Network Theory 1
Analoge Netzwerke

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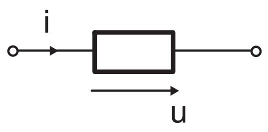
Chapter 1

Introduction and Basics

1.1 Preliminary remarks

Components like resistor, coil etc. are network elements



Two-poles have 2 pins accessible

Two-ports have 4 pins

Two main tasks of network theory:

- Network analysis: gives mathematical description of network properties
- Network synthesis: gives structure and values of components due to given task

1.1 Preliminary remarks

Three solutions steps of network synthesis:

1. Solution step 1: Network characterization by a given characterizing Network function (specification)
2. Solution step 2: Approximation of characterizing Network function by a realizable Network function
3. Solution step 3: Realization of the found realizable network function by a selected circuit (design)

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1.1 Preliminary remarks

Solution step 1:

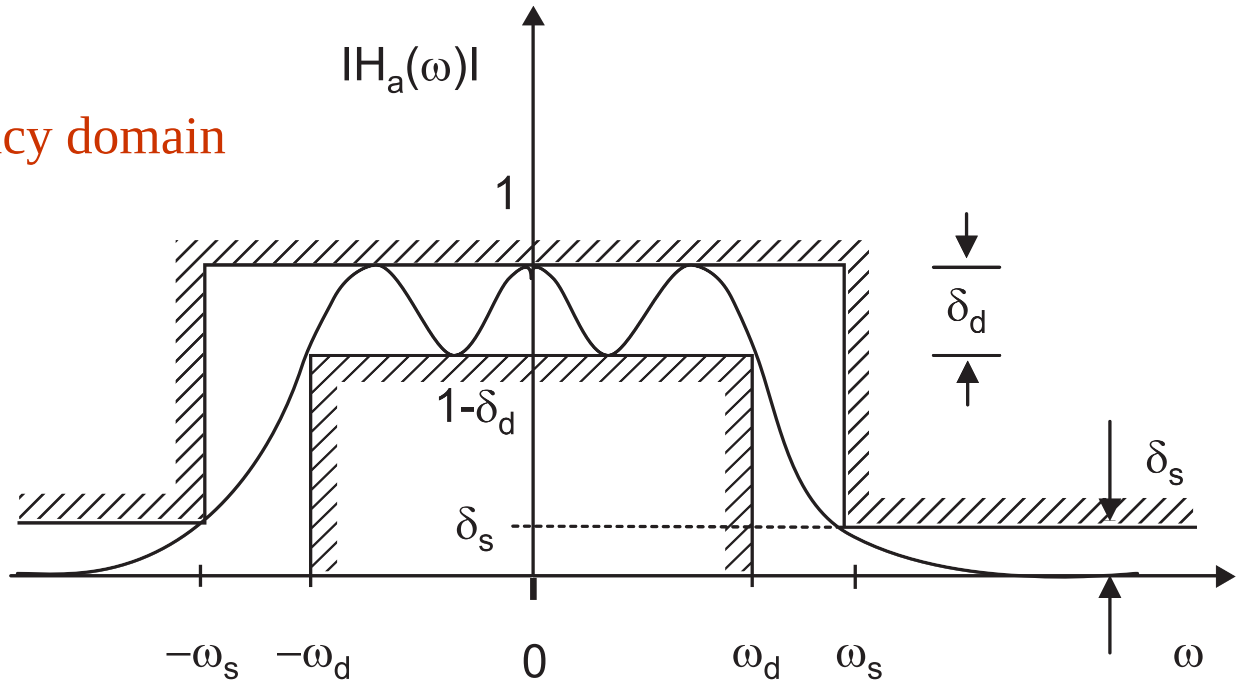
Characterizing network functions can be:

- Impedance function $Z_c(j\omega)$ of a LTI two-pole network (one port network)
- A transfer function $H_c(\omega)$ of a LTI two-port network (or its system function)
- The phase angle $\varphi(\omega)$ or a phase delay $\tau_g(\omega)$ or the magnitude of the transfer function

