Fundamentals of EE 3

Chapter 3.1

Transient processes - The Laplace transform

UNIVERSITÄT DUISBURG ESSEN
Fachgebiet
Nachrichtentechnische Systeme
NTS

3.1.1 Introduction

Use of a complex frequency *p* for time signals. It applies: $p = \sigma + j\omega$

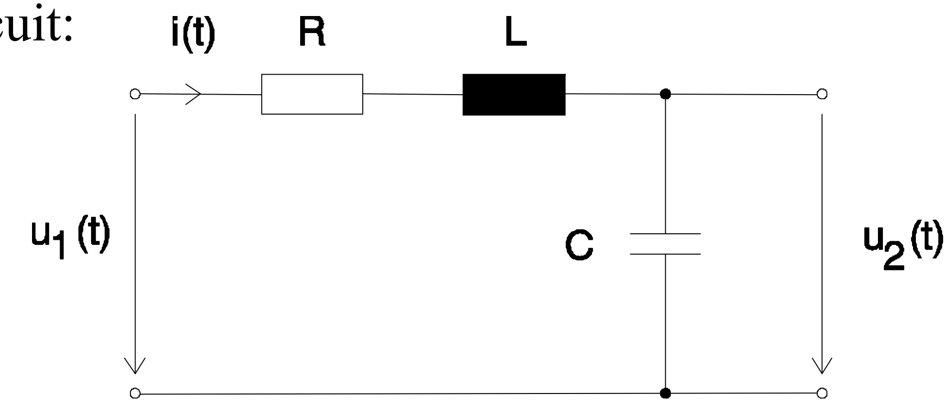
Thus exponential envelopes are representable for sinusoidal functions in addition to those with constant amplitudes:

Example for RLC resonant circuit:

$$u_1(t) = \mathrm{Re}\left\{\underline{\hat{u}}_1 e^{pt}\right\}$$

$$u_2(t) = \mathrm{Re}\left\{\underline{\hat{u}}_2 e^{pt}\right\}$$

$$i(t) = \mathrm{Re}\left\{\underline{\hat{i}} e^{pt}\right\}$$



3.1.1 Introduction

A network analysis shows here that it applies:

$$u_1(t) = i(t)R + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(z)\,dz$$

$$\Rightarrow u_1(t) = i(t)R + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{0} i(z)\,dz + \frac{1}{C}\int_{0}^{t} i(z)\,dz \quad \Big| \quad u_2(0) = \frac{1}{C}\int_{-\infty}^{0} i(z)\,dz$$

Solution can be achived by means of:

- solution of the integro-differential equations
- the Laplace transform

3.1.2 Definition of the Laplace transform

An arbitry function is given: $f(t) = \begin{cases} 0 & \text{for} \quad -\infty \leq t < 0 \\ g(t) & \text{for} \quad 0 < t \leq +\infty \end{cases}$

f(t) exhibits the following characteristics:

1) *f(t)* has only finite jumps in the interval $0 < t_1 \leq t \leq t_2$

2) $\int_{t_1}^{t_2} f(t)dt$ is limited

3) $f(t)$ should drop to zero quickly enough: $\lim_{t \to \infty} f(t) \to e^{-\sigma_0 t}$

If these conditions are fulfilled

the Laplace transform of the function *f(t)* exists:

$$\mathfrak{L}\{f(t)\} = \mathfrak{L}\{g(t)\} = \underline{G}(p) = \int_0^{+\infty} g(t)e^{-pt}dt$$

3.1.2 Definition of the Laplace transform

The relation of original signals and corresponding transform is described by the symbol after DOETSCH:

$$g(t) \circ\!\!-\!\!\bullet \underline{G}(p) \qquad \underline{G}(p) \bullet\!\!-\!\!\circ g(t)$$

Relation with Fourier transform is given for:

- a causal time function and at the same time
- a modified time function due to multiplication with $e^{-\sigma t}$

This leads in the integrand of the Fourier Transf. to a term $s(t) \cdot e^{-\sigma t} e^{-j\omega t} = s(t) e^{-pt}$

$$L\{s(t)\} = \int_0^{\infty} s(t) e^{-pt} dt = S(p)$$

$$L^{-1}\{S(p)\} = \lim_{\omega \to \infty} \frac{1}{2\pi j} \int_{\sigma - j\omega}^{\sigma + j\omega} S(p) e^{pt} dp = s(t)$$

3.1.3 Methods for the determination of the original function from the image function

<u>Direct method:</u>

The original function can be determined according to the relationship:

$$g(t) = \frac{1}{2\pi j} \lim_{\omega \to \infty} \int_{\sigma - j\omega}^{\sigma + j\omega} \underline{G}(p) e^{pt} dp$$

for $t > 0$

The use of this formula requires
however some knowledge of the function theory.

3.1.3 Methods for the determination of the original function from the image function

Use of transformation tables:

Example: In a network it applies to the transform of the current:

$$\mathfrak{L}\{i(t)\} = \frac{U_0}{R}\,\frac{1}{p(1+\tau p)} \quad \text{with} \quad \tau = \frac{L}{R}$$

The original function *i(t)* of the current is looked for.

Solution: From a transformation table one receives:

$$\left(1-e^{-\frac{t}{a}}\right) \circ\!\!-\!\!\bullet \; \frac{1}{p(1+ap)}$$

With $a=\tau$ it applies then to the original function: $i(t) = \frac{U_0}{R}\left(1-e^{-\frac{t}{\tau}}\right)$

3.1.3 Methods for the determination of the original function from the image function

The method of decomposition into partial fractions:

It accepted that itself the image function $\underline{G}(p)$ in a broken rational function:

$$\underline{G}(p) = \frac{\underline{Z}(p)}{\underline{N}(p)}$$

Order of the nominator polynomial $\underline{N}(p)$ **<** Order of the denominator polynomial

The individual partial fractions have thereby the form:

$$\frac{\underline{A}_\nu}{(p - p_\nu)^k} \qquad k = 1, 2, 3, \ldots$$

Now for each partial fraction if the respective original function is intended, then results in itself as sum of these the original function *g(t)*.

