2.1.6 Adjustment of Time and Frequency Functions

Example:

$$s_1(t) = rect\left(\frac{t}{T}\right) \; ; \quad a = u_0 \; ; \quad b = 2$$

$$s_2(t) = rect\left(\frac{t - T_v}{T}\right) = rect\left(\frac{t}{T} - \frac{T_v}{T}\right)$$

$$s_3(t) = a\,rect\left(\frac{t}{bT} - \frac{T_v}{T}\right) = u_0 rect\left(\frac{t}{2T} - \frac{T_v}{T}\right)$$



2.1.6 Adjustment of Time and Frequency Functions

Mirroring & shift:

$$s_2(t) = s_1(-t)$$

$$s_3(t) = s_2(t - T_\nu) = s_1(-(t - T_\nu)) = s_1(T_\nu - t)$$

New sequence: Shift & mirroring:

$$s_4(t) = s_1(t - T_\nu)$$

$$s_5(t) = s_4(-t) = s_1(-t - T_\nu) \neq s_3(t)$$

2.1.6 Adjustment of Time and Frequency Functions

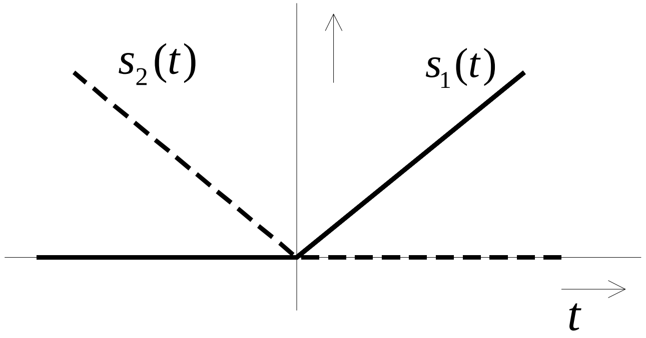
Example with a ramp function *r(t):*

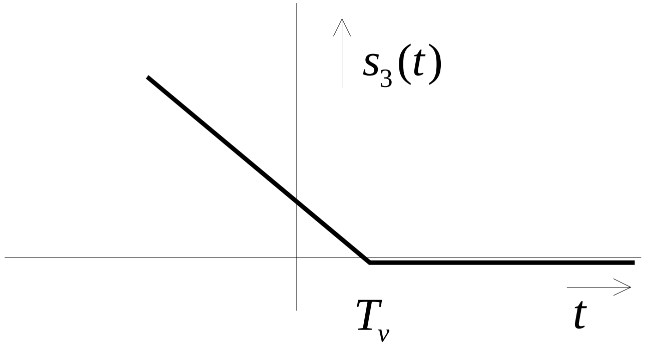
$$r\left(\frac{t}{T}\right)=\frac{t}{T}\cdot\varepsilon(t)$$

$$s_1(t)=r\left(\frac{t}{T}\right)$$

$$s_2(t)=s_1(-t)=r\left(\frac{-t}{T}\right)$$

$$s_3(t)=s_2(t-T_v)=r\left(\frac{T_v-t}{T}\right)$$

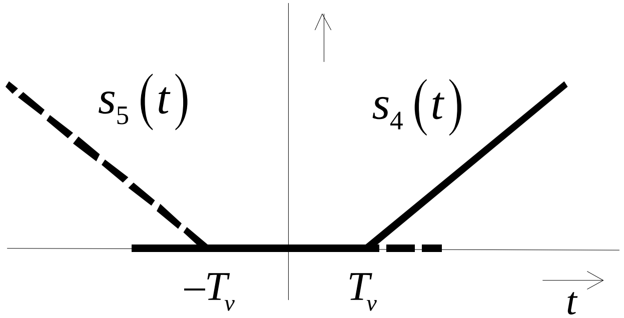




2.1.6 Adjustment of Time and Frequency Functions

$$s_4(t) = s_1(t - T_\nu) = r\left(\frac{t - T_\nu}{T}\right)$$

$$s_5(t) = s_5(-t) = r\left(\frac{-t - T_\nu}{T}\right)$$



There are 4 cases: $\pm t \pm T_\nu$

2.1.6 Adjustment of Time and Frequency Functions

All onsets can be extended to frequency functions:

$$f_1(x) = f_2(y) \quad \text{where} \quad y = f_3(x)$$

$$\Rightarrow f_1(x) = f_2(f_3(x))$$

Example:

$$f_1(\omega) = rect\left(\frac{\omega - \omega_0}{\omega_1}\right)$$

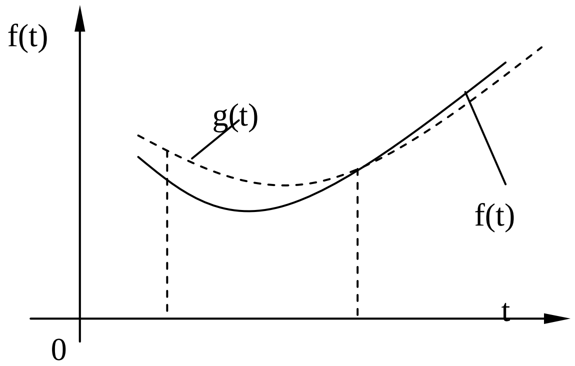
2.2 Description of non-sinusoidal, periodic functions

2.2.1 Approximation of functions

- Motivation: Determination of characteristical functions and parameters, data compression
- Starting point: Given is *f(t)*

  Desired is *g(t)*, approximating *f(t)* in an interval with

$$g(t) = \sum_{i=1}^{n} \alpha_i g_i(t) \text{ by given } g_i(t)$$



2.2.1 Approximation of functions

- Requirement: As small an error of the approximation as possible

- Definition of error function: $\Phi(t) = f(t) - g(t)$

- Mean error: $$\Phi_{m|\min} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f(t) - g(t)] dt_{|\min}$$

- Mean absolute error: $$\Phi_{ma|\min} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} |f(t) - g(t)| dt_{|\min}$$

- Mean square error: $$\Phi_{mq|\min} = \frac{1}{t_2 - t_1} \int_{t_1}^{t_2} [f(t) - g(t)]^2 dt_{|\min}$$

Advantages/disadvantages of the error measures:

- Cancelling of errors is possible in the case of mean error
- Absolute error results in nonlinearity
- Quadratic error measures is most frequent application

2.2.1 Approximation of functions

- Determination of coefficent: α_υ
- Hereby the following steps result:

$$\frac{\partial \phi_{mq}}{\partial \alpha_i} = 0 \quad i = 1, 2, \ldots, n$$

$$\frac{\partial}{\partial \alpha_i}\left[\frac{1}{t_2 - t_1}\int_{t_1}^{t_2}\left[f(t) - \sum_{i=1}^{n} \alpha_i g_i(t)\right]^2 dt\right] = 0$$

$$-\frac{1}{t_2 - t_1}\int_{t_1}^{t_2} 2\left[f(t) - \sum_{i=1}^{n} \alpha_i g_i(t)\right] g_i(t) dt = 0$$

$$\int_{t_1}^{t_2}\left[f(t) \cdot g_i(t)\right] dt = \int_{t_1}^{t_2} g_i(t)\left[\sum_{i=1}^{n} \alpha_i g_i(t)\right] dt$$

This corresponds to a set of equations, which can be solved for the coefficients.

2.2.1 Approximation of functions

$$\int_{t_1}^{t_2} f(t).g_1(t)dt = \alpha_1 \int_{t_1}^{t_2} g_1{}^2(t)dt + \alpha_2 \int_{t_1}^{t_2} g_1(t).g_2(t)dt + \ldots + \alpha_\nu \int_{t_1}^{t_2} g_1(t)g_\nu(t)dt + \ldots + \alpha_n \int_{t_1}^{t_2} g_1(t)g_n(t)dt$$

$$\int_{t_1}^{t_2} f(t).g_2(t)dt = \alpha_1 \int_{t_1}^{t_2} g_1(t)g_2(t)dt + \alpha_2 \int_{t_1}^{t_2} g_2(t).g_2(t)dt + \ldots + \alpha_\nu \int_{t_1}^{t_2} g_2(t)g_\nu(t)dt + \ldots + \alpha_n \int_{t_1}^{t_2} g_2(t)g_n(t)dt$$

.

.

.

$$\int_{t_1}^{t_2} f(t).g_n(t)dt = \alpha_1 \int_{t_1}^{t_2} g_1(t).g_n(t)dt + \alpha_2 \int_{t_1}^{t_2} g_2(t)g_n(t)dt + \ldots + \alpha_\nu \int_{t_1}^{t_2} g_\nu(t)g_n(t)dt + \ldots + \alpha_n \int_{t_1}^{t_2} g_n(t)g_n(t)dt$$

2.2.2 Approximation by means of orthogonaler function systems

- Definition of orthogonal functions in interval *(t_1, t_2)* by means of real functions *$g_i(t)$*. These functions should be continous in the interval.

$$\int_{t_1}^{t_2} g_\mu(t).g_\nu(t)dt = \delta_{\mu\nu} \cdot h_\mu$$

and suitable h_μ

- Chronecker's delta function is used:

$$\delta_{\mu\nu} = \begin{cases} 0 & für \quad \mu \neq \nu \\ 1 & für \quad \mu = \nu \end{cases}$$

- Onset for the approximation:

$$g(t) = \sum_{i=1}^{n} \alpha_i g_i(t)$$

- Thus follows for the coefficients:

$$\alpha_i = \frac{\int_{t_1}^{t_2} f(t) g_i(t) dt}{\int_{t_1}^{t_2} g_i^{\,2}(t) dt}$$

2.2.2 Approximation by means of orthogonaler function systems

One receives an orthonormal function system by means of the definitions

$$G_1(t) = \frac{g_1(t)}{\sqrt{h_1}}, G_2(t) = \frac{g_2(t)}{\sqrt{h_2}}, \ldots, G_\nu(t) = \frac{g_\nu(t)}{\sqrt{h_\nu}}, \ldots, G_n(t) = \frac{g_n(t)}{\sqrt{h_n}}$$

