Fundamentals of EE 3

Chapter 6

Line theory

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6.1 Introduction

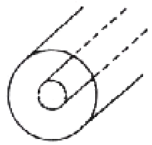
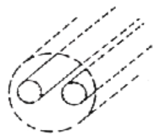
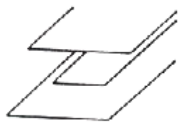
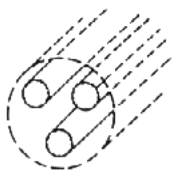
Lines are used in various areas of EE:

In energy producing techniques for the transmission of electrical energy

In information technology for transmission of signals

In RF applications for network analysis

6.1 Vorbemerkungen

Art	Typ	Querschnittsform	Anwendung	Physikalischer Ausbreitungsvorgang	Frequenzbereich
Doppelleitungen und Mehrleitersysteme	Koaxial		Energieübertragung Signalübertragung Schaltelement	Elektrisch : Angelegte Spannung verursacht Ströme in den Leitern	Theoretisch : $0<f<\infty$ Praktisch : keine untere Frequenzgrenze; obere Frequenzgrenze durch Dämpfung und Störwellen
	Parallel draht		Energieübertragung Signalübertragung		
	Streifen		Energieübertragung Signalübertragung Schaltelement		
	Dreh-strom		Energieübertragung (Signalübertragung)		

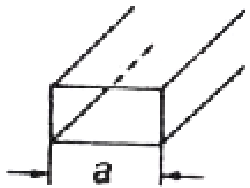
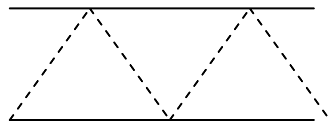
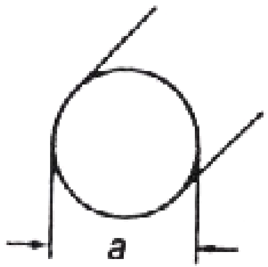
6.1 Vorbemerkungen

Lines can transmit both AC and DC signals

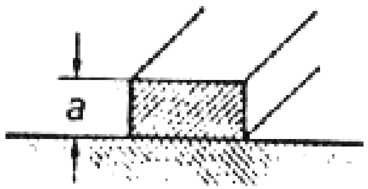
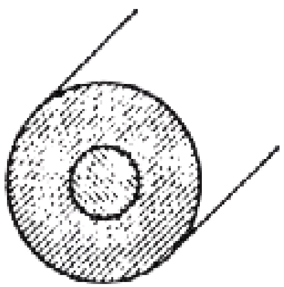
Power losses typically rise with increasing frequency (due to losses in the dielectric medium or by radiation in case of open structures).

For very high frequencies suitable waveguides have to be used.

6.1 Vorbemerkungen

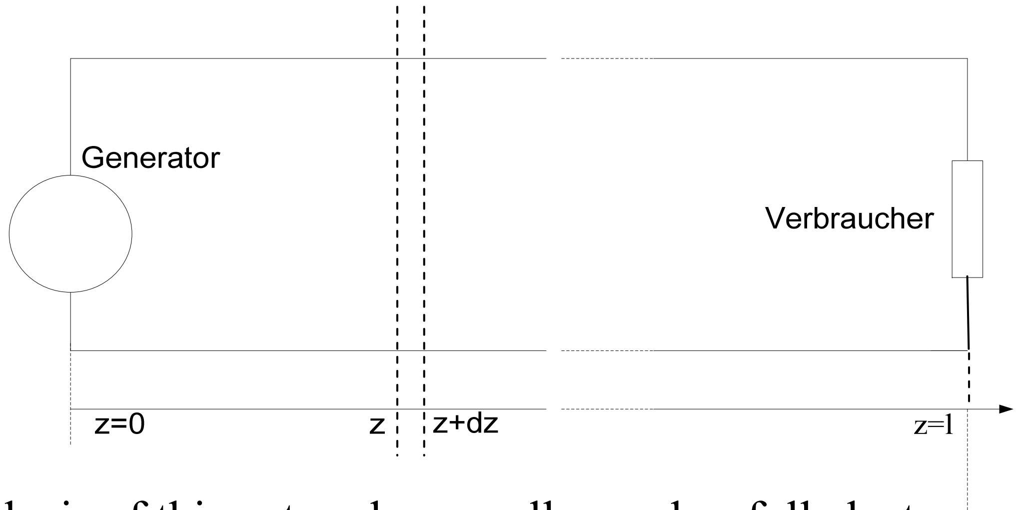
Art	Typ	Querschnittsform	Anwendung	Physikalischer Ausbreitungs-vorgang	Frequenz-bereich
Hohlleiter	Rech-teck		Signalübertragung Schaltelement	Quasioptisch : Elektro-magnetische Wellen breiten sich im Inneren unter fortwährender Reflexion an den Wänden nach den Gesetzen der Optik aus. 	Theoretisch : $f_g < f < \infty$ mit $f_g = \frac{1}{a\sqrt{\mu\varepsilon}}$ Praktisch : Obere Frequenz-grenze durch Störwellen, bzw. Streuung und Dämpfung
	Rund				

6.1 Vorbemerkungen

Art	Typ	Querschnittsform	Anwendung	Physikalischer Ausbreitungs-vorgang	Frequenz-bereich
Dielektrische Wellenleiter	Filme und Streifen auf Sub-straten		Schalt- und Verbindungs-element der planaren und integrierten Optik	Quasioptisch : Elektromagnetische Wellen im Streifen bzw. Faserkern erfahren unter genügend kleinen Winkeln zur Achse Totalreflexion an den Grenzschichten. Ausbreitung ähnlich wie im Hohlleiter	Theoretisch : $0 < f < \infty$ Praktisch : Untere und obere Frequenz-grenzen durch Dämpfung, Strahlung, Störwellen
	Glas-fasern		Optische Signalüber-tragung		

6.2 Beschreibung der Fernleitungen

Here a source is connected to a load by a homogeneous line of lenght L



The analysis of this network normally needs a full electro-magnetic wave analysis.

Instead here quasi stationary methods employing location dependent network analysis is applied.

6.2 Beschreibung der Fernleitungen

For homogenous lines primary line parameter can be introduced:

1) Distributed resistance (due to losses in copper e.g.) :

$$R' = \frac{dR}{dz} = const \qquad [R'] = \frac{\Omega}{m}$$

2) Leakance per unit lenght (due to isolation losses e.g.) :