3.1.3 Methods for the determination of the original function from the image function

The Heaviside expansion theorem:

➡ Conditions: $\underline{G}(p)$ is a broken rational function,
The denominator degree is higher than the nominator degree,
The denominator polynomial has only n simple zeros $(k = 1)$

Thus it applies: $\underline{G}(p) = \dfrac{\underline{Z}(p)}{\underline{N}(p)} = \sum_{\nu=1}^{n} \dfrac{\underline{A}_\nu}{p - p_\nu}$

$$\Leftrightarrow (p - p_\nu) \cdot \underline{G}(p) = \frac{(p - p_\nu)\underline{Z}(p)}{\underline{N}(p)} = (p - p_\nu)\left(\frac{\underline{A}_1}{p - p_1} + \frac{\underline{A}_2}{p - p_2} + \ldots + \frac{\underline{A}_\nu}{p - p_\nu} + \ldots + \frac{\underline{A}_n}{p - p_n}\right)$$

$(p - p_\nu) \cdot \underline{G}(p) = \dfrac{(p - p_\nu) \cdot \underline{Z}(p)}{\underline{N}(p)} = \underline{A}_\nu$ for limiting with $p \to p_\nu$

3.1.3 Methods for the determination of the original function from the image function

- For further solution the L'Hospital rule must be used (if linear factor form is not given):

$$\underline{A}_\nu = \lim_{p \to p_\nu} \frac{\frac{d}{dp}\left[(p-p_\nu)\underline{Z}(p)\right]}{\frac{d}{dp}\left[\underline{N}(p)\right]} = \lim_{p \to p_\nu} \frac{\underline{Z}(p) + \overbrace{(p-p_\nu)}^{0}\underline{Z}'(p)}{\underline{N}'(p)}$$

Thus it applies:

$$\underline{A}_\nu = \lim_{p \to p_\nu} \frac{\underline{Z}(p)}{\underline{N}'(p)}$$

Because of the correspondence

$$e^{p_\nu t} \circ\!\!-\!\!\bullet \frac{1}{p-p_\nu}$$

the original function gives:

$$g(t) = \sum_{\nu=1}^{n} \underline{A}_\nu e^{p_\nu t}$$

3.1.3 Methods for the determination of the original function from the image function

The modified Heaviside expansion theorem:

In case of a pole at the origin the following applies:

$$\underline{G}(p) = \frac{\underline{Z}_1(p)}{p \cdot \underline{N}_1(p)} \quad \texttt{with} \quad p_1 = 0$$

For the original function arises then:

$$g(t) = \frac{\underline{Z}_1(0)}{\underline{N}_1(0)} + \sum_{\nu=2}^{n} \frac{\underline{Z}_1(p_\nu)}{p\underline{N}_1^{'}(p_\nu)} . e^{p_\nu t}$$

Example: In a network is the image function of a branch current is as follows:

$$\mathfrak{L}\{i_Z(t)\} = \frac{U}{L} . \frac{p + 1/\tau}{p\left[p^2(1-k^2) + 2\,p/\tau + 1/\tau^2\right]}$$

3.1.3 Methods for the determination of the original function from the image function

- **Solution:**

$$\underline{Z}_1(p) = p + \frac{1}{\tau}$$

$$\underline{N}_1(p) = p^2(1-k^2) + 2\frac{p}{\tau} + \frac{1}{\tau^2} \quad ; \quad \underline{N}_1^{'}(p) = 2\left[p(1-k^2) + 1/\tau\right]$$

Thus it applies:

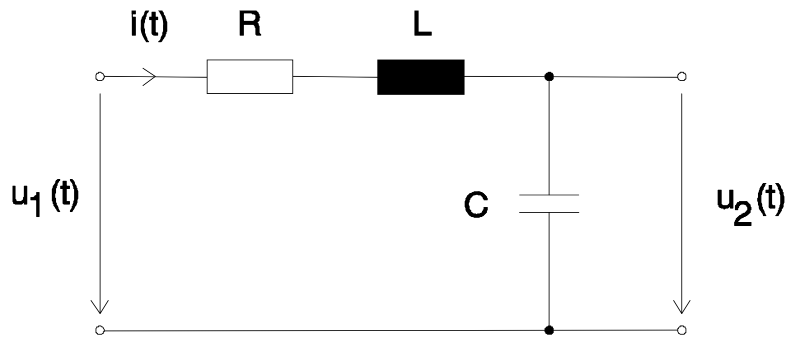
$$i_Z(t) = \frac{U}{L}\left[\frac{\frac{1}{\tau}}{\frac{1}{\tau^2}} + \frac{p_2 + \frac{1}{\tau}}{2\left[p_2(1-k^2) + \frac{1}{\tau}\right]}e^{p_2 t} + \frac{p_3 + \frac{1}{\tau}}{2\left[p_3(1-k^2) + \frac{1}{\tau}\right]}e^{p_3 t}\right]$$