To these applies:

$$\int_{t_1}^{t_2} G_\mu(t) G_\nu(t) dt = \delta_{\mu\nu} = \begin{cases} 0 & \text{für} \quad \mu \neq \nu \\ 1 & \text{für} \quad \mu = \nu \end{cases}$$

Thus a function *f(t)* in the interval can be developed into a set of orthonormal functions by suitable coefficients. The result of the approximation is then a function *G(t)*. In summary it applies:

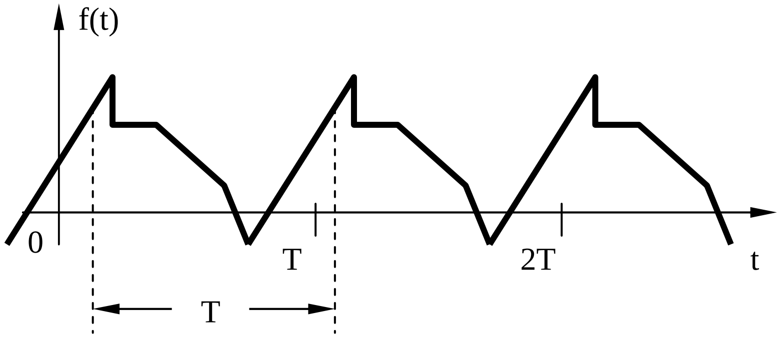
$$f(t) \cong G(t) = \sum_{i=1}^{\nu} A_i G_i(t)$$

The coefficients A_i are the so-called generalized Fourier coefficients:

$$A_i = \int_{t_1}^{t_2} f(t) G_i(t) dt$$

2.2.3 Approximation of periodic, non-sinusoidal functions

Example of a function with the period *T*. This function is interpretable as repetition of one period.



For one period it applies: $f(t) = f(t \pm \nu T) \quad \nu = 0, 1, 2, \ldots,$

After Fourier any function, for which the Dirichlet conditions are fulfilled can be represented in the following trigonometric form:

$$f(t) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} \left[a_\nu \cos(\nu\omega t) + b_\nu \sin(\nu\omega t) \right]$$

2.2.2 Approximation of periodic, non-sinusoidal functions (Fourier series)

Dirichlet conditions (in practice fulfilled)

- Function *f(t)* is either continous in the interval or has finitely many points of discontinuity
- Finite values of *f(t)* exist in the limit, if *t* approaches the point of discontinuity from the right or from the left
- The interval can be divided in such a manner that *f(t)* there is monotonous

Sentence of Dirichlet

- With fulfilment of the Dirichlet conditions the Fourier series converges in the entire interval
- The value of the Fourier series is identical to function *f(t)* in continous areas
- At points of discontinuity the value is alike: $0.5\left[f(t+0)+f(t-0)\right]$
- At end points of the interval the value is alike: $0.5\left[f(t_1+0)+f(t_2-0)\right]$

2.2.3 Fourier series

Analogy to the series expansion of orthogonal transforms

- To the series expansion applies using orthogonal functions

$$\int_{t_1}^{t_2} g_\mu(t).g_\nu(t)dt = \delta_{\mu\nu} \cdot h_\mu$$

$$g(t) = \sum_{i=1}^{n} \alpha_i g_i(t) \quad \text{with} \quad \alpha_i = \frac{\int_{t_1}^{t_2} f(t) g_i(t)dt}{\int_{t_1}^{t_2} g_i^{\,2}(t)dt}$$

- For the used functions the orthogonality can be shown:

$$\int_{t_1=t_0}^{t_2=t_0+T} \sin(\mu\omega t)\sin(\nu\omega t)dt = \int_{t_1=t_0}^{t_2=t_0+T} \cos(\mu\omega t)\cos(\nu\omega t)dt = \frac{T}{2}\delta_{\mu\nu}$$

- Otherwise applies as given above:

$$f(t) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty}\left[a_\nu \cos(\nu\omega t) + b_\nu \sin(\nu\omega t)\right]$$

- Thus it is ensured that Fourier series is the optimum approximation in the square mean sense (also in a terminated series)

2 coefficient sets are necessary, so that even and odd function parts can be represented.

2.2.3 Fourier series

Thus applies to the determination of the Fourier-coeffizients of the trigonometric form:

$$\frac{a_0}{2} = \frac{\int\limits_{t_1=t_0}^{t_2=t_0+T} f(t)\cdot 1 dt}{\int\limits_{t_1=t_0}^{t_2=t_0+T} 1^2\, dt} = \frac{1}{T}\int\limits_{t_1=t_0}^{t_2=t_0+T} f(t)dt$$

This is the DC component (arithm. average value)

$$a_\nu = \frac{\int\limits_{t_1=t_0}^{t_2=t_0+T} f(t)\cos(\mu\omega t)dt}{\int\limits_{t_1=t_0}^{t_2=t_0+T} \cos^2(\mu\omega t)\;\, dt} = \frac{2}{T}\int\limits_{t_1=t_0}^{t_2=t_0+T} f(t)\cos(\mu\omega t)dt$$

$$b_\nu = \frac{\int\limits_{t_1=t_0}^{t_2=t_0+T} f(t)\sin(\mu\omega t)dt}{\int\limits_{t_1=t_0}^{t_2=t_0+T} \sin^2(\mu\omega t)dt} = \frac{2}{T}\int\limits_{t_1=t_0}^{t_2=t_0+T} f(t)\sin(\mu\omega t)dt$$

2.2.4 The polar form of the Fourier series

(Fourier Cosinus series)

By means of the relationship $A\cos(x)+B\sin(x)=\sqrt{A^2+B^2}\cos(x-\arctan(B/A))$
the Fourier series can be rewritten from

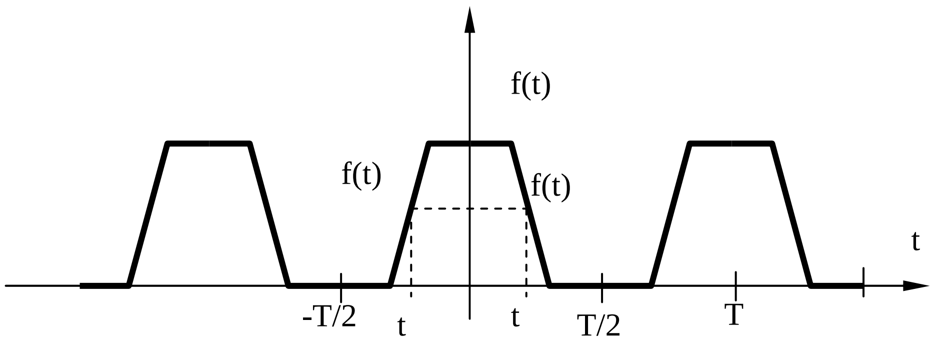
$$f(t)=\frac{a_0}{2}+\sum_{\nu=1}^{\infty}\left[a_\nu\cos(\nu\omega t)+b_\nu\sin(\nu\omega t)\right] \quad \text{to}$$

$$f(t)=d_0+\sum_{\nu=1}^{\infty}d_\nu\cos(\nu\omega t+\psi_\nu) \quad \text{with} \quad d_0=\frac{a_0}{2} \quad \text{and}$$

$$d_\nu=\sqrt{a_\nu^2+b_\nu^2}\ ; \qquad \psi_\nu=-\arctan\left(\frac{b_\nu}{a_\nu}\right) \quad (+/-\pi \ \text{ for negative } a_\nu)$$

2.2.5 Examples for the determination of the Fourier series with symmetrical Functions

- B1: *f(t)* is an even function with $f(-t) = f(t)$ and



$$a_\nu = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos(\nu\omega t)dt$$

$$b_\nu = \frac{2}{T}\int_{t_0=-\frac{T}{2}}^{\frac{T}{2}} f(t)\sin(\nu\omega t)dt = 0$$

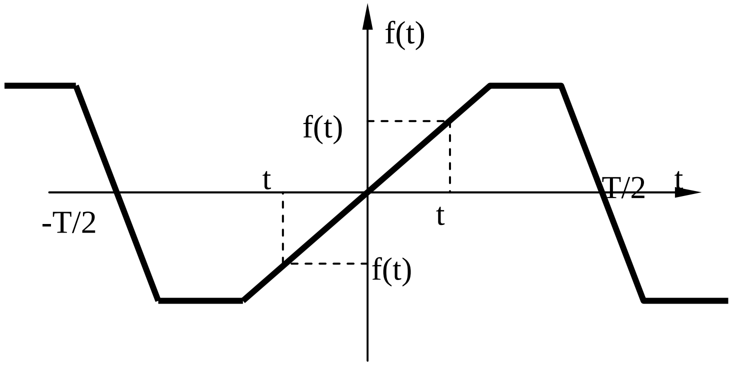
The Fourier thus has the form: $$f(t) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} a_\nu \cos(\nu\omega t)$$

Reason:

Representation of even functions is only possible by other even functions!