$$G' = \frac{dG}{dz} = const \qquad [G'] = \frac{S}{m}$$

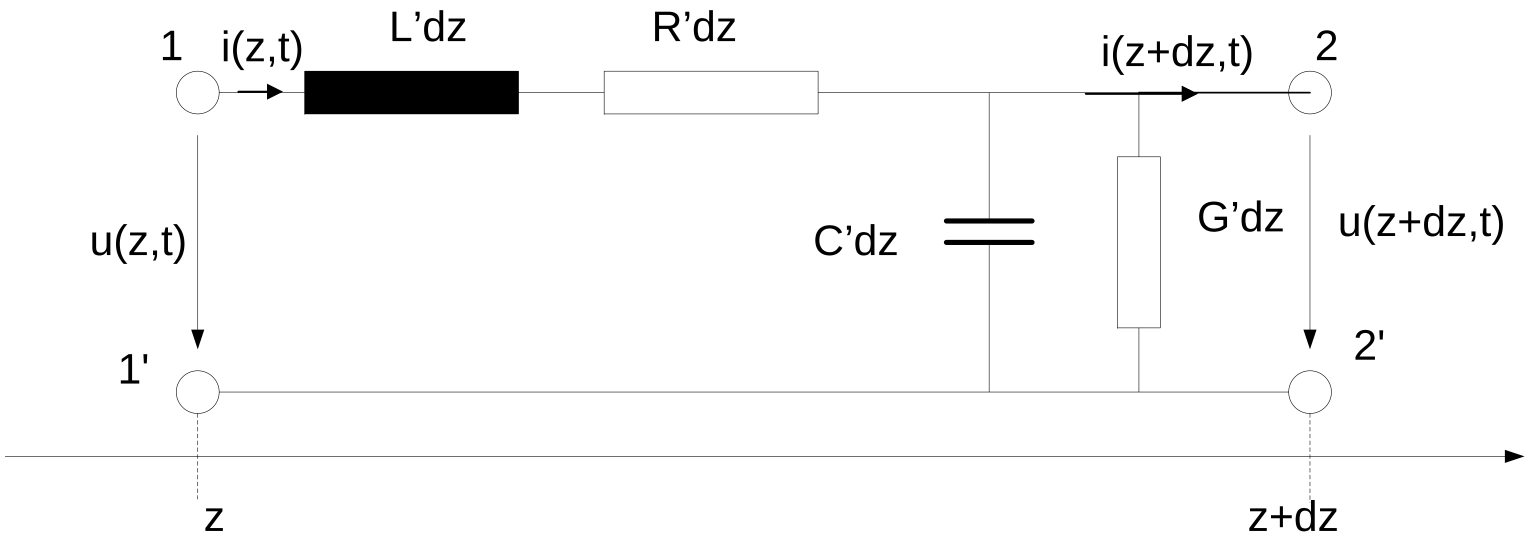
3) Distributed inductance:

$$L' = \frac{dL}{dz} = const \qquad [L'] = \frac{H}{m}$$

6.2 Beschreibung der Fernleitungen

4) Distributed capacitance:

$$C' = \frac{dC}{dZ} = const \qquad [C'] = \frac{F}{m}$$



6.2 Beschreibung der Fernleitungen

The equivalent circuit for a short cable part avoids the use of field parameters E and H. Instead normal current pointers and voltage pointers are used.

Loop and node equations are as follows:

$$u(z+dz,t)-u(z,t)+R'dz\cdot i(z,t)+L'dz\frac{\partial i(z,t)}{\partial t}=0$$

$$i(z+dz,t)-i(z,t)+G'dz\cdot u(z,t)+C'dz\frac{\partial u(z,t)}{\partial t}=0$$

For *u(z+dz, t)* und *i(z+dz, t)* Taylor polynomials of first order hold.

$$u(z+dz,t)=u(z,t)+\frac{\partial u(z,t)}{\partial z}dz$$

$$i(z+dz,t)=i(z,t)+\frac{\partial i(z,t)}{\partial z}dz$$

6.2 Beschreibung der Fernleitungen

Combining Taylor polynomials and loop and node equations give (after cancelling of dz):

$$\frac{\partial u(z,t)}{\partial z} + R'i(z,t) + L'\frac{\partial i(z,t)}{\partial t} = 0 \qquad (1)$$

$$\frac{\partial i(z,t)}{\partial z} + G'u(z,t) + C'\frac{\partial u(z,t)}{\partial t} = 0 \qquad (2)$$

$$\Rightarrow \frac{\partial i(z,t)}{\partial z} = -G'u(z,t) - C'\frac{\partial u(z,t)}{\partial t} \qquad (2a)$$

A linear system of coupled differential equations with constant coefficients result.

For decoupling (1) will be partially derived for z and (2) will be partially derived for t.

6.2 Beschreibung der Fernleitungen

$$\frac{\partial^2 u(z,t)}{\partial z^2} + R'\frac{\partial i(z,t)}{\partial z} + L'\frac{\partial^2 i(z,t)}{\partial t \partial z} = 0 \qquad (3)$$

$$\frac{\partial^2 i(z,t)}{\partial t \partial z} + G'\frac{\partial u(z,t)}{\partial t} + C'\frac{\partial^2 u(z,t)}{\partial t^2} = 0 \qquad (4)$$

$$\Rightarrow \frac{\partial^2 u(z,t)}{\partial z^2} = -R'\frac{\partial i(z,t)}{\partial z} - L'\frac{\partial^2 i(z,t)}{\partial t \partial z} \qquad (3a)$$

$$\Rightarrow \frac{\partial^2 i(z,t)}{\partial t \partial z} = -G'\frac{\partial u(z,t)}{\partial t} - C'\frac{\partial^2 u(z,t)}{\partial t^2} \qquad (4a)$$

(2a): $\frac{\partial i(z,t)}{\partial z} = -G'u(z,t) - C'\frac{\partial u(z,t)}{\partial t}$ is used in (3a):

$$\frac{\partial^2 u(z,t)}{\partial z^2} = -R'(-G'u(z,t) - C'\frac{\partial u(z,t)}{\partial t}) - L'(-G'\frac{\partial u(z,t)}{\partial t} - C'\frac{\partial^2 u(z,t)}{\partial t^2})$$

6.2 Beschreibung der Fernleitungen

$$\frac{\partial^2 u(z,t)}{\partial z^2} = -R'(-G'u(z,t) - C'\frac{\partial u(z,t)}{\partial t}) - L'(-G'\frac{\partial u(z,t)}{\partial t} - C'\frac{\partial^2 u(z,t)}{\partial t^2})$$

$$\frac{\partial^2 u(z,t)}{\partial z^2} = R'G'u(z,t) + (R'C' + L'G')\frac{\partial u(z,t)}{\partial t} + L'C'\frac{\partial^2 u(z,t)}{\partial t^2} \qquad (5)$$

In a similar manner the equation for the current is obtained:

$$\frac{\partial^2 i(z,t)}{\partial z^2} = R'G'i(z,t) + (R'C' + L'G')\frac{\partial i(z,t)}{\partial t} + L'C'\frac{\partial^2 i(z,t)}{\partial t^2} \qquad (6)$$

(5) und (6) are called telegraphic equations.

6.2 Beschreibung der Fernleitungen

General onset and solution of telegraphic equations for sinusoidal voltages and currents

$$\underline{u}(z,t) = \underline{\hat{u}}(z)e^{j\omega t} \qquad (7)$$

$$\underline{i}(z,t) = \underline{\hat{i}}(z)e^{j\omega t} \qquad (8)$$

6.2 Beschreibung der Fernleitungen

The onset now gives:

$$\frac{\partial^2 u(z,t)}{\partial z^2} = R'G'u(z,t) + (R'C' + L'G')\frac{\partial u(z,t)}{\partial t} + L'C'\frac{\partial^2 u(z,t)}{\partial t^2} \qquad (5)$$

$$\frac{\partial^2 i(z,t)}{\partial z^2} = R'G'i(z,t) + (R'C' + L'G')\frac{\partial i(z,t)}{\partial t} + L'C'\frac{\partial^2 i(z,t)}{\partial t^2} \qquad (6)$$

$$\frac{d^2 \underline{\hat{u}}(z)}{dz^2} = (R' + j\omega L')(G' + j\omega C')\underline{\hat{u}}(z) = \gamma^2 \underline{\hat{u}}(z) \qquad (10)$$

$$\frac{d^2 \underline{\hat{i}}(z)}{dz^2} = (R' + j\omega L')(G' + j\omega C')\underline{\hat{i}}(z) = \gamma^2 \underline{\hat{i}}(z) \qquad (11)$$

6.2 Beschreibung der Fernleitungen

The general solution is given by:

$$\underline{\hat{u}}(z) = \underline{\hat{u}}_h e^{-\underline{\gamma} z} + \underline{\hat{u}}_r e^{\underline{\gamma} z} \Leftrightarrow u(z,t) = u_h(z,t) + u_r(z,t) \qquad (12)$$

$$\underline{\hat{i}}(z) = \underline{\hat{i}}_h e^{-\underline{\gamma} z} - \underline{\hat{i}}_r e^{\underline{\gamma} z} \Leftrightarrow i(z,t) = i_h(z,t) - i_r(z,t) \qquad (13)$$

Voltages are added but currents are substracted to due changing propagation directions in the line.