Fundamentals of EE 3

Chapter 3.2

Transient processes - Application of the Laplace transform

3.2. Application of the Laplace transform



$$u_1(t) = i(t)R + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{t} i(z)dz$$

$$\Rightarrow u_1(t) = i(t)R + L\frac{di(t)}{dt} + \frac{1}{C}\int_{-\infty}^{0} i(z)dz + \frac{1}{C}\int_{0}^{t} i(z)dz \quad \Bigg| \quad u_2(0) = \frac{1}{C}\int_{-\infty}^{0} i(z)dz$$

$$U_1(p) = I(p)R + L\left[pI(p) - i(0)\right] + \frac{1}{C}\frac{I(p)}{p} + \frac{u_2(0)}{p}$$

$$= I(p)R + pLI(p) + \frac{1}{pC}I(p) - i(0)L + \frac{u_2(0)}{p}$$

3.2. Application of the Laplace transform

- Dissolve for the transform of current and voltage gives for zero-state condition:

$$U_1(p) = I(p)\cdot(R + pL + 1/pC)$$

$$I(p) = \frac{U_1(p)}{R + pL + 1/pC}$$

$$\Rightarrow i(t) = L^{-1}\left\{\frac{U_1(p)}{R + pL + 1/pC}\right\}$$

For comparison:

NW-Analyses would have given:

$$\underline{\hat{u}}_1 = R\underline{\hat{i}} + pL\underline{\hat{i}} + \frac{1}{pC}\underline{\hat{i}}$$

3.2. Application of the Laplace transform

Procedure for general solution of the network varaibles:

1. NW-analysis in the time intervall → System of integro-differential equations

2. Laplace transform of the integro-differential equation system after specifying the initial values at $t = 0$ and determination of the Laplace Transforms

3. Dissolution after Laplace transforms of the unknown variables by means of algebra rewriting

4. Inverse transform to the desired variables.

Fundamentals of EE 3

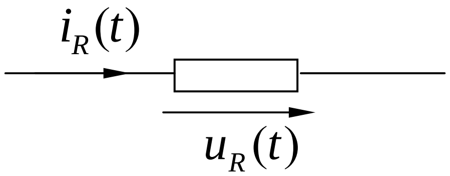
Chapter 3.3

The classical method of directly solving differential equations

3.3.1 Introduction

Repetition to network elements concerning current, voltage, energy and/or instantaneous power *p(t)* :

1) Resisitor :



$$u_R(t) = Ri_R(t)$$

$$W = \int p(t)dt = \int u(t)i(t)dt$$

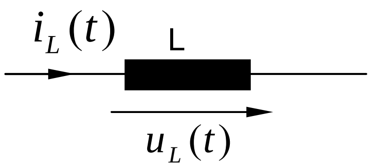
UNIVERSITÄT DUISBURG ESSEN

Fachgebiet
Nachrichtentechnische Systeme

NTS

3.3.1 Introduction

2) Coil:

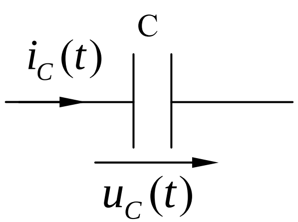


$$u_L(t) = L\frac{di_L(t)}{dt} \quad oder$$

$$i_L(t) = \frac{1}{L}\int_{-\infty}^{t} u_L(\tau)d\tau$$

$$\mathrm{W}_{\mathrm{magn}} = \int_{-\infty}^{t} p(\tau)d\tau = \int_{-\infty}^{t} L\frac{di}{dt}i(t)dt = L\int_{0}^{i_L(t)} idi = \frac{1}{2}Li_L(t)^2$$

3) Capacitor :



$$u_C(t) = \frac{1}{C}\int_{-\infty}^{t} i_C(\tau)d\tau$$

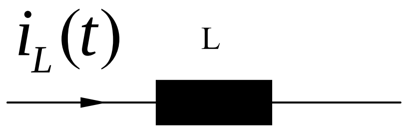
$$i_C(t) = C\frac{du_C(t)}{dt}$$

$$W_{el} = \int_{-\infty}^{t} p(\tau)d\tau = C\int_{o}^{u_C(t)} udu = \frac{1}{2}Cu_C(t)^2$$

3.3.1 Introduction

In the network only finite voltages and currents are possible.

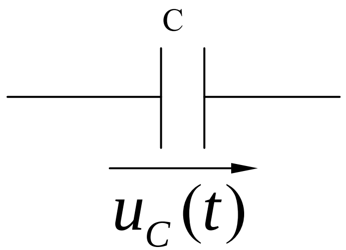
Thus: No instantaneous change of the energy of these elements is possible



i_L cannot change instantaneously

(i_L is constant)

$$i_L(t-\Delta t) = i_L(t+\Delta t)\Big|_{for \quad \Delta t \to 0} \qquad t=0 \to i_L(t=-0) = i_L(t=+0)$$



u_C cannot change instantaneously

(u_C is constant)

$$u_C(t-\Delta t) = u_C(t+\Delta t)\Big|_{for \quad \Delta t \to 0} \qquad t=0 \to u_C(t=-0) = u_C(t=+0)$$

3.3.2 The method of the differential equations

- Kirchhoff‘s equations lead to differential equations
- These equations have only constant coefficients
- Their solutions provide the desired network varaibles
- 1. Step: Solution of the system of the homogeneous differential equations (which represent the natural oscillations of the network)
- 2. Step: Add in each case an individual (particular) solution.

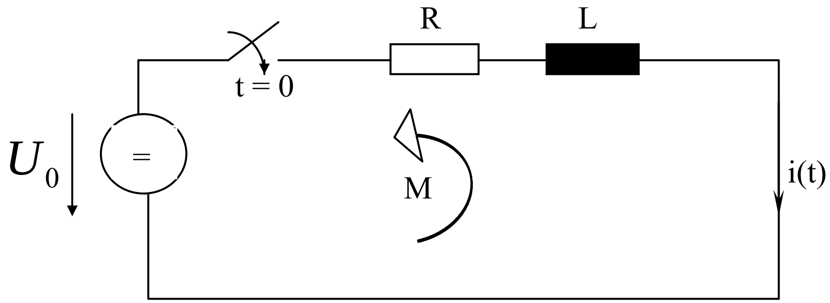
  Thus the general solution of the system of the inhomogenous differential equations is obtained:

$$i(t) = i_h(t) + i_p(t) \qquad u(t) = u_h(t) + u_p(t)$$

For a network with DC or constant sinusoidal excitation, the particular describe the steady-state conditions at the time t - > ∞

3.3.2 The method of the differential equations

Example 1 : Connection of DC voltage to a "RL - circuit".