2.2.5 Examples for the determination of the Fourier series with symmetrical Functions

- B2: *f(t)* is an odd function with $f(t) = -f(-t)$ and



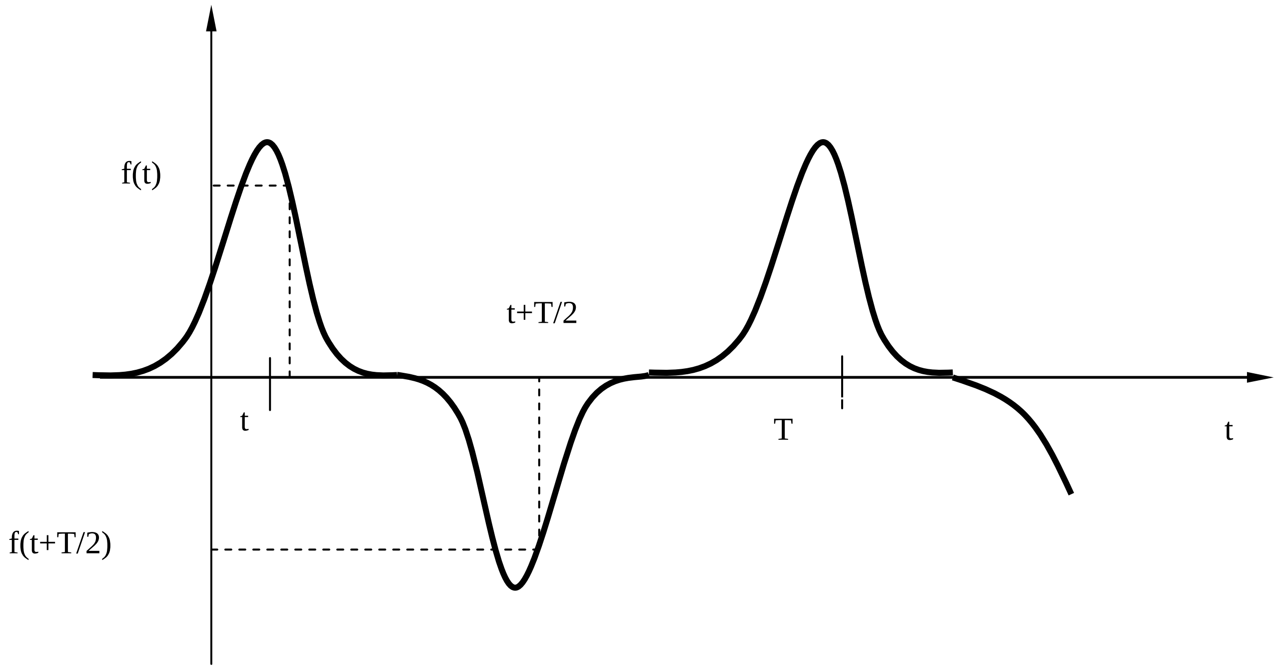
$$\frac{a_0}{2} = a_\nu = 0 \qquad b_\nu = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin(\nu\omega t)dt$$

Thus it results:

$$f(t) = \sum_{\nu=1}^{\infty} b_\nu \sin(\nu\omega t)$$

2.2.5 Examples for the determination of the Fourier series with symmetrical Functions

- B3: $f(t)$ is a completely symmetric function with $f(t) = -f(t + T/2)$



2.2.5 Examples for the determination of the Fourier series with symmetrical Functions

For It applies:
$$a_\nu = \frac{2}{T}\int_0^T f(t)\cos(\nu\omega t)dt$$

or after split-up of the interval:
$$a_\nu = \frac{2}{T}\left[\int_o^{\frac{T}{2}} f(t)\cos(\nu\omega t)dt + \int_{\frac{T}{2}}^{T} f(t)\cos(\nu\omega t)dt\right]$$

for $\nu = 2k$ holds:
$$a_{2k} = \frac{2}{T}\left[\int_o^{\frac{T}{2}} f(t)\cos(2k\omega t)dt + \int_{\frac{T}{2}}^{T} f(t)\cos(2k\omega t)dt\right] = 0$$

for $\nu = 2k+1$ holds:
$$a_{2k+1} = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos\left[(2k+1)\omega t\right]dt$$

Reason: Even-numbered *k* give after *T/2* repeating cosine functions. Cancellation of terms result due to parts of *f(t)* being negative with respect to *T/2* and repeat itself.

2.2.5 Examples for the determination of the Fourier series with symmetrical Functions

- In a similar way the validity of the following statements can be seen (also sin function repeat itself after *T/2*):

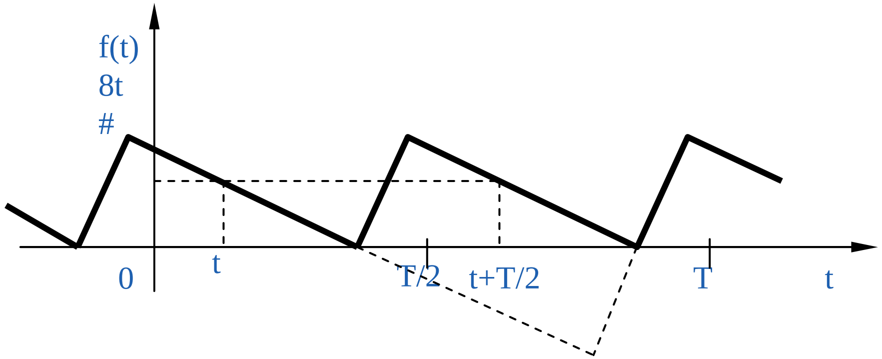
$$b_{2k} = 0 \qquad \text{and} \qquad b_{2k+1} = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin\left[(2k+1)\omega t\right]dt$$

- Therefore only odd-number oscillations occur, for them in this Fourier series $\nu = 2k+1$ applies:

$$f(t) = \sum_{k=1}^{\infty}\left\{a_{2k+1}\cos\left[(2k+1)\omega t\right] + b_{2k+1}\sin\left[(2k+1)\omega t\right]\right\}$$

2.2.5 Examples for the determination of the Fourier series with symmetrical Functions

- B4: The function is completely symmetrically with *f(t) = f(t+ T/2)* . From this follows:



$$a_{2k} = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\cos(2k\omega t)dt \text{ and } a_{2k+1} = 0$$

$$b_{2k} = \frac{4}{T}\int_0^{\frac{T}{2}} f(t)\sin(2k\omega t)dt \;\; and \;\; b_{2k+1} = 0$$

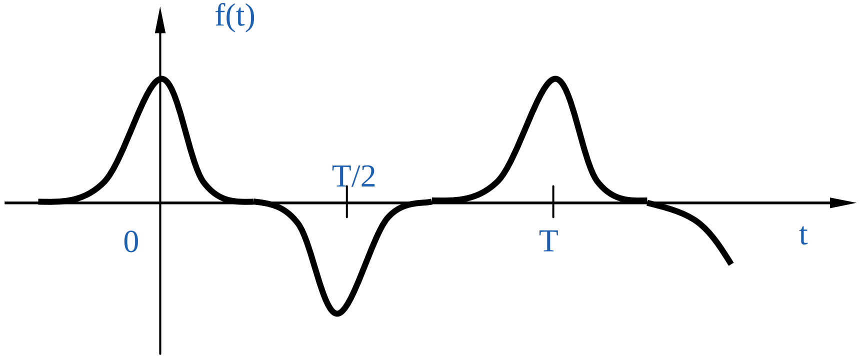
Reason: After *T/2* a repetition of the cos/sin functions with the indices *2k* takes place. Cos/sin functions with indices *2k+1* have in each case a different half wave at *T/2* distance! The Fourier series of *f(t)* then has the following form with only even-numbered coefficients:

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left\{a_{2k}\cos\left[(2k)\omega t\right] + b_{2k}\sin\left[(2k)\omega t\right]\right\}$$

2.2.5 Examples for the determination of the Fourier series with symmetrical Functions

- B5: *f(t)* is <u>even and completely symmetrical</u>: *f(t) = - f(t+ T/2) :*

  Result: Only odd-number cosine oscillations occur.



$$\frac{a_0}{2} = a_{2k} = 0 \quad \text{and} \quad b_\nu = 0$$

$$a_{2k+1} = \frac{8}{T}\int_0^{\frac{T}{4}} f(t)\cos[(2k+1)\omega t]dt$$

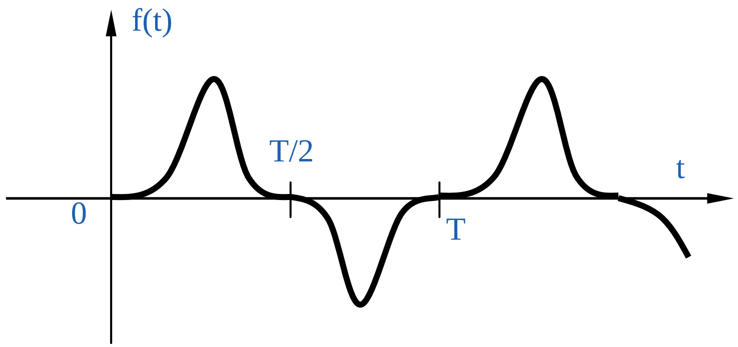
Thus the appropriate Fourier series can be written as:

$$f(t) = \sum_{k=0}^{\infty} a_{2k+1}\cos\left[\left(2k+1\right)\omega t\right]$$

2.2.5 Examples for the determination of the Fourier series with symmetrical Functions

- B6: *f(t)* is odd and completely symmetric with *f(t) = -f(t + T/2)*:

  Here only odd-number sine oscillation occur in the Fourier series



$$\frac{a_0}{2} = a_\nu = 0 \quad \text{and} \quad b_{2k} = 0$$

$$b_{2k+1} = \frac{8}{T} \int_0^{\frac{T}{4}} f(t) \sin[(2k+1)\omega t] dt$$

The Fourier series can thus be written as:

$$f(t) = \sum_{k=0}^{\infty} b_{2k+1} \sin\left[\left(2k+1\right)\omega t\right]$$

2.2.5 Examples for the determination of the Fourier series with symmetrical Functions

- B7: *f(t)* is shifted on the time axis:

  If the shift amounts to $\pm\Delta t$ then it applies with $t' = t \pm \Delta t$:

$$g(t') = f(t \pm \Delta t) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} \left\{ a_\nu \cos[\nu\omega(t \pm \Delta t)] + b_\nu \sin[\nu\omega(t \pm \Delta t)] \right\}$$