The complex pointers $\underline{\hat{u}}_h$, $\underline{\hat{u}}_r$, $\underline{\hat{i}}_h$ *and* $\underline{\hat{i}}_r$ describe voltages and currents of a forward travelling wave (index h) and of a backward travelling wave (index r) respectively.

γ is the propagation constant according to:

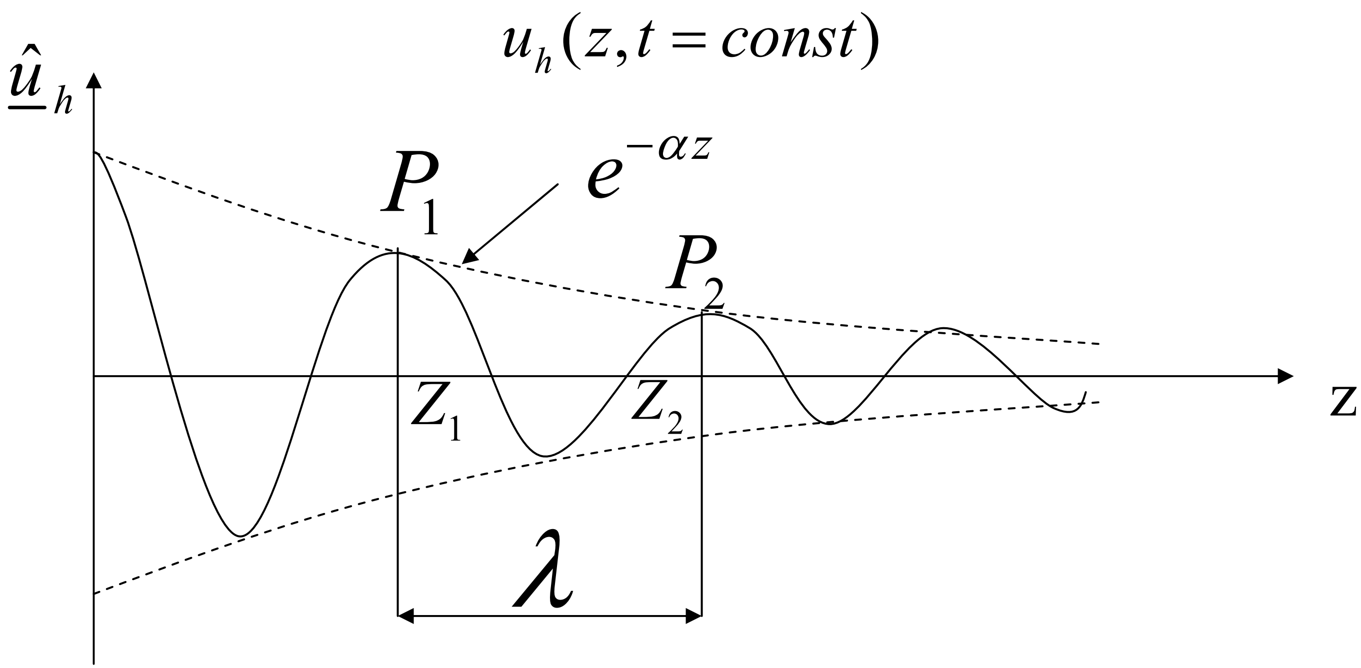
$$\underline{\gamma} = +\sqrt{(R' + j\omega L')(G' + j\omega C')} \qquad (14)$$

6.2 Beschreibung der Fernleitungen

Real and imaginary part of the propagation constant are called attenuation constant and wavelenght constant:

$$\underline{\gamma} = \alpha + j\beta \tag{15}$$

Now the voltage of a forward travelling wave is observed at a fixed time t.



6.2 Beschreibung der Fernleitungen

It holds: $$u_h(z,t) = \mathrm{Re}\{\underline{\hat{u}}_h e^{(-\alpha - j\beta)z} e^{j\omega t}\} = \mathrm{Re}\{\underline{\hat{u}}_h e^{-\alpha z + j(\omega t - \beta z)}\} \qquad (16)$$

$$= \hat{u}_h e^{-\alpha z} \cos(\omega t - \beta z) \qquad (17)$$

Now the coordinates of points P_2 and P_1 at the cable are determined which represent a phase rotation of 360°.

$$(\omega t - \beta z_2)|_{t=const} - (\omega t - \beta z_1)|_{t=const} \overset{!}{=} 2\pi \qquad (18)$$

Using $z_2 - z_1 = \lambda$ it follows $$\beta\lambda = 2\pi \Rightarrow \beta = \frac{2\pi}{\lambda} \qquad (19)$$

Thus for the wave (phase) velocity holds: $$V_{ph} = \frac{\lambda}{T} = \lambda f = \frac{\lambda\omega}{2\pi} = \frac{\omega}{\beta} \qquad (20)$$

6.2 Beschreibung der Fernleitungen

Secondary line parameters

Differential equations (1) und (2) are now shown with complex pointers for voltages and currents:

$$\frac{\partial u(z,t)}{\partial z} + R'i(z,t) + L'\frac{\partial i(z,t)}{\partial t} = 0 \,\Big|\, \underline{i}(t) = \hat{\underline{i}}(z)e^{j\omega t} \qquad (1)$$

$$\frac{\partial i(z,t)}{\partial z} + G'u(z,t) + C'\frac{\partial u(z,t)}{\partial t} = 0 \qquad (2)$$

This gives:

$$-\frac{\partial \hat{\underline{u}}(z)}{\partial z} = (R' + j\omega L')\hat{\underline{i}}(z) \qquad (21)$$

$$-\frac{\partial \hat{\underline{i}}(z)}{\partial dz} = (G' + j\omega C')\hat{\underline{u}}(z) \,\Big|\, \hat{\underline{u}}(z) = \hat{\underline{u}}_h e^{-j\gamma z} + \hat{\underline{u}}_r e^{j\gamma z} \qquad (22)$$

6.2 Beschreibung der Fernleitungen

Now in (21) currents and voltages are replaced by relations (12) and (13) according to waves travelling back and forth.

$$\frac{\partial}{\partial z}\underline{\hat{u}}(z) = -\gamma\underline{\hat{u}}_h e^{-\underline{\gamma}z} + \gamma\underline{\hat{u}}_r e^{\underline{\gamma}z} \qquad \underline{\hat{i}}(z) = \underline{\hat{i}}_h e^{-\underline{\gamma}z} - \underline{\hat{i}}_r e^{\underline{\gamma}z} \qquad \text{(12a+13)}$$

$$-\frac{\partial\underline{\hat{u}}(z)}{\partial z} = -(-\gamma\underline{\hat{u}}_h e^{-\underline{\gamma}z} + \gamma\underline{\hat{u}}_r e^{\underline{\gamma}z}) = (R' + j\omega L')(\underline{\hat{i}}_h e^{-\underline{\gamma}z} - \underline{\hat{i}}_r e^{\underline{\gamma}z}) \qquad \text{(21a)}$$

$$\Rightarrow \underline{\gamma}\cdot\underline{\hat{u}}_h e^{-\underline{\gamma}z} - \underline{\gamma}\cdot\underline{\hat{u}}_r e^{\underline{\gamma}z} = (R' + j\omega L')(\underline{\hat{i}}_h e^{-\underline{\gamma}z} - \underline{\hat{i}}_r e^{\underline{\gamma}z})$$

A comparison performed separately for waves travelling back and forth gives:

$$\underline{\hat{u}}_{h,r} = \frac{R' + j\omega L'}{\underline{\gamma}}\underline{\hat{i}}_{h,r} = \frac{R' + j\omega L'}{\sqrt{(R' + j\omega L')(G' + j\omega C')}}\underline{\hat{i}}_{h,r} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}\underline{\hat{i}}_{h,r} \qquad (23)$$

6.2 Beschreibung der Fernleitungen

(24)

The complex valued factor in (23) und (24) is the so called wave impedance

$$Z_L = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \qquad (25)$$

The values $\underline{\gamma}$ and Z_L are called secondary line parameters.