1.1 Preliminary remarks

Example of a characterizing network function and its approximation

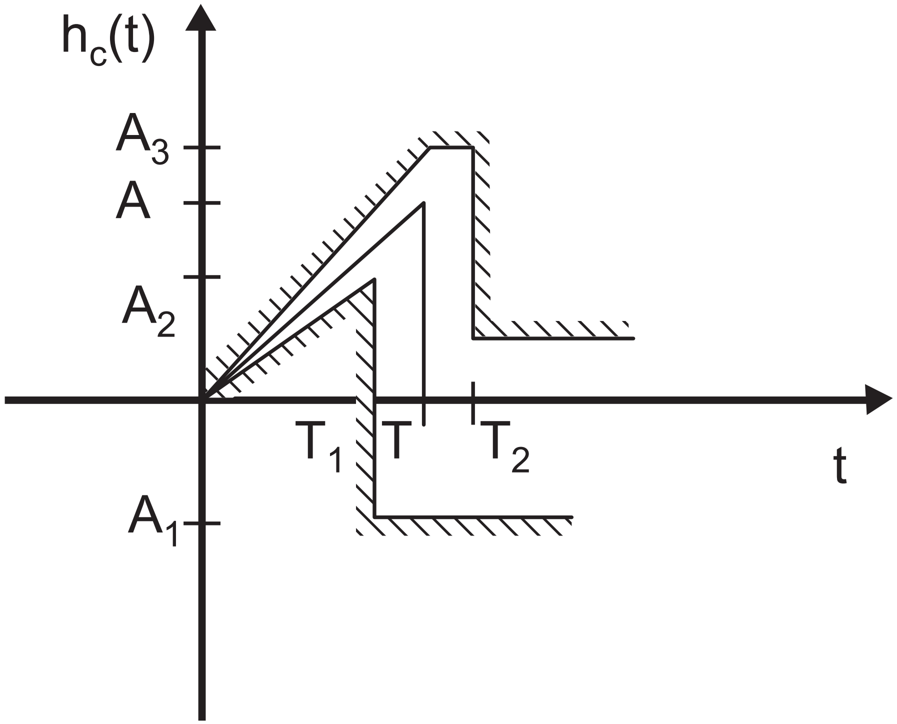
Frequency domain



A tolerance scheme for $|H_a(\omega)|$ of a low-pass

1.1 Preliminary remarks

Time domain



Characterizing impulse response
(Tolerance pattern for an impulse response)

1.1 Preliminary remarks

Solution step 2:

Realization by:

- finite number of linear elements
- time invariant elements (LTI elements)
- passive elements

Additional possibilities:

- active LTI elements
- controlled voltage + current sources

Danger:

- Poles in right p-plane
- Instability (2 cases) ⇨ In best case useful only for signal generators

1.1 Preliminary remarks

To observe:

Rational real fractional network function cannot realize in general

a) Arbitrary phase function $\varphi_c(\omega) = \sphericalangle H_C(\omega)$

b) Arbitrary magnitude $|H_c(\omega)|$

c) Thus also no abitrary $h_c(t)$

Thus differences are always to be expected between desired functions (index c) and realisable functions (index a).

Differences are measured by $H_e(j\omega)$ in a certain range $\omega_1 \leq \omega \leq \omega_2$

$H_e(j\omega)$ will depend on parameters of $H_a(j\omega)$

$$H_e(j\omega) = H_a(\omega) - H_c(\omega)$$

1.1 Preliminary remarks

The typical error function $h_e(t)$ in time domain:

$$h_e(t) = h_a(t) - h_c(t) \quad \text{with approximations intervall} \quad t_1 \le t \le t_2$$

Approximation criteria:

1. CHEBYSHEF criterion or regular approximation:
   Limits max. deviation, example see S.6
2. Criterion of the smallest mean square error:

$$\min \int_{\omega_1}^{\omega_2} \left|H_e(\omega)\right|^2 d\omega \quad \text{or} \quad \min \int_{t_1}^{t_2} h_e^2(t)\ dt$$

Solution of the approximation tasks:

- Simple if approx. function linearly depends on approx. parameters

- Leads to solution of linear system of equations and can be extended by weighting functions

$$\min \int_{\omega_1}^{\omega_2} \left|H_e(\omega)\right|^2 \cdot Q(\omega)\ d\omega \quad \text{or} \quad \min \int_{t_1}^{t_2} h_e^2(t) \cdot q(t)\ dt$$

1.1 Preliminary remarks

Approximation criteria:

3. Criterion of maximum smoothing: The error function and its derivatives should show for orders as high as possible the value of zero at a prescribed location within the approximation interval.