A simpler expression results for the complex coefficients:

$$f(t \pm \Delta t) \text{ results to } \left\{ c_n \cdot e^{\mp j\nu\omega t} \right\}$$

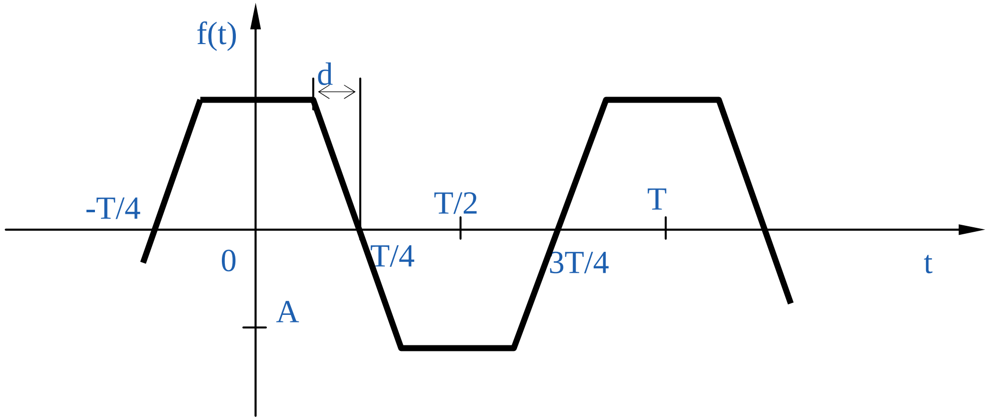
This expression makes it possible, to determine the Fourier series for arbitrary shifts.

It often is of advantage to shift the origin e.g. if thereby symmetrical characteristics of the function result.

2.2.6 Fourier analysis

- It exists the possibility to represent a periodical non-sinusoidal function regarding its "information content" in two versions:

1) In the time intervall (s. the following picture)



2) In the spectral region (frequency range): Representation of the amplitudes a_ν, b_ν and or the cos amplitude d_ν and the phase ψ_ν as a function of the frequency.

2.2.6 Fourier analysis
Further examples

Fourier analysis of the trapezoidal function

This function is even and completely symmetrically with (s. B5)

$$\frac{a_0}{2} = a_{2k} = 0,\ b_\nu = 0 \qquad \text{and} \qquad a_{2k+1} = \frac{8}{T}\int_0^{\frac{T}{4}} f(t)\cos[(2k+1)\omega t]dt$$

To $f(t)$ applies in the first quarter of the period:

$$f(t) = \begin{cases} A = const & \text{für } 0 \le t < \left(\frac{T}{4} - d\right) \\ \frac{A}{d}\left(\frac{T}{4} - t\right) & \text{für } \left(\frac{T}{4} - d\right) < t \le \left(\frac{T}{4} + d\right) \end{cases}$$

Thus it results: $$a_{2k+1} = \frac{8}{T}\left\{ \int_0^{\frac{T}{4}-d} A\cos[(2k+1)\omega t]dt + \int_{\frac{T}{4}-d}^{\frac{T}{4}} \frac{A}{d}(\frac{T}{4} - t)\cos[(2k+1)\omega t]dt \right\}$$

2.2.6 Fourier analysis

The final result then is:

$$a_{2k+1} = \frac{4A}{\pi(2k+1)^2 \omega d} \cos[(2k+1)\omega(\frac{T}{4} - d)]$$

or

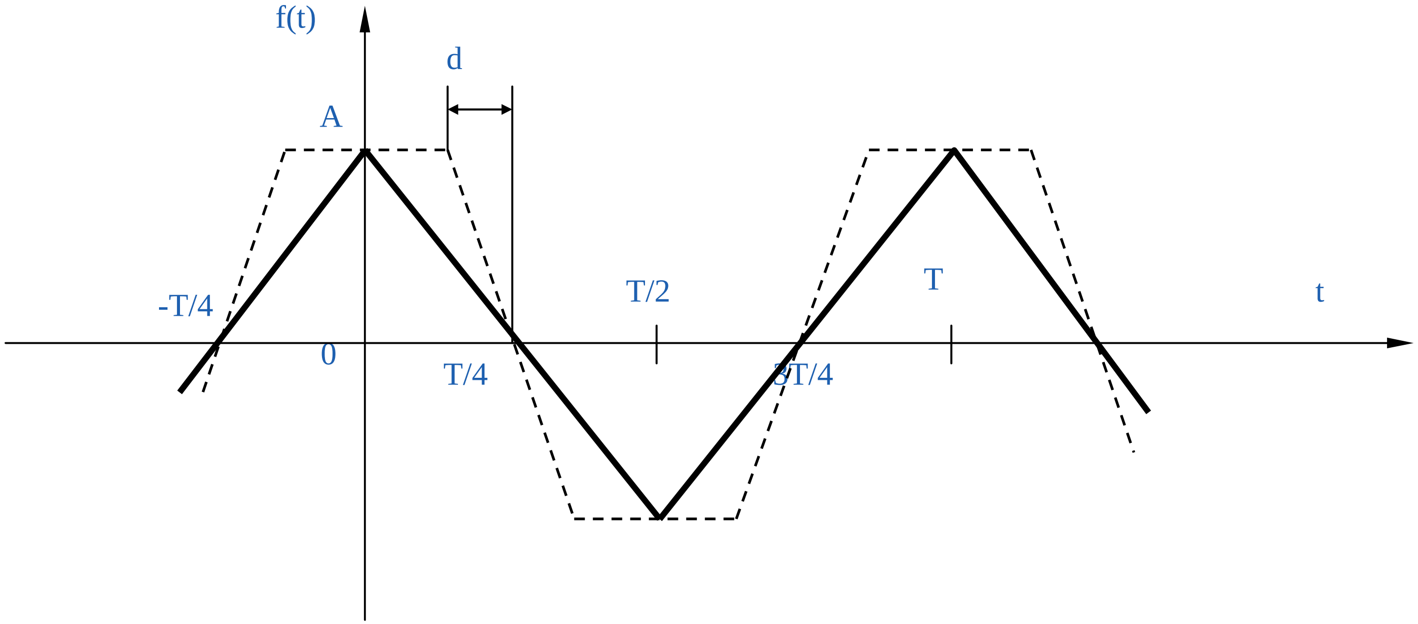
$$a_{2k+1} = \frac{4A \sin[(2k+1)\omega d]}{\pi(2k+1)^2 \omega d} \sin[(2k+1)\frac{\pi}{2}]$$

The Fourier series of the trapezoidal function is thereby:

$$f(t) = \frac{4A}{\pi\omega d}[\sin(\omega d)\cos(\omega t) - \frac{1}{9}\sin(3\omega d)\cos(3\omega t) + \frac{1}{25}\sin(5\omega d)\cos(5\omega t)... + ...)$$

2.2.6 Fourier analysis

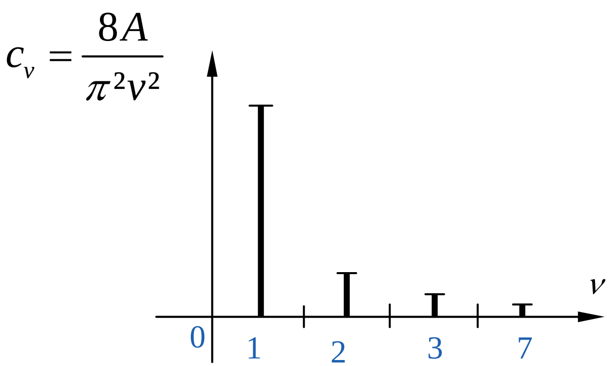
- Special case 1 of the trapezoidal function : *The triangle function (d = T/4)* :

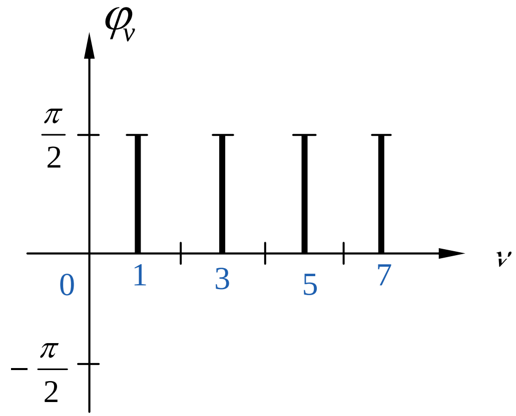


$\pi / 2$

2.2.6 Fourier analysis

- For this function the following amplitudes and phase spectrum results:





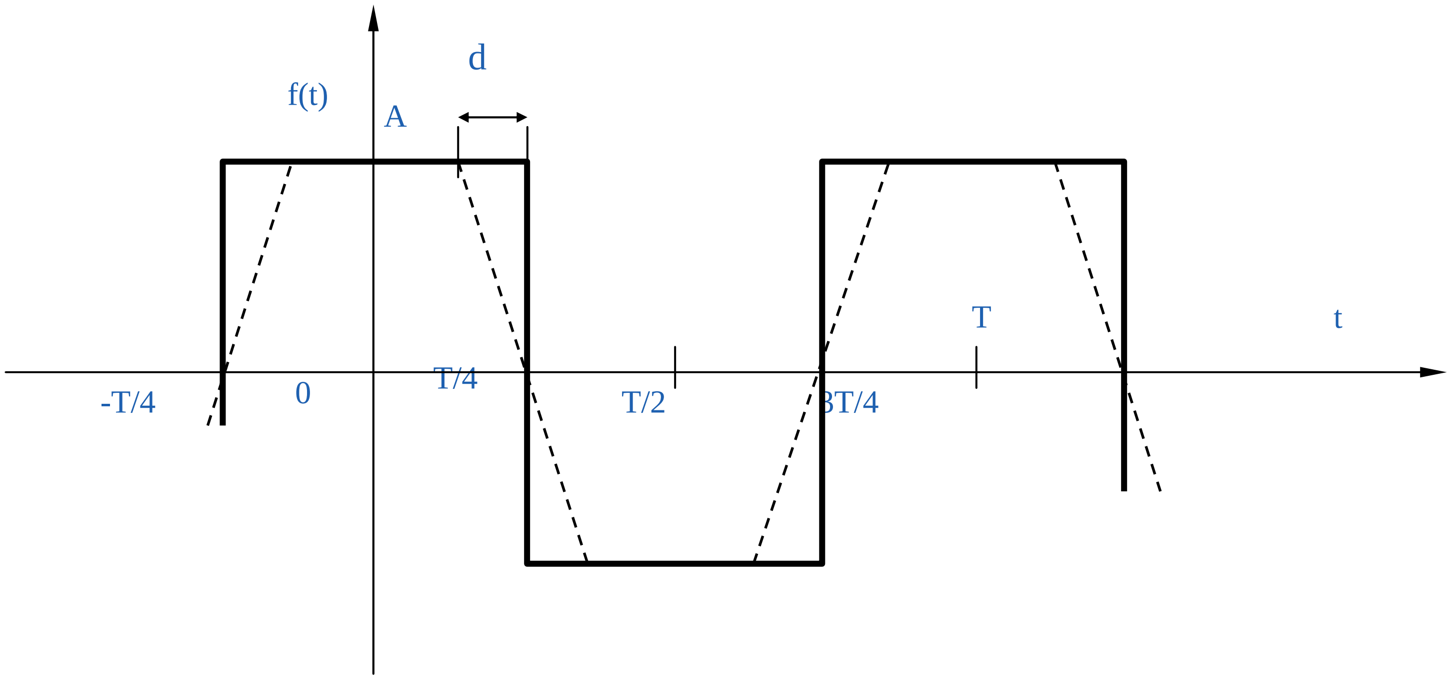
$$\nu = 2k + 1$$

Endergebnisse:

$$a_{2k+1} = \frac{8A}{\pi^2 (2k+1)^2}, \quad b_k = 0, \quad \psi_{2k+1} = 0\ , \ \ \varphi_{2k+1} = \psi_{2k+1} + \pi / 2$$

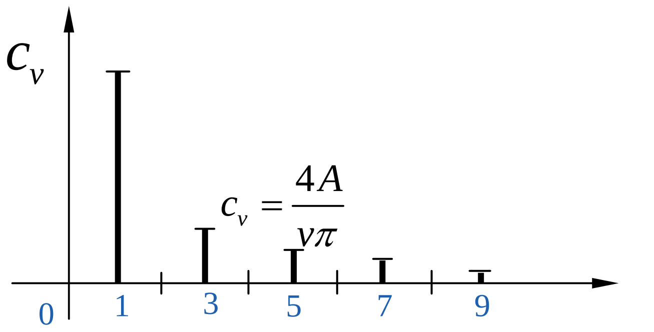
2.2.6 Fourier analysis

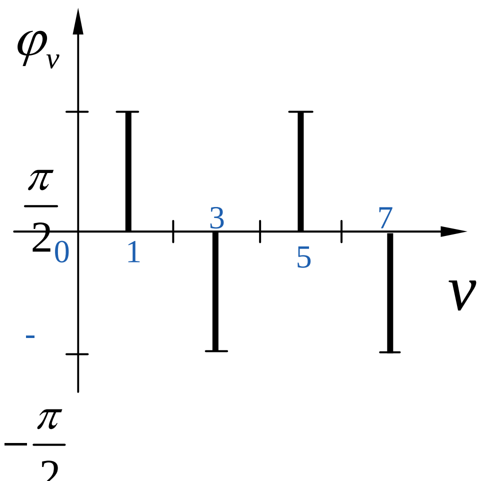
- Special case 2 of the trapezoidal function : *Rectangle function with* $d \to 0$



2.2.6 Fourier analysis

- Limiting performed by means of the rule of Bernoulli L'Hospital:





$$\lim_{d\to 0}\frac{\sin[(2k+1)\omega d]}{(2k+1)\omega d}=\lim_{d\to 0}\frac{\{\sin[(2k+1)\omega d]\}'}{\{(2k+1)\omega d\}'}=\lim_{d\to 0} si((2k+1)\omega d)=si(0)=1$$

Thus it follows: $a_{2k+1}=\frac{4A}{\pi(2k+1)}\sin[(2k+1)\frac{\pi}{2}],\quad b_k=0$

$$c_{2k+1}=\frac{4A}{\pi(2k+1)},\quad \varphi_{2k+1}=\pm\frac{\pi}{2}\sin[(2k+1)\frac{\pi}{2}]$$

2.2.7 The complex form of the Fourier series

- Generally it applies to the Fourier series: $f(t) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} \left[a_\nu \cos(\nu\omega t) + b_\nu \sin(\nu\omega t) \right]$

⇨ In addition it applies: $\cos(\nu\omega t) = \frac{e^{j\nu\omega t} + e^{-j\nu\omega t}}{2}$ $\sin(\nu\omega t) = \frac{e^{j\nu\omega t} - e^{-j\nu\omega t}}{2j}$

Thus it is obtained

$$f(t) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} \left[a_\nu \frac{e^{j\nu\omega t} + e^{-j\nu\omega t}}{2} + b_\nu \frac{e^{j\nu\omega t} - e^{-j\nu\omega t}}{2j} \right]$$

or:

$$f(t) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} \left[\frac{a_\nu + jb_\nu}{2} e^{-j\nu\omega t} + \frac{a_\nu - jb_\nu}{2} e^{j\nu\omega t} \right]$$

2.2.7 The complex form of the Fourier series

- Now also negative values for ν are included.

With the abbreviations

$$\underline{c}_0 = \frac{a_0}{2},$$

$$\underline{c}_\nu = \frac{a_\nu - jb_\nu}{2} \text{ for positive } \nu \qquad \text{Also: } \underline{c}_\nu = \underline{c}_{-\nu}^*$$

$$\underline{c}_\nu = \frac{a_\nu + jb_\nu}{2} \text{ for negative } \nu$$

one receives pairs of coefficients.

This can be written in the very compact representation of the Fourier series in complex form:

$$f(t) = \sum_{\nu=-\infty}^{\infty} \underline{c}_\nu e^{j\nu\omega t}$$

2.2.7 The complex form of the Fourier series

In addition it applies: $a_\nu = 2\,\mathrm{Re}(c_\nu) \quad b_\nu = -2\,\mathrm{Im}(c_\nu)\ \forall\, \nu > 0$

- For the complex coefficients thereby the conditional equations result:

$$\underline{c}_0 = \frac{a_0}{2} = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)dt,$$

$$\underline{c}_\nu = \frac{a_\nu - jb_\nu}{2} = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)[\cos(\nu\omega t) - j\sin(\nu\omega t)]dt = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-j\nu\omega t}dt,$$

$$\underline{c}_\nu = \frac{1}{T}\int_{t_0}^{t_0+T} f(t)e^{-j\nu\omega t}, \quad \nu = 0,1,2,...$$

2.2.8 Interpretation of Fourier-coefficients

Usually the following representations of the Fourier series are used:

$$f(t) = \frac{a_0}{2} + \sum_{\nu=1}^{\infty} \left[a_\nu \cos(\nu\omega t) + b_\nu \sin(\nu\omega t) \right]$$

or

$$f(t) = \sum_{\nu=-\infty}^{\infty} c_\nu e^{j\nu\omega t}$$

or

$$f(t) = d_0 + \sum_{\nu=1}^{\infty} d_\nu \cos(\nu\omega t + \psi_\nu)$$

<u>Appointment off here</u>: c_ν instead of $\underline{c}_\nu$ for better overview of formulas.

2.2.8 Interpretation of Fourier-coefficients

Summarizing the Fourie series gives the following sets of parameters:

- DC component of the signal: $\frac{a_0}{2} = c_0 = d_0$
- Peak values or amplitudes of the Fourier components: a_ν, b_ν and d_ν
- Zero-phase (or phase) of the cosine oscillations: ψ_ν
- Basic oscillation at the fundamental frequency: $d_1 \cdot \cos(\omega t + \psi_1)$

2.2.9 Application of the Fourier series to a network

Given is a cosine-type voltage $u\,(t) = \hat{u}\cos(\nu\omega t + \varphi_{u\nu})$

A complex pointer is usually assigned like this: $\underline{\hat{u}} = \hat{u}\, e^{j\varphi_{u\nu}}$

Now one can set: $u(t) = \sum_{\nu=-\infty}^{\infty} \underline{u}_\nu e^{j\nu\omega t}$ with $\underline{u}_\nu = \begin{cases} \frac{1}{2}\underline{\hat{u}}_\nu^*, \; \textit{für} & \nu \le -1 \\ u_0 \quad \textit{für} & \nu = 0 \\ \frac{1}{2}\underline{\hat{u}}_\nu \quad \textit{für} & \nu \ge 1 \end{cases}$

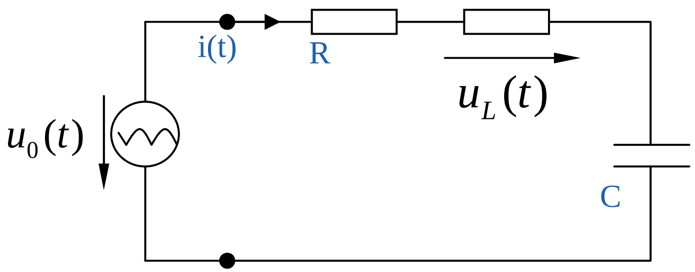
Also for electrical networks one uses the representation (for voltages and currents) in cosine form:

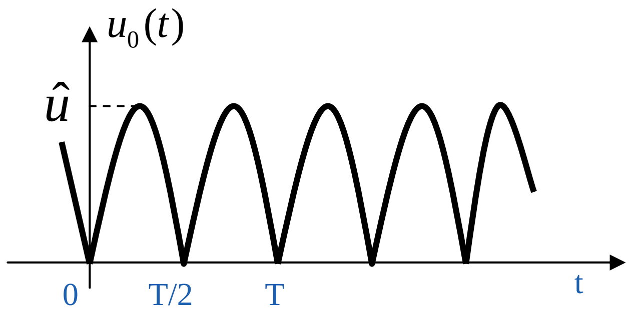
$$u(t) = u_0 + \sum_{\nu=1}^{\infty} \hat{u}_\nu \cos(\nu\omega t + \varphi_{u\nu})$$

$$i(t) = i_0 + \sum_{\nu=1}^{\infty} \hat{i}_\nu \cos(\nu\omega t + \varphi_{i\nu})$$

2.2.9 Application of the Fourier series to a network

Example of a network: Series oscillator circuit





Here is given: $u_0(t) = \hat{u}\left|\sin(\omega t)\right|$ mit $\omega = 2\pi/\mathrm{T}$

From this follows: $u_0(t) = \dfrac{2\hat{u}}{\pi} - \sum_{k=1}^{\infty} \dfrac{4\hat{u}}{\pi(4k^2-1)} \cos(2k\omega t)$

To determine are: $i(t)$ and $u_L(t)$

2.2.9 Application of the Fourier series to a network

- Solution :

  - Use of impedance for each frequency $k\omega$: $\underline{Z}_k = \dfrac{\underline{\hat{u}}_k}{\underline{\hat{i}}_k}$

  - Rewriting of the Fourier series of $u_0(t)$ in complex form with:

$$\underline{\hat{u}}_k = -\frac{4\hat{u}}{\pi(4k^2-1)}$$

$$u_0(t) = -\sum_{k=-\infty}^{\infty} \frac{2\hat{u}}{\pi(4k^2-1)} e^{j2k\omega t}$$

2.2.9 Application of the Fourier series to a network

- Impedance and/or current of the series oscillator circuit for a certain frequency $k\omega$:

$$\underline{Z}_k = R + jX_k \quad \text{with} \quad X_k = k\omega L - \frac{1}{k\omega C} \quad \text{and} \quad i_k = u_k / Z_k$$

Then applies to the current:

$$i(t) = \sum_{k=-\infty}^{\infty} i_k = -\sum_{k=-\infty}^{\infty} \frac{2\hat{u}}{\pi(4k^2-1)} \cdot \frac{1}{R + jX_{2k}} e^{j2k\omega t}$$

Using the Euler' formula it results:

$$i(t) = -\sum_{k=-\infty}^{\infty} \frac{2\hat{u}}{\pi(4k^2-1)} \cdot \frac{1}{R + jX_{2k}} .[\cos(2k\omega t) + j\sin(2k\omega t)]$$

2.2.9 Application of the Fourier series to a network

- After rewriting the equation, it results:

$$i(t) = -\sum_{k=-\infty}^{\infty} \frac{2\hat{u}}{\pi(4k^2-1)} \left[\frac{R\cos(2k\omega t) + X_{2k}\sin(2k\omega t)}{R^2 + X_{2k}^2} + j\frac{R\sin(2k\omega t) - X_{2k}\cos(2k\omega t)}{R^2 + X_{2k}^2} \right]$$

If the characteristics of the functions (cos, sin) for +/- k are used, then it applies with $X_{-k} = -X_k$:

$$i(t) = -\sum_{k=0}^{\infty} \frac{4\hat{u}}{\pi(4k^2-1)} \frac{R\cos(2k\omega t) + X_{2k}\sin(2k\omega t)}{R^2 + X_{2k}^2}$$

2.2.9 Application of the Fourier series to a network

One receives the voltage at the coil by means of : $u_L(t) = L\frac{di(t)}{dt}$

$$u_L(t) = \sum_{k=0}^{\infty} \frac{4\hat{u}}{\pi(4k^2-1)} \frac{2k\omega L[R\sin(2k\omega t) - X_{2k}\cos(2k\omega t)]}{R^2 + X_{2k}^2}$$

The results can be represented also in polar form:

$$\Leftrightarrow \quad u_L(t) = \sum_{k=0}^{\infty} \frac{4\hat{u}}{\pi(2k^2-1)} \cdot \frac{2k\omega L}{\sqrt{R^2 + X_{2k}{}^2}} \cos[(2k\omega t) + \arctan(\frac{R}{X_{2k}})]$$

$$\Leftrightarrow \quad i(t) = \sum_{k=0}^{\infty} \frac{4\hat{u}}{\pi(2k^2-1)} \cdot \frac{1}{\sqrt{R^2 + X_{2k}{}^2}} \cos[(2k\omega t) - \arctan(\frac{X_{2k}}{R})]$$

2.2.10 Formulation of Parseval's equation

- Two generally non-sinusoidal periodic functions are regarded
- The functions $f_1(t)$ and $f_2(t)$ have the same period T:
- The 2 appropriate Fourier series formulas then look as follows:

$$f_1(t) = \sum_{\nu=-\infty}^{\infty} \underline{C}_\nu e^{j\nu\omega t} \quad \text{with} \quad \underline{C}_\nu = \frac{1}{T}\int_{t_0}^{t_0+T} f_1(t) e^{-j\nu\omega t} dt$$

and

$$f_2(t) = \sum_{\mu=-\infty}^{\infty} \underline{D}_\mu e^{j\mu\omega t} \quad \text{with} \quad \underline{D}_\mu = \frac{1}{T}\int_{t_0}^{t_0+T} f_2(t) e^{-j\mu\omega t} dt$$

- To the product of both functions applies: $f_1(t) \cdot f_2(t) = \sum_{\nu=-\infty}^{\infty} \underline{C}_\nu e^{j\nu\omega t} . \sum_{\mu=-\infty}^{\infty} \underline{D}_\mu e^{j\mu\omega t}$

and at the same time as they are periodical:

$$f_1(t) \cdot f_2(t) = \sum_{k=-\infty}^{\infty} \underline{E}_k e^{jk\omega t} \quad \text{with} \quad \underline{E}_k = \frac{1}{T}\int_{t_0}^{t_0+T} f_1(t) f_2(t) e^{-jk\omega t} dt$$

2.2.10 Formulation of Parseval's equation

- Further applies: $$\underline{E}_k = \frac{1}{T}\int_{t_0}^{t_0+T}\left[\sum_{\nu=-\infty}^{\infty}\underline{C}_\nu e^{j\nu\omega t}\cdot\sum_{\mu=-\infty}^{\infty}\underline{D}_\mu e^{j\mu\omega t}\right]e^{-jk\omega t}dt$$

$$\underline{E}_k = \sum_{\nu=-\infty}^{\infty}\underline{C}_\nu\left[\sum_{\mu=-\infty}^{\infty}\underline{D}_\mu \frac{1}{T}\int_{t_0}^{t_0+T} e^{j(\nu+\mu-k)\omega t}dt\right]$$

and/or

$$\underline{E}_k = \sum_{\nu=-\infty}^{\infty}\underline{C}_\nu I \quad with \quad I = \sum_{\mu=-\infty}^{\infty}\underline{D}_\mu \frac{1}{T}\int_{t_0}^{t_0+T} e^{j(\nu+\mu-k)\omega t}dt$$

- One can show that I is different from zero only in case of: $\nu+\mu-k=0$
- (due to orthogonality of *cos(nx)* and *sin(nx)*)

Thus the integrand becomes identical to 1 and it applies: $$\mathrm{I} = \sum_{\mu=-\infty}^{\infty}\underline{\mathrm{D}}_\mu \frac{1}{\mathrm{T}}\cdot\mathrm{T} = \sum_{\mu=-\infty}^{\infty}\underline{\mathrm{D}}_\mu$$

2.2.10 Formulation of Parseval's equation

For the fourier-coefficients of the product $f_1(t)\,f_2(t)$ then results

$$\underline{E}_k = \sum_{\nu=-\infty}^{\infty} \underline{C}_\nu \sum_{\mu=-\infty}^{\infty} \underline{D}_\mu = \sum_{\nu=-\infty}^{\infty} \underline{C}_\nu \underline{D}_{k-\nu}$$

due to $\nu + \mu - k = 0$

or $\mu = k - \nu$

Determination of the DC component $\underline{E}_0$ of the product $f_1(t) \cdot f_2(t)$ using $k = 0$:

$$E_0 = \frac{1}{T} \int_{t_0}^{t_0+T} f_1(t).f_2(t)dt = \sum_{\nu=-\infty}^{\infty} \underline{C}_\nu . \underline{D}_{-\nu}$$

There are various applications of this relationship (determination of the integral in the time or frequency range)!