Grundlagen der Elektrotechnik 3

Chapter 6.3

Dispersion properties of secondary line parameters

6.3.1 Analyse des Ausbreitungsmaßes

It holds: $\underline{\gamma} = \alpha + j\beta = +\sqrt{(R' + j\omega L')(G' + j\omega C')} = \sqrt{R' + j\omega L'}\sqrt{G' + j\omega C'}$

$$\underline{\gamma}^2 = \alpha^2 + j2\alpha\beta - \beta^2 = (R' + j\omega L')(G' + j\omega C')$$

$$= R'G' + j\omega(L'G' + R'C') - \omega^2 L'C'$$

From this the following relations are determined for preparation of α and β calculation:

$$\mathrm{Re}\{\underline{\gamma}^2\} = \alpha^2 - \beta^2 = R'G' - \omega^2 L'C'$$

$$|\underline{\gamma}|^2 = \alpha^2 + \beta^2 = \left|(R' + j\omega L')\right|\left|(G' + j\omega C')\right|$$

$$= \sqrt{(R'^2 + \omega^2 L'^2)(G'^2 + \omega^2 C'^2)}$$

6.3.1 Analyse des Ausbreitungsmaßes

Thus it gives:

$$\mathrm{Re}\{\underline{\gamma}^2\} + \left|\gamma^2\right| = 2\alpha^2 = R'G' - \omega^2 L'C' + \sqrt{(R'^2 + \omega^2 L'^2)(G'^2 + \omega^2 C'^2)}$$

$$-\mathrm{Re}\{\underline{\gamma}^2\} + \left|\gamma^2\right| = 2\beta^2 = -(R'G' - \omega^2 L'C') + \sqrt{(R'^2 + \omega^2 L'^2)(G'^2 + \omega^2 C'^2)}$$

$$\alpha = \sqrt{\frac{1}{2}(R'G' - \omega^2 L'C') + \frac{1}{2}\sqrt{(R'^2 + \omega^2 L'^2)(G'^2 + \omega^2 C'^2)}} \qquad (26)$$

$$\beta = \sqrt{\frac{1}{2}(\omega^2 L'C' - R'G') + \frac{1}{2}\sqrt{(R'^2 + \omega^2 L'^2)(G'^2 + \omega^2 C'^2)}} \qquad (27)$$

6.3.2 Analyse des Fernleitungs-Wellenwiderstandes

The wave impedance is given by:

$$\underline{Z}_L = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}}$$

As any complex value it can be given in algebraic form oder in polar form:

$$\underline{Z}_L = \mathrm{Re}\{\underline{Z}_L\} + j\,\mathrm{Im}\{\underline{Z}_L\} = |\underline{Z}_L|\, e^{j\varphi_{Z_L}}$$

For the magnitude holds:

$$\left|\underline{Z}_L\right|^2 = \frac{|R' + j\omega L'|}{|G' + j\omega C'|} = \frac{\sqrt{R'^2 + (\omega L')^2}}{\sqrt{G'^2 + (\omega C')^2}} \quad \Rightarrow \quad |\underline{Z}_L| = \sqrt[4]{\frac{R'^2 + \omega^2 L'^2}{G'^2 + \omega^2 C'^2}} \qquad (28)$$

6.3.2 Analyse des Fernleitungs-Wellenwiderstandes

The angle φ_{ZL} is here determined by the angle of: $\underline{Z}_L$

It holds:

$$\varphi_{Z_L} = \frac{1}{2}\arctan\left[\frac{\omega(G'L'-R'C')}{R'G'+\omega^2 L'C'}\right] \qquad (29)$$

Due to:

$$\frac{R'+j\omega L'}{G'+j\omega C'} = \frac{(R'+j\omega L')(G'-j\omega C')}{G'^2-(\omega C')^2} = \frac{R'G'+\omega^2 L'C'+j\omega(G'L'-R'C')}{G'^2-(\omega C')^2}$$

6.3.3 Verlustlose Fernleitungen

For loss-less lines hold: R‘ = 0 G‘ = 0

Such lines exhibit no damping, thus it holds: $\alpha = 0$ (30)

From
$$\mathrm{Re}\{\underline{\gamma}^2\} = \alpha^2 - \beta^2 = R'G' - \omega^2 L'C'$$

It can be determined: $\beta = \omega\sqrt{L'C'}$ (31)

The phase velocity then gives according to (20): $V_{ph} = \frac{\omega}{\beta} = \frac{1}{\sqrt{L'C'}}$ (32)

For loss-less lines the line impedance is real valued!

This is important for line applications.

$$\underline{Z}_L = Z_L = \sqrt{\frac{j\omega L'}{j\omega C'}} = \sqrt{\frac{L'}{C'}} \quad (33)$$

6.3.4 Verlustarme/stark verlustbehaftete Fernleitungen

For low lossy lines hold:

$$\omega L' \gg R' \quad \omega C' \gg G'$$

$$\underline{\gamma} = \sqrt{(R' + j\omega L')(G' + j\omega C')} = j\omega\sqrt{L'C'(1 - \frac{jR'}{\omega L'})(1 - \frac{jG'}{\omega C'})}$$

Grundlagen der Elektrotechnik 3

Kapitel 6.4

Properties of lines of limited length

6.4 Berechnung der Fernleitungen endlicher Länge

In this chapter lines are dealt with as networks. Thus methods of network theory can be applied.

Such lines are represented as follows:

- Lines itself by 2 parallel lines
- Concentrated components and its connections by thinner lines

6.4.1 Fernleitungen als Zweitore

According to chapter 5.2 it holds:

$$\underline{\hat{u}}(z) = \underline{\hat{u}}_h e^{-\underline{\gamma} z} + \underline{\hat{u}}_r e^{+\underline{\gamma} z} \qquad (45)$$

$$\underline{\hat{i}}(z) = \underline{\hat{i}}_h e^{-\underline{\gamma} z} - \underline{\hat{i}}_r e^{+\underline{\gamma} z} \qquad (46)$$

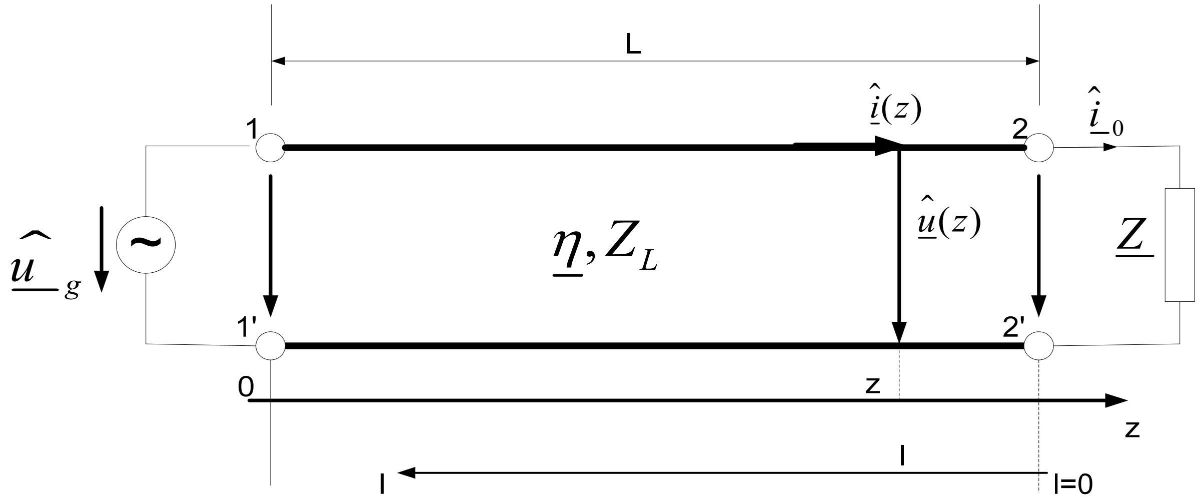
with

$$\underline{\hat{u}}_{h,r} = \underline{Z}_L \underline{\hat{i}}_{h,r} \qquad (47)$$

6.4.1 Fernleitungen als Zweitore

In the following the calaculations are performed with complex pointers which are dependent on line position.