4. Interpolation criterion: Adjusting the approximation parameters in such a way that the error function at given discrete points within the approximation interval is zero.

   The number of these discrete interpolation points corresponds to the number of approximation parameters.

   Disadvantage: Large deviations at many other locations!

1.1 Preliminary remarks

Solution step 3: Circuit realization

- Several realization possibilities exist depending on certain defaults:

  - a passive or active network
  - a pure reactance network
  - an active or passive RC network
  - a network with passive symmetrical bridges
  - arrangements of pieces of line (for MW frequencies)

- Networks composed with a minimum of components should also be preferred

Chapter 1

Introduction

1.2 LTI Concentrated Network Elements

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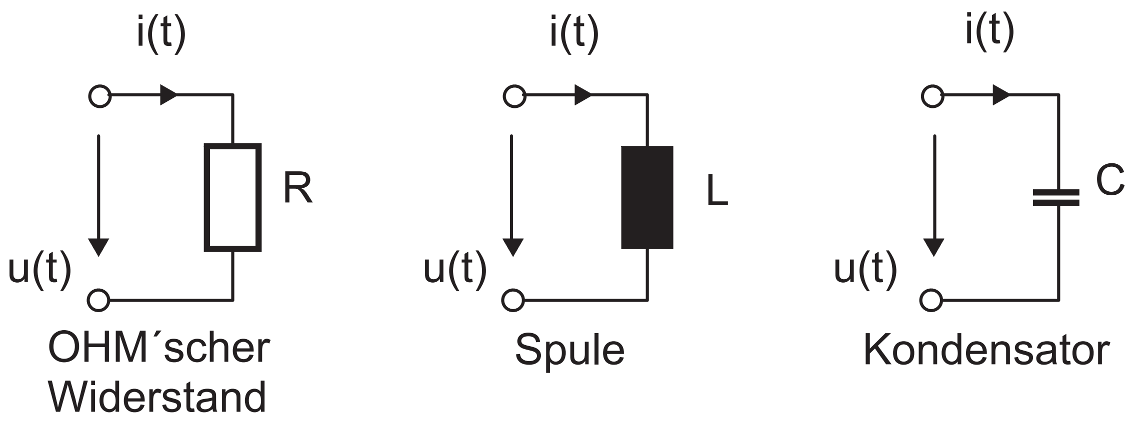
1.2 LTI concentrated network elements

1.2.1 The Ohm's resistance:

The symbol and the defining reference directions:



$$u(t) = R \cdot i(t)$$

LAPLACE-Transform:

$$U(p) = R \cdot I(p)$$

1.2 LTI concentrated network elements

1.2.2 The capacity C

The relationship between the current *i(t)* and voltage *u(t)* of an electric capacitor is described as:

$$i(t) = C \cdot \frac{du(t)}{dt}$$

as well as

$$u(t) = u(t_0) + \frac{1}{C} \cdot \int_{t_0}^{t} i(\tau) \; d\tau$$

Laplace transform:

$$I(p) = p \cdot C \cdot U(p)$$

Admittance: $Y(p) = p \cdot C$

Impedance: $Z(p) = 1 / Y(p)$

1.2 LTI concentrated network elements

1.2.3 The inductivity L

$$u(t) = L \cdot \frac{di(t)}{dt}$$

and

$$i(t) = i(t_0) + \frac{1}{L} \cdot \int_{t_0}^{t} u(\tau)\ d\tau$$

Laplace transform:

$$U(p) = pL \cdot I(p)$$

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1.2.4 The ideal transformer:

The transmission characteristics of a lossless transformer with leakage field in the time domain:

$$\begin{cases} u_1(t) = L_1 \cdot \dfrac{di_1}{dt} + M \cdot \dfrac{di_2}{dt} \\ u_2(t) = M \cdot \dfrac{di_1}{dt} + L_2 \cdot \dfrac{di_2}{dt} \end{cases}$$

with

L_1 : primary inductivity

L_2 : secondary inductivity

Laplace transform with zero initial condition gives:

$$\begin{cases} U_1(p) = pL_1 \cdot I_1(p) + pM \cdot I_2(p) \\ U_2(p) = pM \cdot I_1(p) + pL_2 \cdot I_2(p) \end{cases}$$

1.2 LTI concentrated network elements

The relation between the ideal transformer and its transmission characteristics:

$$\begin{cases} u_1(t) = \ddot{u} \cdot u_2(t) \\ i_1(t) = -\dfrac{1}{\ddot{u}} \cdot i_2(t) \end{cases}$$

where $\ddot{u}$ is transformer constant

$$\ddot{u} = \frac{w_1}{w_2} = \sqrt{\frac{L_1}{L_2}}$$

in Laplace transforms:

$$\begin{cases} U_1(p) = \ddot{u} \cdot U_2(p) \\ I_1(p) = -\dfrac{1}{\ddot{u}} \cdot I_2(p) \end{cases}$$

1.2.5 The ideal Gyrator

→ Changes resistance into conductance and vice versa or capacitors to coils

- Voltage-current relationship of ideal gyrators:

$$i_1(t) = g \cdot u_2(t)$$
$$i_2(t) = -g \cdot u_1(t)$$

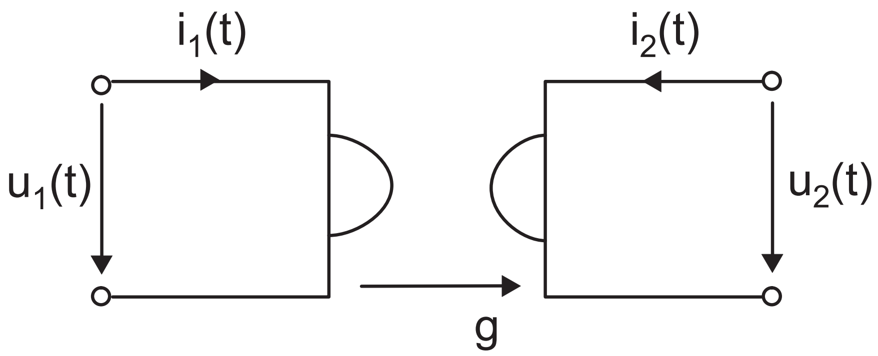
Laplace transform →

$$I_1(p) = g \cdot U_2(p)$$
$$I_2(p) = -g \cdot U_1(p)$$

(g is gyrator´s value)

1.2 LTI concentrated network elements

Symbol of an ideal gyrator with the reference directions



$$I_1(p) = g \cdot U_2(p)$$
$$I_2(p) = -g \cdot U_1(p)$$

$$\Rightarrow \frac{U_1(p)}{I_1(p)} = \frac{I_2(p)/(-g)}{gU_2(p)} = -\frac{I_2(p)}{g^2 U_2(p)}$$

$$Z_e(p) = \frac{1}{g^2} \cdot \frac{1}{Z_a(p)}$$

with

$$Z_e(p) = \frac{U_1(p)}{I_1(p)}$$ the input impedance

$$Z_a(p) = -\frac{U_2(p)}{I_2(p)}$$ the load impedance

1.2 LTI concentrated network elements

Examples A and B with $g = \dfrac{1}{R_g} = 1mS \quad C = 1\mu F \quad R = R_g = 1k\Omega$

$$\text{A: } Z_a(p) = R \Rightarrow Z_e(p) = \frac{1}{g^2} \cdot \frac{1}{Z_a(p)} = \frac{1}{10^{-6}S^2 \cdot 1k\Omega} = 10^3\Omega$$

$$\text{B: } Z_a(p) = 1/pC \Rightarrow Z_e(p) = \frac{1}{g^2} \cdot \frac{1}{Z_a(p)}$$

$$= \frac{1}{10^{-6}S^2 \cdot 1/(p \cdot 10^{-6}F)} = \frac{p10^{-6}F}{10^{-6}S^2} = p1H$$