2.2.10 Formulation of Parseval's equation

All complex Fourier-coefficients possess the characteristic:

$$\underline{C}_\nu = \underline{C}^*_{-\nu} \quad \text{and} \quad \underline{D}_\nu = \underline{D}^*_{-\nu}$$

Thus the following formula can be rewritten based on

$$E_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f_1(t).f_2(t)dt = \sum_{\nu=-\infty}^{\infty} \underline{C}_\nu.\underline{D}_{-\nu} = \sum_{\nu=-\infty}^{\infty} \underline{C}_{-\nu}.\underline{D}_\nu$$

to:

$$E_0 = \sum_{\nu=-\infty}^{\infty} \mathrm{Re}\left\{\underline{C}^*_\nu \cdot \underline{D}_\nu\right\} = \sum_{\nu=-\infty}^{\infty} \mathrm{Re}\left\{\underline{C}_\nu \cdot \underline{D}^*_\nu\right\}$$

➡ ***This is Parseval's equation!***

2.2.11 Power of non-sinusoidal periodic network functions

The electrical energy E per period (or the power P) concerning an Ohm's resistance amounts to:

$$P = E/T = \frac{1}{T}\cdot\frac{1}{R}\int_{t_0}^{t_0+T} u^2(t)dt = \frac{1}{T}\cdot R\int_{t_0}^{t_0+T} i^2(t)dt$$

Application of Parseval's equation for the special case

$$f_1(t) = f_2(t) = f(t) = i(t) \qquad \text{with} \quad f_1(t)\, f_2(t) = f^2(t) = i(t)$$

results in: $E_0 = \frac{1}{T}\int_{t_0}^{t_0+T} f^2(t)dt = C_0 \cdot C_0 + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left\{\underline{C}_\nu . \underline{C}_\nu^*\right\}$

$$\Rightarrow \quad P = E_0 R = R(C_0^2 + 2\sum_{\nu=1}^{\infty}\left|\underline{C}_\nu\right|^2)$$

2.2.11 Power of non-sinusoidal periodic network functions

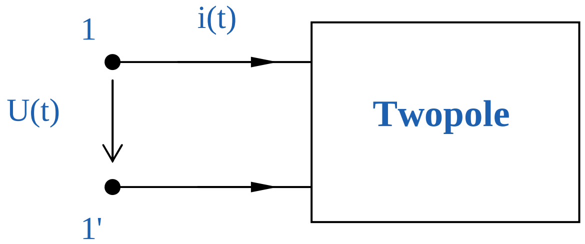
With $c_0 = \frac{a_0}{2} = C_0 \, , c_\nu = C_\nu = \frac{a_\nu - jb_\nu}{2} \quad \nu \geq 1$ it follows:

$$\frac{1}{T}\int_{t_0}^{t_0+T} f^2(t)dt = c_0^2 + 2\sum_{\nu=1}^{\infty}\left|c_\nu\right|^2 = \left(\frac{a_0}{2}\right)^2 + \sum_{\nu=1}^{\infty}\frac{a_\nu^2 + b_\nu^2}{2}$$

If *f(t)* has character of a voltage or a current, the appropriate spectral parameters a_ν, b_ν and c_ν describe the effective power conditions of the appropriate network elements.

2.2.11 Power of non-sinusoidal periodic network functions

Now a two-pole with non-sinusoidal periodic network functions *u(t)* and *i(t) is* regarded:



$$u(t) = f_1(t) = \sum_{\nu=-\infty}^{\infty} \underline{C}_\nu e^{j\nu\omega t}$$

and

$$i(t) = f_2(t) = \sum_{\nu=-\infty}^{\infty} \underline{D}_\mu e^{j\varpi\omega t}$$

For the power it applies:

$$P = \frac{1}{T}\int_{t_0}^{t_0+T} u(t)i(t)dt = \frac{1}{T}\int_{t_0}^{t_0+T} f_1(t)\, f_2(t)dt = C_0 D_0 + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left\{\underline{C}_\nu^{\,*}\underline{D}_\nu\right\}$$

$$= C_0 D_0 + 2\sum_{\nu=1}^{\infty} \mathrm{Re}\left\{\underline{C}_\nu \underline{D}_\nu^{*}\right\}$$

2.2.11 Power of non-sinusoidal periodic network functions

With consideration of the relations

$$C_0 = U_0 \quad , \quad D_0 = I_0 \quad , \quad \underline{C}_\nu = \frac{1}{2}\underline{\hat{u}}_\nu \quad , \quad \underline{D}_\nu = \frac{1}{2}\underline{\hat{i}}_\nu$$

follows:

$$P = U_0 I_0 + \frac{1}{2}\sum_{\nu=1}^{\infty} \mathrm{Re}\left\{\underline{\hat{u}}_\nu^* \cdot \underline{\hat{i}}_\nu\right\} = U_0 I_0 + \frac{1}{2}\sum_{\nu=1}^{\infty} \mathrm{Re}\left\{\underline{\hat{u}}_\nu \cdot \underline{\hat{i}}_\nu^*\right\}$$

Thus the total power is to be determined over the sum of all individual powers for each spectral line!

2.2.12 Assessing of deviations from the sinusoidal form of periodic functions

Definition of the <u>rms value</u> of a periodic function:

$$f(t)_{eff} = \sqrt{\frac{1}{T}\int_{t_0}^{t_0+T} f^2(t)\,dt}$$

Parseval's equation permits the determination of the rms value using Fourier coefficients (and/or the associated rms values):

$$f(t)_{eff} = \sqrt{\sum_{\nu=-\infty}^{\infty} |c_\nu|^2} = \sqrt{\left(\frac{a_0}{2}\right)^2 + \sum_{\nu=1}^{\infty}\left[\left(\frac{a_\nu}{\sqrt{2}}\right)^2 + \left(\frac{b_\nu}{\sqrt{2}}\right)^2\right]}$$

2.2.12 Assessing of deviations from the sinusoidal form of periodic functions

- The rms value for a periodic voltage *u(t)* amounts to:

$$U_{eff} = \sqrt{U_0^2 + \sum_{\nu=1}^{\infty} \left(\frac{\hat{u}_\nu}{\sqrt{2}} \right)^2} = \sqrt{\sum_{\nu=0}^{\infty} U_{eff\ \nu}^2}$$

U_0 : DC component of *u(t)*

$\hat{u}_\nu$: Peak value

$U_{eff\ \nu} = \underline{\hat{u}}_\nu / \sqrt{2}$: Rms value of the component ν (at frequency: $\nu\omega$)

- The rms value for a periodic current amounts to:

$$I = I_{eff} = \sqrt{I_0^2 + \sum_{\nu=1}^{\infty} \left(\frac{\hat{i}_\nu}{\sqrt{2}} \right)^2} = \sqrt{I_0^2 + \sum_{\nu=1}^{\infty} I_{eff\ \nu}^2} = \sqrt{\sum_{\nu=0}^{\infty} I_{eff\ \nu}^2}$$

2.2.12 Assessing of deviations from the sinusoidal form of periodic functions

For pure alternating currents (AC), without any DC component applies: $a_0 = 0$

Otherwise: f(t) contains both (DC and AC components)

Thus $\frac{a_0}{2} \neq 0$ ⇨ The <u>oscillation content</u> *s* describes the amount of AC in the total signal.

$$s = \frac{\sqrt{\sum_{\nu=1}^{\infty} U_{eff\,\nu}^2}}{U_{eff}} = \frac{\sqrt{\sum_{\nu=1}^{\infty} U_{eff\,\nu}^2}}{\sqrt{\sum_{\nu=0}^{\infty} U_{eff\,\nu}^2}}$$

2.2.12 Assessing of deviations from the sinusoidal form of periodic functions

The deviation from the sinusoidal form can be described by the basic oscillation amount g:

$$g = \frac{U_{eff\,1}}{U_{eff\,\sim}} = \frac{U_{eff\,1}}{\sqrt{\sum_{\nu=1}^{\infty} U^2_{eff\,\nu}}}$$

The harmonic content k or distortion factor k amounts to:

$$k = \frac{\sqrt{\sum_{\nu=2}^{\infty} U^2_{eff\,\nu}}}{U_{eff\,\sim}} = \frac{\sqrt{\sum_{\nu=2}^{\infty} U^2_{eff\,\nu}}}{\sqrt{\sum_{\nu=1}^{\infty} U^2_{eff\,\nu}}} \quad \Rightarrow \quad g^2 + k^2 = 1$$

2.2.12 Assessing of deviations from the sinusoidal form of periodic functions

Additionally there are further definitions named shape factor and amplitude factor :

Form factor:

$$k_f = \frac{\sqrt{\sum_{\nu=0}^{\infty} U_{eff\,\nu}^2}}{\frac{1}{T}\int_0^T |u(t)|\,dt}$$

Crest factor for signals without DC component:

$$k_a = \frac{u(t)_{\max}}{\sqrt{\sum_{\nu=1}^{\infty} U_{eff\,\nu}^2}}$$

For purely sinusoidal form one finds:

$$k_f = \frac{\pi}{2\sqrt{2}} \approx 1{,}11 \quad \text{and} \quad k_a = \sqrt{2} = 1{,}41$$

2.2.13 Additional properties of the Fourier series

- Linearity

  $k \cdot s(t)$ gives a series with $k \cdot c_v$

  $a \cdot s_1(t) + b \cdot s_2(\mathrm{t})$ gives a series with $a \cdot c_{v1} + b \cdot c_{v2}$

- Time-shift

  $s(t - t_1)$ gives a series with $c_v \cdot e^{-jv\omega t_1}$

- Reflection

  $s(-t)$ gives a series with c_v^*

Fundamentals of Electrical Engineering 3

Chapter 2.3

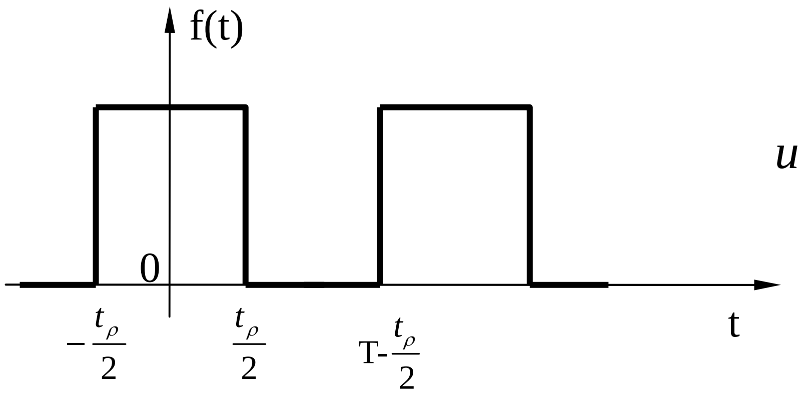
Description of aperiodic time operations by means of the Fourier transform

2.3.1 Prefaces

Onset: Development of the Fourier transform from the Fourier series by transfer of periodic functions to aperiodic impulses

<u>Example</u> : A periodic rectangular impulse is regarded

u(t) is an even function



$$u(t) = \begin{cases} U_0 & \text{for} \quad -\frac{t_i}{2} < t < \frac{t_i}{2} \\ 0 & \text{otherwise} \end{cases}$$

A Fourier analysis of this signal is to be accomplished.