A line of length L is considered which is connected to a voltage source and an arbitrary load.



6.4.1 Fernleitungen als Zweitore

At a certain location z at the line complex pointers are given as shown before:

$$\underline{\hat{u}}(z) = \underline{\hat{u}}_h e^{-\underline{\gamma} z} + \underline{\hat{u}}_r e^{\underline{\gamma} z} \quad (48)$$

$$\underline{Z}_L \underline{\hat{i}}(z) = \underline{\hat{u}}_h e^{-\underline{\gamma} z} - \underline{\hat{u}}_r e^{\underline{\gamma} z} \quad (49)$$

The pointers $\underline{\hat{u}}_h$ and $\underline{\hat{u}}_r$ represent the voltages at the beginning of the line.

Sometimes voltages and currents are only known at the end of the line. The corresponding relations are considered now.

6.4.1 Fernleitungen als Zweitore

At first a transform of the coordinate is performed. $z = L - l$

Then it holds:

$$\underline{\hat{u}}(l) = \underline{\hat{u}}_h e^{-\underline{\gamma}(L-l)} + \underline{\hat{u}}_r e^{\underline{\gamma}(L-l)} \quad (50)$$

$$\underline{Z}_L \underline{\hat{i}}(l) = \underline{\hat{u}}_h e^{-\underline{\gamma}(L-l)} - \underline{\hat{u}}_r e^{\underline{\gamma}(L-l)} \quad (51)$$

Using the complex pointers at the end of the line it follows:

$$\underline{\hat{u}}_0 = \underline{\hat{u}}(l=0) = \underline{\hat{u}}_h e^{-\underline{\gamma}L} + \underline{\hat{u}}_r e^{\underline{\gamma}L}$$

$$\underline{Z}_L \underline{\hat{i}}_0 = \underline{Z}_L \underline{\hat{i}}(l=0) = \underline{\hat{u}}_h e^{-\underline{\gamma}L} - \underline{\hat{u}}_r e^{\underline{\gamma}L}$$

6.4.1 Fernleitungen als Zweitore

By addition and subtraction of the 2 previous equations the amplitudes of forth and back travelling waves are determined:

$$\underline{\hat{u}}_h = \frac{1}{2}\left(\underline{\hat{u}}_0 + \underline{Z}_L \underline{\hat{i}}_0\right) e^{\underline{\gamma} L} \qquad (52)$$

$$\underline{\hat{u}}_r = \frac{1}{2}\left(\underline{\hat{u}}_0 - \underline{Z}_L \underline{\hat{i}}_0\right) e^{-\underline{\gamma} L} \qquad (53)$$

Thus equations (50) and (51) can be rewritten:

$$\underline{\hat{u}}(l) = \frac{1}{2}\left(\underline{\hat{u}}_0 + \underline{Z}_L \underline{\hat{i}}_0\right) e^{\underline{\gamma} l} + \frac{1}{2}\left(\underline{\hat{u}}_0 - \underline{Z}_L \underline{\hat{i}}_0\right) e^{-\underline{\gamma} l} = \underline{\hat{u}}_h(l) + \underline{\hat{u}}_r(l) \qquad (54)$$

$$\underline{Z}_L \underline{\hat{i}}(l) = \frac{1}{2}\left(\underline{\hat{u}}_0 + \underline{Z}_L \underline{\hat{i}}_0\right) e^{\underline{\gamma} l} - \frac{1}{2}\left(\underline{\hat{u}}_0 - \underline{Z}_L \underline{\hat{i}}_0\right) e^{-\underline{\gamma} l} = \underline{\hat{u}}_h(l) - \underline{\hat{u}}_r(l) \qquad (55)$$

6.4.1 Fernleitungen als Zweitore

Now expressions are rearranged like this:

$$\underline{\hat{u}}(l) = \underline{\hat{u}}_0 \frac{e^{\underline{\gamma} l} + e^{-\underline{\gamma} l}}{2} + \underline{Z}_L \underline{\hat{i}}_0 \frac{e^{\underline{\gamma} l} - e^{-\underline{\gamma} l}}{2}$$

This relation can be simplified by means of hyperbolic functions.

Note: A change of a sign in (57):

$$\underline{\hat{u}}(l) = \underline{\hat{u}}_0 \cosh(\underline{\gamma} l) + \underline{Z}_L \underline{\hat{i}}_0 \sinh(\underline{\gamma} l) \tag{56}$$

$$\underline{\hat{i}}(l) = \frac{\underline{\hat{u}}_0}{\underline{Z}_L} \sinh(\underline{\gamma} l) + \underline{\hat{i}}_0 \cosh(\underline{\gamma} l) \tag{57}$$

6.4.1 Fernleitungen als Zweitore

Equations (56) and (57) can be written in matrix form:

$$\begin{pmatrix} \underline{\hat{u}}(l) \\ \underline{\hat{i}}(l) \end{pmatrix} = \begin{pmatrix} \cosh(\underline{\gamma} l) & \underline{Z}_L \sinh(\underline{\gamma} l) \\ \frac{1}{\underline{Z}_L} \sinh(\underline{\gamma} l) & \cosh(\underline{\gamma} l) \end{pmatrix} \cdot \begin{pmatrix} \underline{\hat{u}}_0 \\ \underline{\hat{i}}_0 \end{pmatrix} \qquad (58)$$

Obviously this is a representation of the line properties in the form of a chain matrix of a corresponding network.

The elements of the matrix represent the chain parameters.

As any passive network such a piece of line is a reciprocal and symmetric two-port.

6.4.1 Fernleitungen als Zweitore

$$\underline{\overleftrightarrow{a}} = \begin{pmatrix} \cosh(\underline{\gamma} l) & \underline{Z}_L \sinh(\underline{\gamma} l) \\ \frac{1}{\underline{Z}_L} \sinh(\underline{\gamma} l) & \cosh(\underline{\gamma} l) \end{pmatrix} \qquad (59)$$

6.4.1 Fernleitungen als Zweitore

Equations (56) und (57) can be rewritten for voltage $\underline{\hat{u}}_0$ and current $\underline{\hat{i}}_0$ at the end of the line due to :

$$\underline{Z} = \frac{\underline{\hat{u}}_0}{\underline{\hat{i}}_0}$$

$$\frac{\underline{\hat{u}}(l)}{\underline{\hat{u}}_0} = \cosh(\underline{\gamma} l) + \frac{\underline{Z}_L}{\underline{Z}} \sinh(\underline{\gamma} l)$$

$$\frac{\underline{\hat{i}}(l)}{\underline{\hat{i}}_0} = \frac{\underline{Z}}{\underline{Z}_L} \sinh(\underline{\gamma} l) + \cosh(\underline{\gamma} l)$$