At the time $t = 0$ the network is connected to a source. The current $i(t)$ is computed with the initial condition:

$$i(t=-0)=0$$

Solution : For $t \geq 0$ applies:

$$Ri(t)+L\frac{di(t)}{dt}=U_0$$

The differential equation is a usual, linear differential equation of first order with constant coefficients, which is inhomogenous.

3.3.2 The method of the differential equations

A) Solution of the homogeneous differential equation:

$$Ri(t) + L\frac{di(t)}{dt} = 0 \quad \Leftrightarrow \quad \frac{di(t)}{dt} + \frac{R}{L}i(t) = 0$$

On-set: $i_h(t) = Ke^{-\frac{t}{\tau}} \rightarrow \frac{di_h(t)}{dt} = -\frac{K}{\tau}e^{-\frac{t}{\tau}}$

$$-\frac{K}{\tau}e^{-\frac{t}{\tau}} + \frac{R}{L}Ke^{-\frac{t}{\tau}} = (\frac{-1}{\tau} + \frac{R}{L})Ke^{-\frac{t}{\tau}} = 0 \text{ with } \tau = \frac{L}{R}$$

τ is the time constant of the circuit

3.3.2 The method of the differential equations

$$\frac{di(t)}{dt} + \frac{1}{\tau} i(t) = 0 \quad \longrightarrow \quad i_h(t) = K e^{-\frac{R}{L}t}$$

B) Individual (particular) solution of the inhomogenous differential equation:

After infinitely long time $(t \to \infty)$ the current of the coil L shows its steady-state value. Then the voltage disappears, i.e. for the current i(t) holds:

$$i_p(t) = \frac{U_0}{R} \neq f(t) \quad for \quad t \to \infty$$

This corresponds to the case: $di_L(t)/dt = 0 \ , \ d.h \ : \ u_L = 0$

3.3.2 The method of the differential equations

C) Complete solution:

$$i(t) = i_h(t) + i_p(t) = Ke^{-\frac{R}{L}t} + \frac{U_0}{R}$$

Initial condition: The coil current $i(t=-0) = i(t=+0) = 0$ cannot change instantaneously

$$0 = \underbrace{Ke^{-\frac{R}{L}.0}}_{1} + \frac{U_0}{R} \rightarrow \quad K = -\frac{U_0}{R}$$

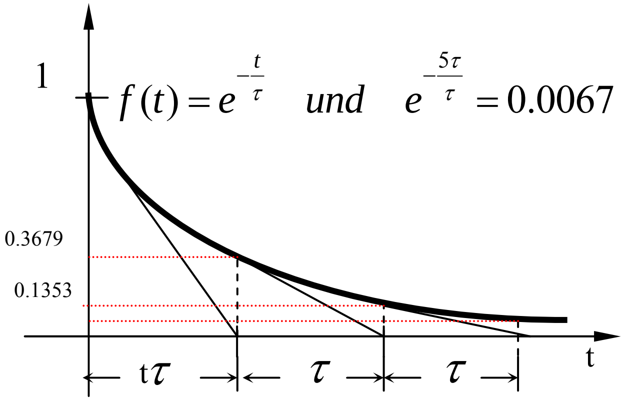
For the current *i(t)* thus results:

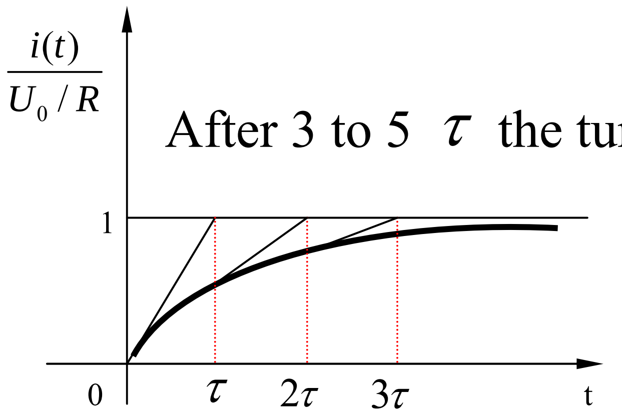
$$i(t) = \frac{U_0}{R}(1 - e^{-\frac{R}{L}t})$$