2.3.1 Prefaces

Solution:

$$c_0 = \frac{1}{T} \int_{t=-\frac{t_i}{2}}^{t=+\frac{t_i}{2}} U_0 dt = \frac{U_0 t_i}{T}$$

$$\underline{c}_\nu = \left. \frac{a_\nu - jb_\nu}{2} \right|_{b_{\nu=0}} = \frac{a_\nu}{2} = \frac{1}{T} \int_{t=-\frac{t_i}{2}}^{t=\frac{t_i}{2}} U_0 \cos(\nu\omega t) dt = \frac{U_0}{T} \left. \frac{\sin(\nu\omega t)}{\nu\omega} \right|_{-\frac{t_i}{2}}^{\frac{t_i}{2}}$$

$$\text{with} \quad \omega = \frac{2\pi}{T}$$

$$\underline{c}_\nu = \frac{U_0}{T} \frac{\sin\left(\frac{2\pi\nu}{T}\frac{t_i}{2}\right) - \sin\left[\frac{2\pi\nu}{T}\left(-\frac{t_i}{2}\right)\right]}{\nu\omega} = \frac{U_0 t_i}{T} \frac{\sin\left(\nu\pi\frac{t_i}{T}\right)}{\nu\pi\frac{t_i}{T}}$$

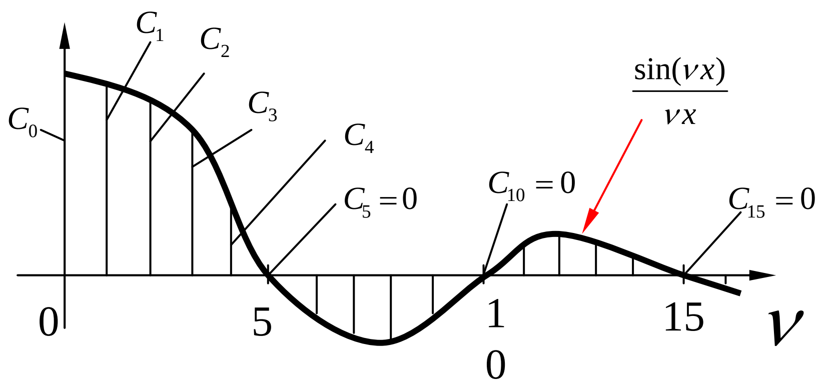
2.3.1 Prefaces

and:

$$a_\nu = 2\underline{c}_\nu = 2\frac{U_0 t_i}{T}\frac{\sin\left(\nu\pi\frac{t_i}{T}\right)}{\nu\omega\frac{t_i}{T}} = 2\frac{U_0 t_i}{T}si(\nu\omega\frac{t_i}{T})$$

Thus it applies:

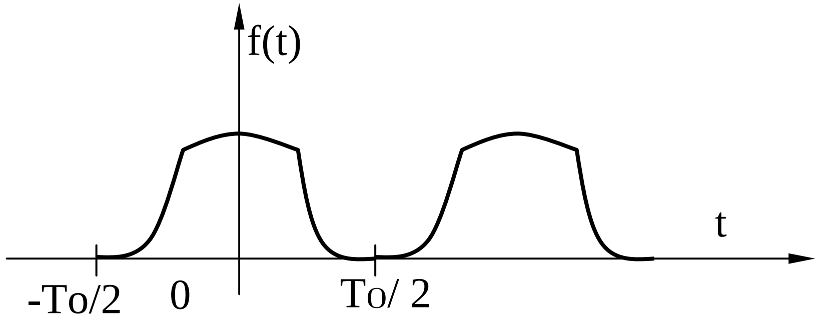
$$u(t) = \frac{U_0 t_i}{T}\sum_{\nu=-\infty}^{\nu=\infty} \mathrm{si}\left(\nu\pi\frac{t_i}{T}\right)e^{j\nu\omega t}$$



Sketch of the spectrum of *u(t)* for the case $\frac{t_i}{T} = 0,2$

2.3.2 The Fourier integral

In the following the Fourier series of a periodic function *f(t)* is examined. Here the period is: $T_0 = \frac{2\pi}{\omega_0}$



$$\underline{f}(t) = \sum_{\nu=-\infty}^{\infty} \underline{c}_\nu e^{j\nu\omega_0 t}$$

The following conditions are assumed:

1) *f(t)* shall be continous
2) In each finite period $-T_0/2 \le t \le T_0/2$ the function shall meets the Dirichlet' conditions

3) With infinite period *f(t)* shall be absolutely integrable

2.3.2 The Fourier integral

The following representation starts from a periodic signal which is transformed into a non-periodical of signal.

Note: Enlargement of the period is made by: $\lim_{T_0 \to \infty}$

Each term in the complex Fourier series corresponds to a line in spectrum. Distances between lines amount to: $\omega_0 = 2\pi / T_0$

In an interval $\Delta\omega$ around any point of frequency ω thereby lie in the following number of lines:

$$m = \frac{\Delta\omega}{\omega_0} = \frac{T_0}{2\pi}\Delta\omega$$

With sufficient small intervals then only small difference of the *m* individual terms of the complex Fourier series result.

Consequence: Summarizing of these terms is permitted!

2.3.2 The Fourier integral

In each interval with *m* lines applies thereby applies concerning its contribution to the row:

$$m \cdot \underline{c}_{v} e^{jv\omega_0 t} = \frac{T_0}{2\pi} \Delta\omega \cdot \underline{c}_{v} e^{jv\omega_0 t}$$

For $T_0 \rightarrow \infty$ one can select infinite small intervals (if *m* should remain unchanged)

Thus arises for the contribution of each interval to the row:

$$\frac{T_0}{2\pi} d\omega \cdot \underline{c}_{v} e^{jv\omega_0 t} \quad \text{with} \quad v\omega_0 \rightarrow \omega \quad \text{in which it holds:} \quad \frac{T_0}{2\pi} \underline{c}_{v} e^{j\omega t} d\omega$$

In addition can be written shortening: $T_0 \underline{c}_{v} = F(\omega)$

Altogether it results : $$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} F(\omega) e^{j\omega t} d\omega$$

2.3.2 The Fourier integral

Thus it also applies: $$f(t) = \frac{1}{2\pi}\int_0^{\infty} \underline{F}(\omega)e^{j\omega t}d\omega + \frac{1}{2\pi}\int_0^{\infty} \underline{F}^*(\omega)e^{-j\omega t}d\omega$$

or

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \underline{F}(\omega)e^{j\omega t}d\omega \quad mit \quad \underline{c}_\nu = \frac{1}{T}\int_{t=t_0}^{t=t_0+T} f(t)e^{-j\nu\omega t}$$

Now it consequently follows because of $T_0\underline{c}_\nu = F(\omega)$:

The Fourierspektrum and/or the Fourier transform $\underline{\mathfrak{F}}\{f(t)\}$

represents the function *f(t)* as follows:

$$\underline{\mathfrak{F}}\{f(t)\} = \underline{F}(\omega) = \int_{-\infty}^{+\infty} f(t)e^{-j\omega t}dt$$

The corresponding symbol:

$$f(t) \circ\!\!-\!\!\!-\!\!\bullet \underline{F}(\omega)$$

2.3.3 The Fourier inverse transform

The function *f(t)* can thus be represented by means of its spectrum:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \underline{F}(\omega) e^{j\omega t} d\omega$$

The inverse transform (from frequncy domain to the time domain is in short written as follows:

$$\underline{F}(\omega) \bullet\!\!-\!\!\circ \; f(t)$$

$\underline{F}(\omega)$ has not the property of an amplitude (as in the Fourier series), but it is an amplitude density function with the dimension:

$$\text{Amplitude x Time or } \frac{\text{Amplitude}}{\text{Frequency}}$$

The existence of the Fourier integral is ensured if *f(t)* is absolutely integrable:

$$\int_{-\infty}^{+\infty} |f(t)| dt \le S = const$$

2.3.4 Interpretation and summary

Now a signal *s(t)* is considered, from which by ideal BP filtering only portions within a certain frequency band are extracted. The filtering takes place so narrow-banded that therein the spectrum (and the exponential function) changes only insignificantly. For this extracted portion *g(t)* follows:

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\int_{-\infty}^{0} S(\omega)e^{j\omega t}d\omega + \frac{1}{2\pi}\int_{0}^{+\infty} S(\omega)e^{j\omega t}d\omega$$

$$\approx \frac{\Delta\omega}{2\pi}(S(-\omega_0)e^{-j\omega_0 t} + S(\omega_0)e^{j\omega_0 t}) = \frac{\Delta\omega}{\pi}\mathrm{Re}\{S(\omega_0)e^{j\omega_0 t}\}$$

$$= \frac{\Delta\omega}{\pi}\mathrm{Re}\{|S(\omega_0)|e^{j\arg(S(\omega_0))}e^{j\omega_0 t}\} \quad \text{wegen} \quad S(-\omega_0) = S^*(\omega_0)$$

$$= \frac{\Delta\omega}{\pi}|S(\omega_0)|\cos(\omega_0 t + \arg(S(\omega_0)))$$

2.3.4 Interpretation and summary

- The Fourier transform (under given conditions) thus is a measure for the amplitude and the phase of a signal component with reference to the regarded bandwith concerning the regarded frequency.
- The method of the Fourier transform permits:

  1) To describe a signal known in the time domain equivalently in the frequency domain

  2) To determine the function in the time domain from a known Fourier transform
- The Fourier transform is an important and one of the most powerful tools of electrical engineering, control engineering, physics (optics, mechanics and many more).
- This method forms at the same time the basis of the Laplace transform, Z-transform and the discrete Fourier transform.