6.4.1 Fernleitungen als Zweitore

Special case: For a lossless line holds:

$$\underline{\gamma} l\Big|_{\alpha=0} = j\beta l, \quad \cosh(\underline{\gamma} l)\Big|_{\alpha=0} = \cos(\beta l)$$

$$\sinh(\underline{\gamma} l)\Big|_{\alpha=0} = j\sin(\beta l) \qquad \underline{Z}_L = Z_L \text{ (rein reell)}$$

Thus it gives: $$\hat{\underline{u}}(l) = \hat{\underline{u}}_0 \cos(\beta l) + j\underline{Z}_L \hat{\underline{i}}_0 \sin(\beta l) \qquad (60)$$

$$\hat{\underline{i}}(l) = j\frac{\hat{\underline{u}}_0}{\underline{Z}_L}\sin(\beta l) + \hat{\underline{i}}_0 \cos(\beta l) \qquad (61)$$

For the chain matrix follows:

$$\underline{\overleftrightarrow{a}} = \begin{pmatrix} \cos(\beta l) & j\underline{Z}_L \sin(\beta l) \\ \dfrac{j}{\underline{Z}_L}\sin(\beta l) & \cos(\beta l) \end{pmatrix} \qquad (62)$$

6.4.2 Eingangsimpedanz einer Fernleitung der Länge “ l ”

Now the input impedance of a line at the location “l” is determined using equations (56) and (57) in the form of numerator and denominator polynom:

$$\underline{Z}_E(l) = \frac{\underline{\hat{u}}(l)}{\underline{\hat{i}}(l)} = \frac{\underline{\hat{u}}_0 \cosh(\underline{\gamma} l) + \underline{Z}_L \underline{\hat{i}}_0 \sinh(\underline{\gamma} l)}{\frac{\underline{\hat{u}}_0}{\underline{Z}_L}\sinh(\underline{\gamma} l) + \underline{\hat{i}}_0 \cosh(\underline{\gamma} l)} = \frac{\frac{\underline{\hat{u}}_0}{\underline{\hat{i}}_0} + \underline{Z}_L \frac{\underline{\hat{i}}_0 \sinh(\underline{\gamma} l)}{\underline{\hat{i}}_0 \cosh(\underline{\gamma} l)}}{\frac{\underline{\hat{u}}_0}{\underline{Z}_L \underline{\hat{i}}_0} \frac{\sinh(\underline{\gamma} l)}{\cosh(\underline{\gamma} l)} + 1}$$

This leads to:

$$\underline{Z}_E(l) = \underline{Z}_L \frac{\underline{Z} + \underline{Z}_L \tanh(\underline{\gamma} l)}{\underline{Z}_L + \underline{Z} \tanh(\underline{\gamma} l)} \qquad \text{mit } \underline{Z} = \frac{\underline{\hat{u}}_0}{\underline{\hat{i}}_0} \tag{63}$$

Special case:

A loss-less line for which holds:

$$\underline{Z}_E(l) = Z_L \frac{\underline{Z} + jZ_L \tan(\beta l)}{Z_L + j\underline{Z} \tan(\beta l)} \tag{64}$$

6.4.3 Der Reflexionsfaktor

<u>The voltage reflection factor</u>

The factor describes relation of voltage amplitudes for forth and back travelling waves in the form:

$$\underline{r}(l) = \frac{\hat{\underline{u}}_r(l)}{\hat{\underline{u}}_h(l)} = \frac{\hat{\underline{u}}_r e^{-\gamma l}}{\hat{\underline{u}}_h e^{\gamma l}} \qquad (65)$$

Using eq. (54) one can write:

$$\hat{\underline{u}}(l) = \frac{1}{2}\left(\hat{\underline{u}}_0 + \underline{Z}_L \hat{\underline{i}}_0\right) e^{\underline{\gamma} l} + \frac{1}{2}\left(\hat{\underline{u}}_0 - \underline{Z}_L \hat{\underline{i}}_0\right) e^{-\underline{\gamma} l} \qquad (54a)$$

Thus it gives:

$$\underline{r}(l) = \frac{\hat{\underline{u}}_0 - \underline{Z}_L \hat{\underline{i}}_0}{\hat{\underline{u}}_0 + \underline{Z}_L \hat{\underline{i}}_0} e^{-2\underline{\gamma} l}$$

6.4.3 Der Reflexionsfaktor

At location *l* = *0* the reflection factor is:

with $\underline{Z} = \frac{\hat{\underline{u}}_0}{\hat{\underline{i}}_0}$

$$\underline{r}(l=0) = \underline{r}_0 = \frac{\hat{\underline{u}}_0 - \underline{Z}_L \hat{\underline{i}}_0}{\hat{\underline{u}}_0 + \underline{Z}_L \hat{\underline{i}}_0} = \frac{\underline{Z} - \underline{Z}_L}{\underline{Z} + \underline{Z}_L} \quad (66)$$

Thus it gives:

$$\underline{r}(l) = \underline{r}_0 e^{-2\underline{\gamma} l} \quad (67)$$

Note: Values of the reflection factor are within -1 and +1 !

Special case For a loss-less line it holds:

$$\underline{r}(l) = \underline{r}_0 e^{-2j\beta l} \quad (68)$$

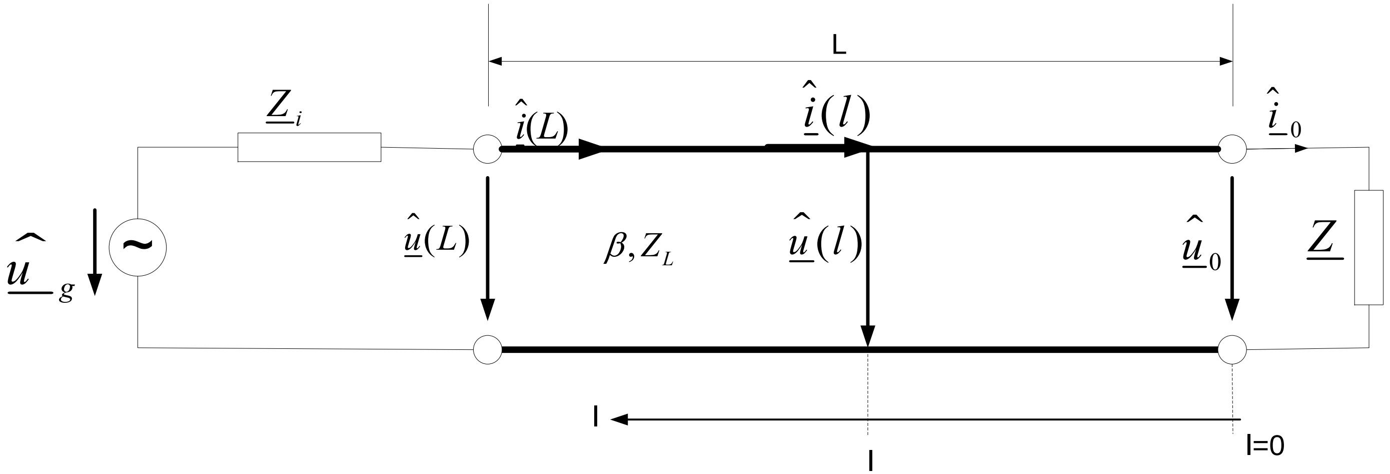
6.4.4 Verlustlose Fernleitungen

Loss-less lines

Ofte quite short lines are used so that losses can be neglected in the first degree.

Such lines are described by the propagation constant “ß” and a purely real-valued wave impedance “Z_L” and the length “L” .

This is represented in the following drawing.