for t > 0

3.3.2 The method of the differential equations

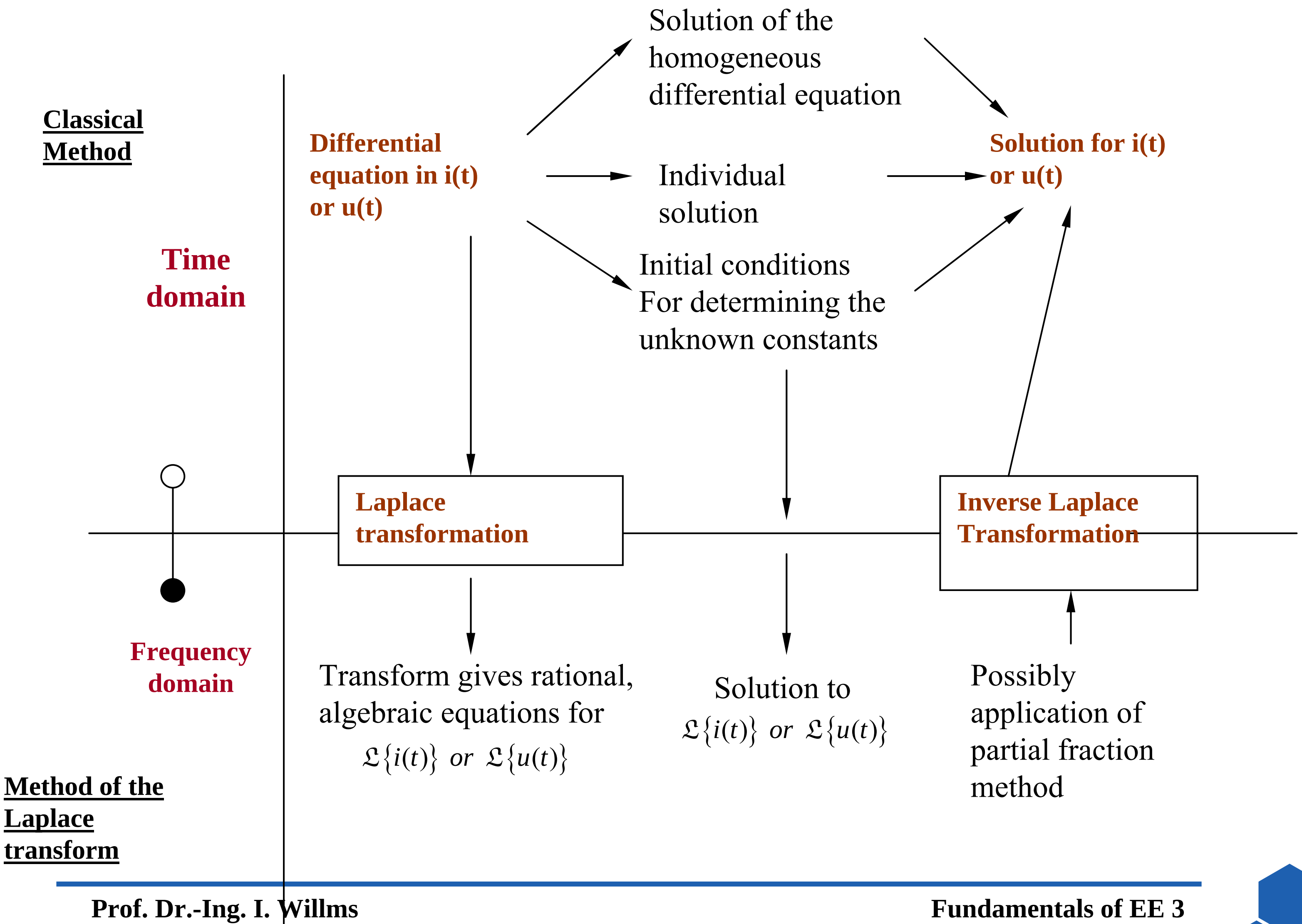
<u>Remarks on the exponential function:</u> $f(t) = e^{-\frac{t}{\tau}}$





After 3 to 5 τ the turn-on transient is practically terminated

3.3.3 Overview



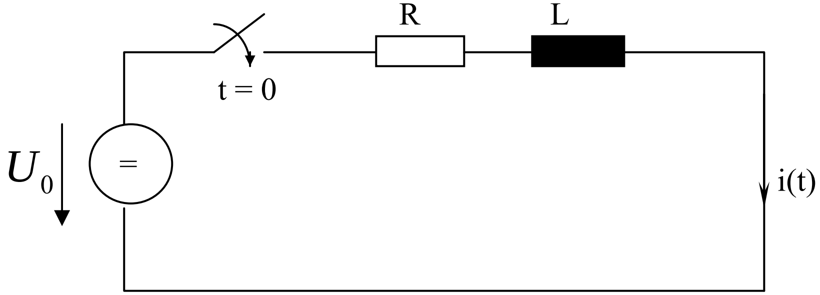
Fundamentals of EE 3

Chapter 3.4

The method of solving differential equations by the Laplace transform

3.4 The method of the Laplace transform

- Example 1:



Initial condition: $i(t=-0)=0$

$$\Rightarrow \frac{di(t)}{dt}+\frac{1}{\tau}i(t)=\frac{U_0}{L} \text{ mit } \tau=\frac{L}{R}$$

$$\mathfrak{L}\left\{\frac{di(t)}{dt}\right\}+\frac{1}{\tau}\mathfrak{L}\{i(t)\}=\frac{U_0}{L}\mathfrak{L}\{\varepsilon(t)\}$$

$$p\mathfrak{L}\{i(t)\}-\underbrace{i(0)}_{0}+\frac{1}{\tau}\mathfrak{L}\{i(t)\}=\frac{U_0}{L}\frac{1}{p}$$

$$\mathfrak{L}\{i(t)\}\left(p+\frac{1}{\tau}\right)=\frac{U_0}{L}\frac{1}{p}$$

$$\mathfrak{L}\{i(t)\}=\frac{U_0}{L}\frac{1}{p(p+1/\tau)}$$

3.4 The method of the Laplace transform

The inverse transform can be performed by means of the modified Heaviside` theorem:

$$\mathfrak{L}^{-1}\left\{\underline{I}(p)=\frac{\underline{Z}_1(p)}{p\underline{N}_1(p)}\right\}=i(t)=\frac{\underline{Z}_1(0)}{\underline{N}_1(0)}+\sum_{\nu=2}^{n}\frac{\underline{Z}_1(p_\nu)}{p_\nu\underline{N}_1'(p_\nu)}e^{p_\nu t} \quad \text{with} \quad p_1=0 \text{ und } n=2$$

Thus applies:

$$\underline{Z}_1(p)=\frac{U_0}{L},\quad \underline{N}_1(p)=p+\frac{1}{\tau},\quad p_2=-\frac{1}{\tau},\quad \underline{N}_1'(p)=1$$

