Convolution

The convolution procedure

\[
y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t - \tau) \cdot h(\tau) \, d\tau = \int_{-\infty}^{\infty} x(\tau) \cdot h(t - \tau) \, d\tau
\]

(1)

is one of the most important analytic tools used by communication engineers. In this leaflet, we explain the procedure of executing a convolution and we show some laws which simplify or even help to avoid extensive calculations in special cases.

Important for the convolution procedure is the correct interpretation of the integrand of the convolution integral. The figure below illustrates the procedure, using two very simple time-limited functions \(x(t)\) and \(h(t)\).

It is important to note that the integrand must be interpreted in terms of the integration variable (here \(\tau\))! First we plot both signals as a function of the integration variable \(\tau\) which requires nothing else than a change of the abscissa scaling from \(t\) to \(\tau\). Then we fold (mirror) one of the functions (in this case \(x(\tau)\)) at the vertical axis to get from \(x(\tau) \Rightarrow x(-\tau)\). Finally both functions, the folded and shifted function \(x(t - \tau)\) and \(h(\tau)\) are plotted as a function of the integration variable \(\tau\). Since \(t\) is an independent variable with respect to the integration with values in the interval \(-\infty < t < \infty\), the location of \(x(t - \tau)\) in the figure is dependent on the arbitrary chosen point in time \(t\), which can be determined using a special point of \(x(t)\).

\(x(0)\) for example is the distinct point in time where the function begins to rise. We find the same point for the mirrored and shifted version \(x(t - \tau)\) where \(t - \tau = 0\) or \(t - \tau = 0\), respectively. This special point is marked in the figure above. It shows, that the function \(x(t - \tau)\)
is plotted for exactly the point in time, which is scaled on the horizontal axis in terms of $\tau$ (in the plot above for $t > 0$).

Another specific point could be for example the maximum of $x(t)$ at $t = T$. For the folded and shifted function $x(t - \tau)$ we find the maximum $x(T)$ for $t - \tau = T$, thus for $t - T = \tau$ which can be identified on the horizontal axis in terms of $\tau$ (see last figure above).

For all $t < t_1$ the functions $x(t - \tau)$ and $h(\tau)$ do not overlap; hence the integrand of the convolution integral $x(t - \tau) \cdot h(\tau) \equiv 0$ and consequently the integral is 0.

Thus, we have for $t < t_1$: $y(t) \equiv 0 \forall t < t_1$.

For all $t$ with $t_1 < t < t_2 + T$ both functions partly overlap (see figure below).

For $t_1 \leq t \leq t_1 + T$: the function $h(\tau) \equiv 0 \forall \tau < t_1$, hence we obtain for the product $x(t - \tau) \cdot h(\tau) \equiv 0 \forall \tau < t_1$.

For $t > t_1$ we have $x(t - \tau) \equiv 0$ hence, the product is also zero for all $\tau > t$.

We thus have to integrate in the range from $\tau = t_1$ up to $\tau = t$.

\[
y(t) = \int_{-\infty}^{0} x(t - \tau) \cdot h(\tau) \, d\tau = b \cdot \int_{t_1}^{t} a \cdot \frac{t - \tau}{T} \, d\tau = \frac{a \cdot b}{2T} (t - t_1)^2
\]

The result of the integration is proportional to the shaded area in the figure.

For $t_1 + T \leq t \leq t_2$: the functions completely overlap.

In this case the lower integration limit is determined by the left edge of $x(t - \tau)$, the upper limit by the right edge. Hence, for $t_1 + T \leq t \leq t_2$ we obtain:

\[
y(t) = \int_{-\infty}^{0} x(t - \tau) \cdot h(\tau) \, d\tau = b \cdot \int_{t_1 - T}^{t} a \cdot \frac{t - \tau}{T} \, d\tau = a \cdot b \cdot \frac{T}{2}
\]

The integral is proportional to the shaded triangular area in the figure above and is obviously constant for every $t$ in this time interval $t_1 + T \leq t \leq t_2$ because a time shift within this interval does not change the shaded area.

\[
y(t) = a \cdot b \cdot \frac{T}{2} \quad \text{for } t_1 + T \leq t \leq t_2
\]

For $t_2 \leq t \leq t_2 + T$

the functions again only partially overlap and we obtain (see figure below)

$$y(t) = \int_{-\infty}^{\infty} x(t-\tau) \cdot h(\tau) \, d\tau = b \cdot \int_{t-T}^{t_2} a \cdot \frac{t - \tau}{T} \, d\tau = \frac{a \cdot b \cdot T}{2} \cdot \left[ 1 - \left( \frac{t - t_2}{T} \right)^2 \right]$$

(5)

The integral is again proportional to the shaded area in the figure.

Finally for $t > t_2 + T$ the functions do not overlap any more, hence the product $x(t-\tau) \cdot h(\tau) \equiv 0$ and the integral is zero for all $t$ in this time interval.

We have only given the results of integration in the above integrals and the given results have in fact not been obtained by carrying out the integration procedure. An alternative procedure shall be explained in the following:

From the fact that the integrands of all integrals above are linearly dependent on the integration variable ($a$ straight line in $\tau$) and that at least one of the integral limits is dependent on $t$ we deduce, that the result of integration leads to piecewise parabolic functions except in the section $t_1 + T \leq t \leq t_2$ where the parabolic terms cancel because both of the limits are dependent on $t$.