6.4.4 Verlustlose Fernleitungen

Special case1   Load is equal to the wave impedance $\underline{Z} = Z_L$

In this often desired case (wave impedance matching, power matching) no back travelling waves exist due to:

$$\underline{r}_0 = \left. \frac{\underline{Z} - Z_L}{\underline{Z} + Z_L} \right|_{\underline{Z}=Z_L} = 0$$

For the input impedance holds (s. eq.(64))

$$\underline{Z}_E(l) = Z_L \left. \frac{\underline{Z} + jZ_L \tan(\beta l)}{Z_L + j\underline{Z} \tan(\beta l)} \right|_{\underline{Z}=Z_L} = Z_L$$

The impedance is depending on *l* and the phase constant.

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6.4.4 Verlustlose Fernleitungen

Special case 2: Eine am Ende kurzgeschlossene Fernleitung $\underline{Z} = 0$

Now it holds:

$$\underline{r}_0 = \left.\frac{\underline{Z} - \underline{Z}_L}{\underline{Z} + \underline{Z}_L}\right|_{\underline{Z}=0} = -1 \qquad \underline{r}(l) = -e^{-2j\beta l}$$

This means a total reflection, voltages of both waves cancel out each other.

$$\underline{\hat{u}}_h(0) = -\underline{\hat{u}}_r(0) \qquad \underline{\hat{u}}(0) = \underline{\hat{u}}_0 = 0$$

But the current is douibled according to eq. (55) :

$$\underline{Z}_L \underline{\hat{i}}(l) = \frac{1}{2}\left(\underline{\hat{u}}_0 + \underline{Z}_L \underline{\hat{i}}_0\right) e^{\underline{\gamma} l} - \frac{1}{2}\left(\underline{\hat{u}}_0 - \underline{Z}_L \underline{\hat{i}}_0\right) e^{-\underline{\gamma} l}\Big|_{l=0\,;\,\underline{\hat{u}}_0=0}$$

$$\Rightarrow \underline{Z}_L \underline{\hat{i}}(0) = \underline{Z}_L \underline{\hat{i}}_0 = \frac{1}{2}\underline{Z}_L \underline{\hat{i}}_0 + \frac{1}{2}\underline{Z}_L \underline{\hat{i}}_0 \quad \text{mit } \underline{\hat{i}}_0 = \frac{2\underline{\hat{u}}_h e^{-\underline{\beta} L}}{\underline{Z}_L}$$

6.4.4 Verlustlose Fernleitungen

For the input impedance holds: $\underline{Z}_E(l) = jZ_L \tan(\beta l)$

For a small value of *l* moreover holds

$$l \ll \lambda, \ \beta l = \frac{2\pi}{\lambda} \ll 1 \qquad \Rightarrow \tan(\beta l) \approx \beta l = \omega\sqrt{L'C'}l$$

which gives:

$$\underline{Z}_E(l) = j\sqrt{\frac{L'}{C'}}\omega\sqrt{L'C'}l = j\omega L'l$$

Thus a short short-circuited line acts as an <u>inductance *" L'l"*</u>.

6.4.4 Verlustlose Fernleitungen

Special case 3 A line with open loop: $\underline{Z} \rightarrow \infty$

Here it holds: $\underline{r}_0 = 1$

There is total reflection with cancelling of currents but doubling of the voltage at the end of the line.

$$\underline{\hat{u}}(l) = \frac{1}{2}\left(\underline{\hat{u}}_0 + \underline{Z}_L \underline{\hat{i}}_0\right) e^{\underline{\gamma} l} + \frac{1}{2}\left(\underline{\hat{u}}_0 - \underline{Z}_L \underline{\hat{i}}_0\right) e^{-\underline{\gamma} l} \Big|_{l=0;\underline{\hat{i}}_0=0}$$

$$\Rightarrow \underline{\hat{u}}(0) = \underline{\hat{u}}_0 = \frac{1}{2}\underline{\hat{u}}_0 + \frac{1}{2}\underline{\hat{u}}_0$$

For the input impedance holds: $$\underline{Z}_E(l) = \frac{\underline{Z}_L}{j\tan(\beta l)} \qquad (70)$$

This gives for a short line of length l: $$\underline{Z}_E(l) = \sqrt{\frac{L'}{C'}} \frac{1}{j\omega\sqrt{L'C'}l} = \frac{1}{j\omega C' l}$$

This line acts as a capacitor with the capacity“ $C'l$ ”.

6.4.4 Verlustlose Fernleitungen

Special case 4: $\lambda/4$ -Transformer with the length $L=\lambda/4$

Here it holds: $\beta L = \frac{2\pi}{\lambda}\frac{\lambda}{4} = \frac{\pi}{2}$

This gives for the reflection factor:

$$\underline{r}(L) = \underline{r}_0 e^{-j2\beta L} = \underline{r}_0 e^{-j\pi} = -\underline{r}_0$$

The output reflection factor is transformed to the input with a shift of 180°.
For the input impedance holds:

$$\underline{Z}_E(l) = Z_L \left. \frac{\underline{Z} + jZ_L \tan(\beta l)}{Z_L + j\underline{Z}\tan(\beta l)} \right|_{\tan(\beta l)\to\infty,\, l=L} = \frac{Z_L^2}{\underline{Z}}$$

6.4.4 Verlustlose Fernleitungen

This property can be used by choosing: $\underline{Z} = R$

This gives an impedance matching as follows $\underline{Z}_i = R_i$

if Z_L fulfils the condition: $Z_L = \sqrt{R_i R}$ (71)

It holds: $\underline{Z}_E(L) = \frac{Z_L^2}{\underline{Z}} = \frac{R_i R}{R} = R_i \quad \underline{Z}_E(0) = Z_L \left. \frac{\underline{Z} + jZ_L \tan(\beta l)}{Z_L + j\underline{Z}\tan(\beta l)} \right|_{l=0} = \underline{Z} = R$

Thus impedance matching at both ends of the line is given!

Special case 5 : $\lambda / 2$ -Transformator $L = \lambda / 2$

For this case holds: $\beta L = \pi$

The reflection factor in this case is at the input:

$$\underline{r}(L) = \underline{r}_0 e^{-j2\beta L} = \underline{r}_0 e^{-j2\pi} = \underline{r}_0$$

6.4.4 Verlustlose Fernleitungen

For the input impedance holds:

$$\underline{Z}_E(l) = Z_L \left. \frac{\underline{Z} + jZ_L \tan(\beta l)}{Z_L + j\underline{Z} \tan(\beta l)} \right|_{\tan(\beta l)=0, l=L} = \underline{Z}$$

So λ/2 transformer changes in no respect the situations at input and output.

Please note:

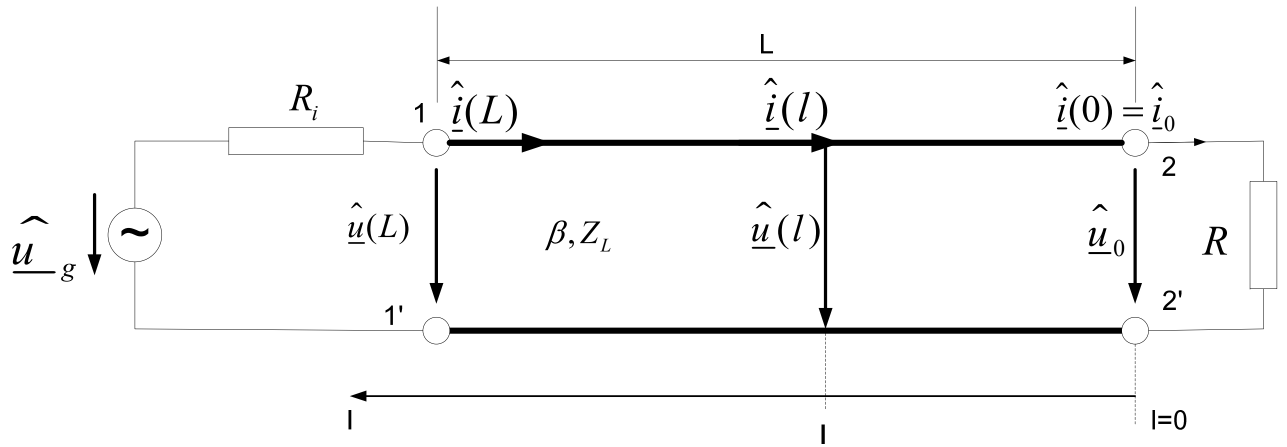
λ/4 and λ/2 transformers exhibit the described properties only for a single frequency.