This gives for the current *i(t)*:

$$i(t)=\frac{U_0}{L}\left[\frac{1}{1/\tau}+\frac{1}{-1/\tau}e^{-\frac{t}{\tau}}\right]=\frac{U_0}{L}\tau\left(1-e^{-\frac{t}{\tau}}\right) \quad \text{with} \quad \tau=\frac{L}{R}$$

$$i(t)=\frac{U_0}{R}(1-e^{-\frac{t}{\tau}}) \quad \text{for } t>0$$

3.4 The method of the Laplace transform

For the inverse transform also the convolution theorem might have been used:

$$\underline{G}_1(p)\cdot\underline{G}_2(p)=\mathfrak{L}\{g_1(t)*g_2(t)\} \qquad g_2(t)*g_1(t)=\int_0^t g_2(x)g_1(t-x)dx=\int_0^t g_1(x)g_2(t-x)dx$$

$$\mathfrak{L}^{-1}\left\{\frac{1}{p}\right\}=1 \text{ for } t\geq 0 \quad \textit{and} \quad \mathfrak{L}^{-1}\left\{\frac{1}{p+1/\tau}\right\}=e^{-\frac{t}{\tau}}$$

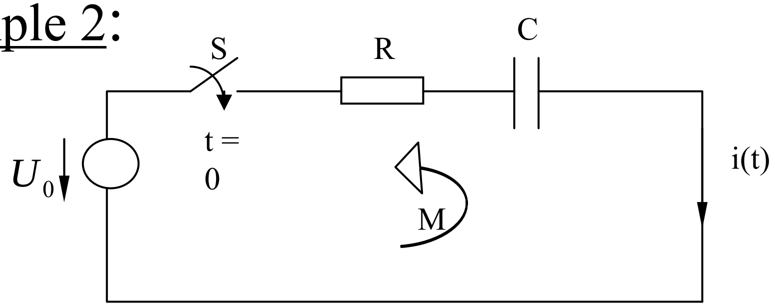
Thus it follows: $\mathfrak{L}^{-1}\{i(t)\}=\mathfrak{L}^{-1}\left\{\frac{U_0}{L}\frac{1}{p}\cdot\frac{1}{(p+1/\tau)}\right\}$

$$=\frac{U_0}{L}\int_0^t 1\cdot e^{-\frac{x}{\tau}}dx=\frac{U_0}{L}\left[\frac{e^{-\frac{x}{\tau}}}{-\frac{1}{\tau}}\right]\Bigg|_0^t=\frac{U_0}{L}(-\tau)\left(e^{-\frac{t}{\tau}}-1\right)$$

$$=\frac{U_0}{R}(1-e^{-\frac{t}{\tau}})$$

3.4 The method of the Laplace transform

Example 2:



At $t = 0$ the capacitor is charged to Q_0

1) General relations:

$$Ri(t) + \frac{Q(t)}{C} = U_0 \quad mit \quad Q(t) = Q_0 + \int_0^t i(x)dx$$

$$i(t)R + \frac{Q_0 + \int_0^t i(x)dx}{C} = U_0$$

$$\Rightarrow \quad i(t) + \frac{1}{RC}\int_0^t i(x)dx = \frac{U_0}{R} - \frac{Q_0}{RC}$$

3.4 The method of the Laplace transform

2) Solution with the Laplace transform:

$$i(t)+\frac{1}{RC}\int_0^t i(x)dx=\frac{U_0}{R}-\frac{Q_0}{RC}$$

$$\mathfrak{L}\{i(t)\}+\frac{1}{RC}\underbrace{\frac{1}{p}\mathfrak{L}\{i(t)\}}_{\text{Integration theorem}}=\left(\frac{U_0}{R}-\frac{Q_0}{RC}\right)\mathfrak{L}\{\varepsilon(t)\}$$

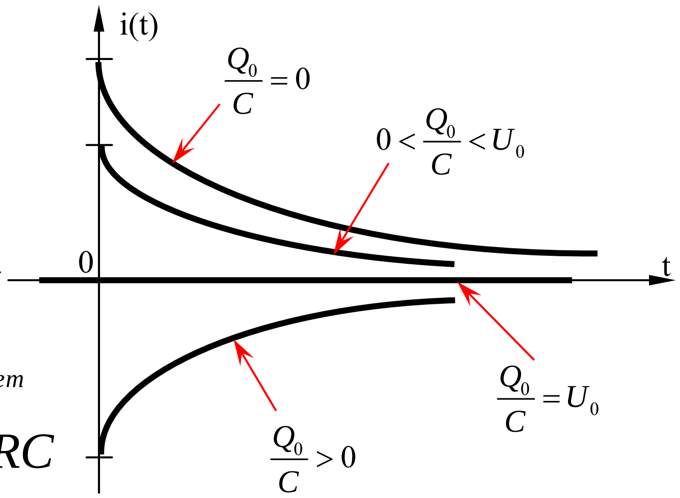
$$\mathfrak{L}\{i(t)\}\left(1+\frac{1}{RCp}\right)=\left(\frac{U_0}{R}-\frac{Q_0}{RC}\right)\frac{1}{p}$$

$$\mathfrak{L}\{i(t)\}=\left(\frac{U_0}{R}-\frac{Q_0}{RC}\right)\frac{RCp}{1+RCp}\frac{1}{p}=\left(\frac{U_0}{R}-\frac{Q_0}{RC}\right)\underbrace{\frac{1}{{}^1/_{RC}+p}}_{\text{Attenuation theorem}}$$