With this considerations we can plot the convolution result at least qualitatively as shown below.

From this (or similar) consideration(s), we can deduce some general rules to determine the convolution result and avoid extensive computations.

For $t_1 \leq t \leq t_1 + T$: we have a parabola $y(t) = c \cdot (t - t_1)^2$ with its vertex touching the $t$-axes at $t = t_1$. The unknown constant $c$ is determined by calculating the area of a triangle at $t = t_1 + T$, i.e.: $y(t_1 + T) = c \cdot T^2 = \frac{a \cdot b \cdot T}{2}$. Hence: $c = \frac{a \cdot b}{2T}$.

For $t_2 \leq t \leq t_2 + T$: we have a parabola $y(t) = d - c \cdot (t - t_2)^2$ with its vertex at $t = t_2$, $y = b = \frac{a \cdot b \cdot T}{2}$ and thus, the unknown constant $b$ follows from $y(t_2) = d = \frac{a \cdot b \cdot T}{2}$.

Clearly, this is a much more simple solution than the calculation of all the integrals above.
Convolution laws.

1. **The convolution of two rect-functions** always results in a trapezium (if the rect-functions have different widths) or a triangle (if the rect-functions have the same width).
   The maximum value of the trapezium or triangle can be easily computed by \( a \cdot b \cdot T_s \), where \( a, b \) describe the heights of the two rect-functions and \( T_s \) the width of the more narrow rect-function.
   The lower and upper limits in time of the convolution result are determined by some more generally valid laws stated later.

2. **The convolution of a rect-function with a polygon** (consisting of piecewise straight lines) always leads to a function consisting of constant parts and first or second order terms (proportional to \( t \) and/or \( t^2 \)), depending on the slope of the polygon.

3. **The convolution of time-limited functions** \( x(t), y(t) \) with
   \[ x(t) \neq 0 \quad \forall t \in [t_{x1}, t_{x2}]; \quad x(t) \equiv 0 \quad \text{elsewhere} \quad \text{and} \]
   \[ y(t) \neq 0 \quad \forall t \in [t_{y1}, t_{y2}]; \quad y(t) \equiv 0 \quad \text{elsewhere} \]
   leads to a time-limited function \( z(t) = x(t) \ast y(t) \) with
   \[ z(t) \neq 0 \quad \forall t \in [t_{x1} + t_{y1}, t_{x2} + t_{y2}]; \quad z(t) \equiv 0 \quad \text{elsewhere} \]
   That means,
   the lower limit in time \( t_{z1} \) of \( z(t) \) is the sum of the lower limits \( t_{x1} + t_{y1} \).
   the upper limit in time \( t_{z2} \) of \( z(t) \) is the sum of the upper limits \( t_{x2} + t_{y2} \).
   the width of \( z(t) \) is the sum of widths of \( x(t) \) and \( y(t) \). The convolution result equals 0 at the lower and upper limit in time and below the lower limit in time and above upper limit in time.

4. From the above laws clearly follows, that **convolution of two causal functions** results in a causal function.

5. **The convolution of two axis symmetrical functions** \( x(t), y(t) \) having their symmetry axis at \( t_{xs}, t_{ys} \) results in a function \( z(t) = x(t) \ast y(t) \), which is axis symmetrical with symmetry axis at \( t_{zs} = t_{xs} + t_{ys} \).
   If \( x(t) \) is even or odd with respect to \( t_{xs} \) and \( y(t) \) is even or odd with respect to \( t_{ys} \), then the convolution result is for:
   \( x(t) \) even and \( y(t) \) even \( \Rightarrow z(t) \) is even with respect to \( t_{zs} = t_{xs} + t_{ys} \)
   \( x(t) \) even and \( y(t) \) odd \( \Rightarrow z(t) \) is odd with respect to \( t_{zs} = t_{xs} + t_{ys} \)
   \( x(t) \) odd and \( y(t) \) even \( \Rightarrow z(t) \) is odd with respect to \( t_{zs} = t_{xs} + t_{ys} \)
   \( x(t) \) odd and \( y(t) \) odd \( \Rightarrow z(t) \) is even with respect to \( t_{zs} = t_{xs} + t_{ys} \).

6. **Convolution of** \( x(t) \) **with** \( x(-t) \)
   i.e. \( y(t) = x(t) \ast x(-t) \) results always in an even function!

7. **Convolution of a Dirac pulse** \( \delta(t - t_0) \) **with an arbitrary function** \( h(t) \) shifts the function to the right by \( t_0 \) if \( t_0 > 0 \) or to the left if \( t_0 < 0 \).
   Hence: \( \delta(t - t_0) \ast h(t) = h(t - t_0) \) and also \( \delta(t - t_0) \ast \delta(t - t_1) = \delta(t - t_0 - t_1) \).

8. The convolution of two ordinary functions - even if they show discontinuities - always results in a function without discontinuities! But if one function contains Dirac-pulses and the other has discontinuities the result shows discontinuities!