To achieve a certain operating bandwidth several transformers with stepped wave impedance values are cascaded.

6.4.4 Verlustlose Fernleitungen

Example: Determination of the voltage along the line of λ/4 transformer.

The following parameters are given: $R_i = 25\Omega,\ R = 100\Omega,\ L = \lambda/4$



Solution:

The transformer requires: $Z_L = \sqrt{R_i R} = \sqrt{25\Omega \cdot 100\Omega} = 50\Omega$

At the location $l = 0$ holds: $\underline{r}_0 = \frac{\underline{Z} - Z_L}{\underline{Z} + Z_L} = \frac{100\Omega - 50\Omega}{100\Omega + 50\Omega} = \frac{1}{3}$

6.4.4 Verlustlose Fernleitungen

At the pins 1-1' it holds (see special case 4)

$$\underline{r}(L) = -\underline{r}_0 = -\frac{1}{3} = \frac{\hat{\underline{u}}_r(L)}{\hat{\underline{u}}_h(L)}$$

For the input voltage $\hat{\underline{u}}(L)$ it holds:

$$\hat{\underline{u}}(L) = \frac{\hat{\underline{u}}_g}{2} = \hat{\underline{u}}_h(1+\underline{r}(L)) = \frac{2}{3}\hat{\underline{u}}_h \quad \Rightarrow \quad \hat{\underline{u}}_h = \frac{3}{4}\hat{\underline{u}}_g$$

Due to $\hat{\underline{u}}_r + \hat{\underline{u}}_h = \hat{\underline{u}}_g/2$ follows: $\hat{\underline{u}}_r = -\frac{\hat{\underline{u}}_g}{4}$

This leads to the following relation for the voltage: (72a)

$$\hat{\underline{u}}(l) = \hat{\underline{u}}_h e^{-j\beta L} e^{j\beta l} + \hat{\underline{u}}_r e^{j\beta L} e^{-j\beta l} = \frac{3}{4}\hat{\underline{u}}_g j e^{j\beta l} - \frac{1}{4}\hat{\underline{u}}_g(-j)e^{-j\beta l} = j\frac{3}{4}\hat{\underline{u}}_g\left(e^{j\beta l} + \frac{1}{3}e^{-j\beta l}\right)$$

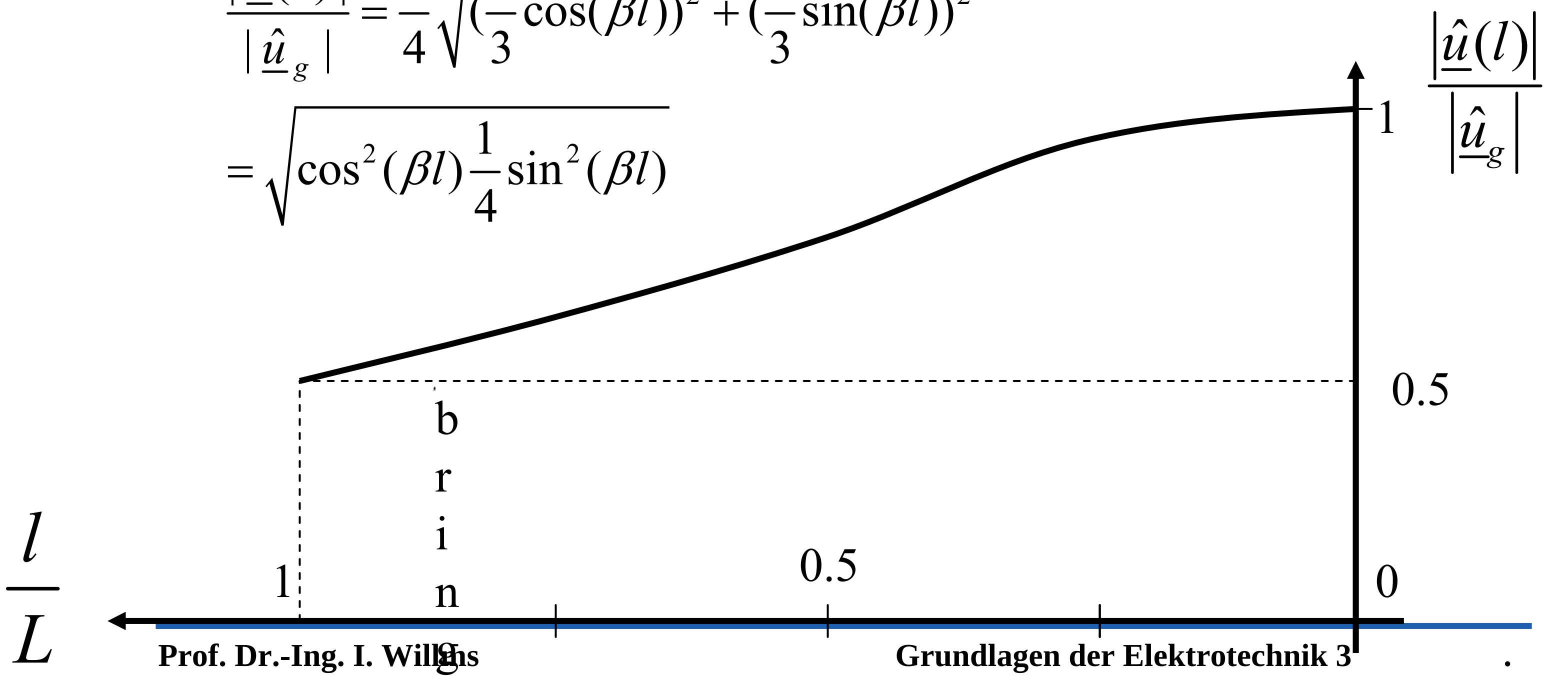
6.4.4 Verlustlose Fernleitungen

$$\underline{\hat{u}}(0) = \underline{\hat{u}}_0 = j\underline{\hat{u}}_g$$

Using eq. (72a) the relation for the voltage amplitude along the line can be set-up:

$$\frac{|\underline{\hat{u}}(0)|}{|\underline{\hat{u}}_g|} = \frac{3}{4}\sqrt{(\frac{4}{3}\cos(\beta l))^2 + (\frac{2}{3}\sin(\beta l))^2}$$

$$= \sqrt{\cos^2(\beta l)\frac{1}{4}\sin^2(\beta l)}$$



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6.4.4 Verlustlose Fernleitungen

- Infolge der endlich großen Reflektionsfaktoren (+1/3 und -1/3) für $l = 0$ und $l = L$ ergeben sich daher auf dem λ/4-Transformator reflektierte Wellen!
- Vor und nach diesem kurzen Leitungsstück existiert jedoch Wellenanpassung – ohne Reflektion von Wellen!
- Damit kann ein derartiger Transformator zur Anpassung von Leitungen unterschiedlichen Wellenwiderstands eingesetzt werden.
- Damit wird auch eine Leistungsanpassung erreicht – die Leistung am Eingang des λ/4-Transformator wird ohne Verluste an den Ausgang dieses Leitungsstücks weitergegeben.