$$i(t)=\left(\frac{U_0}{R}-\frac{Q_0}{RC}\right)e^{-t/RC}$$

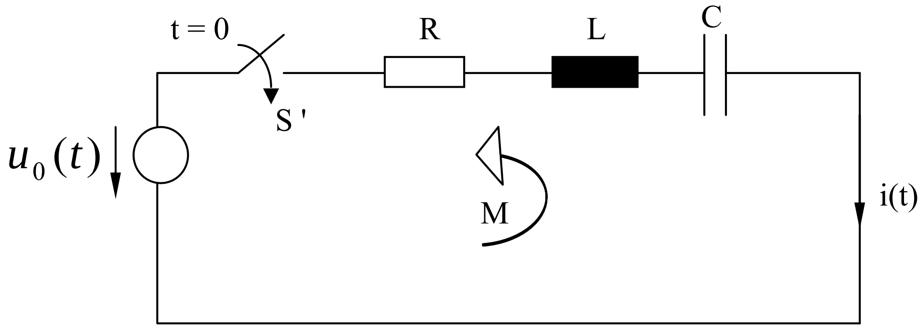
for $t > 0$ also $\tau = RC$

Diagram of the results:



3.4 The method of the Laplace transform

Example 3 : In this example a series resonant circuit is connected to a source, with cases a, b and c by an ideal switch S:



a) AC voltage: $u_0(t) = \hat{u}_0 \cos(\omega t + \varphi_u)$

b) DC voltage: $u_0(t) = U_0$

c) Mixed case: $u_0(t) = U_0 + \hat{u}_0 \cos(\omega t + \varphi_u)$

Desired for all 3 cases: *i(t)* for *t > 0*

0) General relations

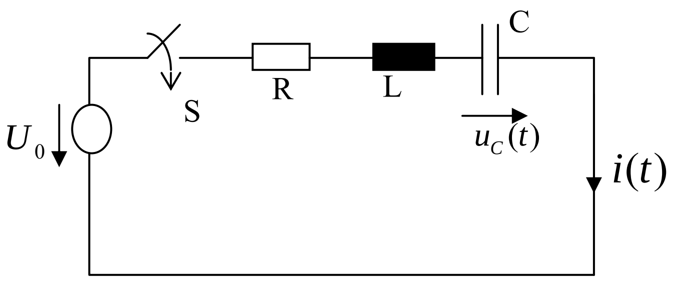
The loop equation reads:

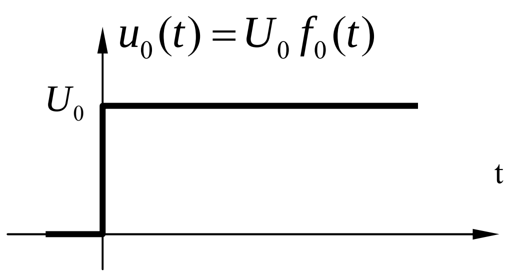
$$R \cdot i(t) + L\frac{di(t)}{dt} + u_C(t) = u_0(t)$$ and also $$i(t) = C\frac{du_C(t)}{dt}$$ applies.

$$RC\frac{du_C(t)}{dt} + LC\frac{d^2u_C(t)}{dt} + u_C(t) = u_0(t)$$

3.4 The method of the Laplace transform

- Solution for case 3 (D < 1) and source case b) (DC source)





3.4 The method of the Laplace transform

- Solution by means of the Laplace Transform for case 3 (D < 1) and source case b) (DC source)

The inhomogenous differential equation is subjected to the Laplace Transform:

$$\frac{d^2u_C(t)}{dt^2}+2\omega_0 D\frac{du_C(t)}{dt}+\omega_0{}^2u_C(t)=\omega_0{}^2U_0 \qquad \text{for } t>0$$

$$\circ\!-\!\!\bullet$$

$$p^2\mathfrak{L}\{u_C(t)\}-pu_C(0)-u_C^{'}(0)+2\omega_0 D\left[p\mathfrak{L}\{u_C(t)\}-u_C(0)\right]+\omega_0{}^2\mathfrak{L}\{u_C(t)\}=\frac{\omega_0{}^2U_0}{p}$$

With the initial conditions $u_C(0)=0 \quad und \quad u_C^{'}(0)=0$

it follows:

$$p^2\mathfrak{L}\{u_C(t)\}+2\omega_0 Dp\mathfrak{L}\{u_C(t)\}+\omega_0^2\mathfrak{L}\{u_C(t)\}=\frac{\omega_0^2U_0}{p}$$

3.4 The method of the Laplace transform

Thus it applies to the Laplace transform of $u_C(t)$:

$$\mathfrak{L}\{u_C(t)\} = \frac{\omega_0^2 U_0}{p(p^2 + 2\omega_0 Dp + \omega_0{}^2)}$$

The original function can be obtained using the modified theorem of Heaviside:

$$\mathfrak{L}\{u_C(t)\} = \frac{\underline{Z}_1(p)}{p\underline{N}_1(p)} \quad \text{with} \quad \underline{Z}_1(p) = \omega_0{}^2 U_0 \;, \quad \underline{N}_1(p) = p^2 + 2\omega_0 Dp + \omega_0{}^2$$

$$\frac{d\underline{N}_1(p)}{dp} = 2p + 2\omega_0 D \;, \quad p_1 = 0, p_2 = \omega_0(-D + j\sqrt{1-D^2})$$

$$p_3 = \omega_0(-D - j\sqrt{1-D^2})$$

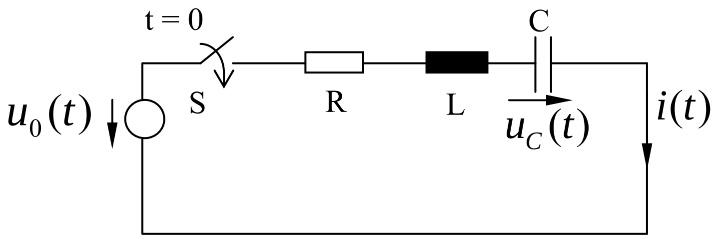
Thus the original function results to:

$$u_C(t) = \frac{\underline{Z}_1(0)}{\underline{N}_1(0)} + \sum_{v=2}^{3} \frac{\underline{Z}_1(p_v)}{p_v \underline{N}_1^{'}(p_v)} \cdot e^{p_v t} = \frac{\omega_0{}^2 U_0}{\omega_0{}^2} + \frac{\omega_0{}^2 U_0}{p_2 2(p_2 + \omega_0 D)} e^{p_2 . t} + \frac{\omega_0{}^2 U_0}{p_3 2(p_3 + \omega_0 D)} e^{p_3 . t}$$

3.4 The method of the Laplace transform

- Solution for case 3 (D < 1) and source case a) (AC source)

Here only the case of conjugated complex poles (i.e. D < 1) is delt with.
For the inhomogenous differential equation then holds:



$$\frac{d^2u_C(t)}{dt^2} + 2\omega_0 D \frac{du_C(t)}{dt} + \omega_0{}^2 u_C(t) = \omega_0{}^2 u_0(t)$$

for $t > 0$, also $u_0(t) = \hat{u}_0 \cos(\omega t + \varphi_u)$

3.4 The method of the Laplace transform

- Solution by means of the Laplace transform for case 3 (D < 1) and source case a) (AC source)

Following is the differential equation for $u_C(t)$:

$$\frac{d^2u_C(t)}{dt^2} + 2\omega_0 D \frac{du_C(t)}{dt} + \omega_0{}^2 u_C(t) = \omega_0{}^2 u_0(t)$$

After Laplace transforming of this equation it results:

$$p^2\mathfrak{L}\{u_C(t)\} - pu_C(0) - u_C{}'(0) + 2\omega_0 D\left[p\mathfrak{L}\{u_C(t)\} - u_C(0)\right] + \omega_0{}^2\mathfrak{L}\{u_C(t)\} = \omega_0{}^2\mathfrak{L}\{u_0(t)\}$$

With the initial conditions: $u_C(0) = 0 \; und \; u_C{}'(0) = 0$ it follows:

$$\underline{p}^2\mathfrak{L}\{u_C(t)\} + 2\omega_0 D\left[\underline{p}\mathfrak{L}\{u_C(t)\}\right] + \omega_0{}^2\mathfrak{L}\{u_C(t)\} = \omega_0{}^2\mathfrak{L}\{u_0(t)\}$$

3.4 The method of the Laplace transform

Thus it applies to the image function of $u_C(t)$:

$$\mathfrak{L}\{u_C(t)\} = \frac{\omega_0{}^2 \mathfrak{L}\{u_0(t)\}}{p^2 + 2\omega_0 D p + \omega_0{}^2}$$

Concerning the source it holds:

$$\mathfrak{L}\{u_0(t)\} = \mathfrak{L}\{\hat{u}_0 \cos(\omega t + \varphi_u)\} = \hat{u}_0 \frac{p\cos\varphi_u - \omega \sin\varphi_u}{p^2 + \omega^2}$$

Combining the equations given above leads then to:

$$\mathfrak{L}\{u_C(t)\} = \hat{u}_0 \omega_0{}^2 \frac{p\cos\varphi_u - \omega \sin\varphi_u}{\left(p^2 + \omega^2\right)\left(p^2 + 2\omega_0 D p + \omega_0{}^2\right)}$$

3.4 The method of the Laplace transform

The image function must then be back-transformed then into the original domain e.g. with help of the method of the decomposition into partial fractions. Here it applies:

$$\mathfrak{L}\{u_C(t)\} = \frac{\underline{Z}(p)}{\underline{N}(p)}$$

with $\underline{Z}(p) = \hat{u}_0\omega_0(p\cos\varphi_u - \omega\sin\varphi_u)$

and $\underline{N}(p) = (p^2+\omega^2)(p^2+2\omega_0 Dp+\omega_0{}^2)$

Due to the periodic case (D < 1) it applies:

$$p_1 = j\omega\,,\; p_2 = -j\omega$$

$$\underline{p}_{3,4} = \sigma \pm j\omega_1 = \omega_0(-D \pm j\sqrt{1-D^2})$$

For the first derivative of the denominator it applies:

$$\frac{d\underline{N}(p)}{dp} = 4p^3 + 6\omega_0 Dp^2 + 2(\omega_0{}^2+\omega^2)p + 2\omega_0 D\omega^2$$

3.4 The method of the Laplace transform

Thus finally for the original function holds:

$$u_C(t)=\hat{u}_0\omega_0^2\left\{\frac{j\omega\cos\varphi_u-\omega\sin\varphi_u}{4(j\omega)^3+6\omega_0D(j\omega)^2+2(\omega_0^2+\omega^2)j\omega}e^{j\omega t}+\frac{-j\omega\cos\varphi_u-\omega\sin\varphi_u}{4(-j\omega)^3+6\omega_0D(-j\omega)^2+2(\omega_0^2+\omega^2)(-j\omega)}e^{-j\omega t}\right.$$

$$\left.+\frac{p_1\cos\varphi_u-\omega\sin\varphi_u}{4p_1^3+6\omega_0Dp_1^2+2(\omega_0^2+\omega^2)p_1+2\omega_0D\omega^2}e^{p_1t}+\frac{p_2\cos\varphi_u-\omega\sin\varphi_u}{4p_2^3+6\omega_0Dp_2^2+2(\omega_0^2+\omega^2)p_2+2\omega_0D\omega^2}e^{p_2t}\right\}$$

Final problem:

Further simplification to real values is quite complicated !