1.2 LTI concentrated network elements

1.2.6 Independent sources:

There are two different kinds of independent sources:

1. The independent (ideal) voltage supply:

$$u_q(t) \circ\!\!-\!\!\bullet\; U_q(p)$$

independent of $i(t) \circ\!\!-\!\!\bullet\; I(p)$

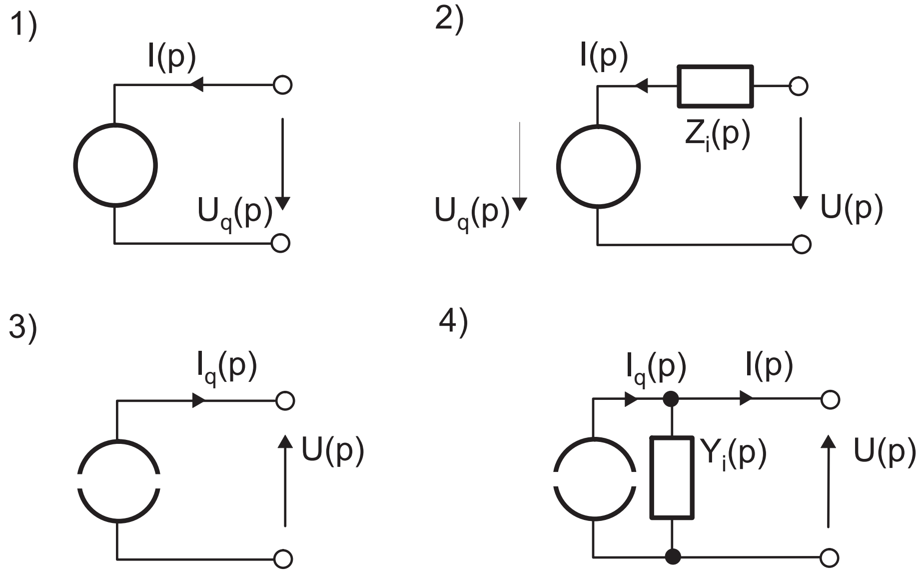
2. The independent (ideal) current supply:

$$i_q(t) \circ\!\!-\!\!\bullet\; I_q(p)$$

independent of $u(t) \circ\!\!-\!\!\bullet\; U(p)$

1.2 LTI concentrated network elements

Symbols of uncontrolled sources and reference directions



1) Independent (ideal) voltage supply

2) Voltage supply with an internal resistance

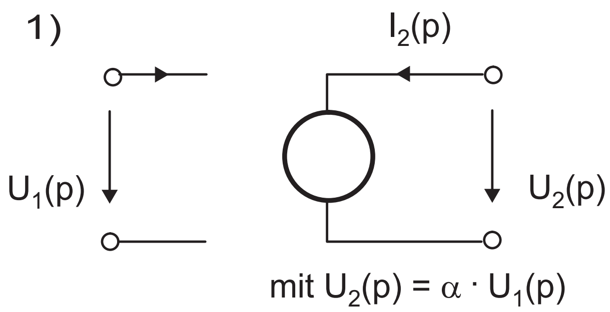
3) Independent (ideal) current source

4) Current source with an internal resistance

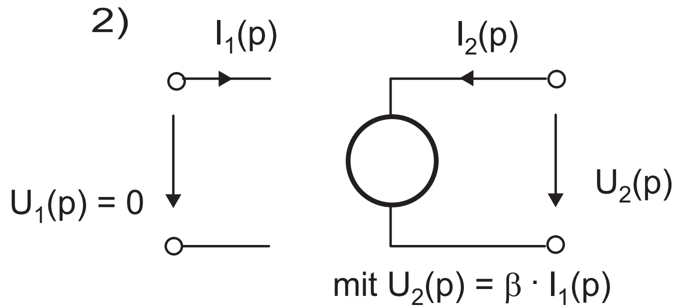
1.2 LTI concentrated network elements

1.2.7 Dependent sources:

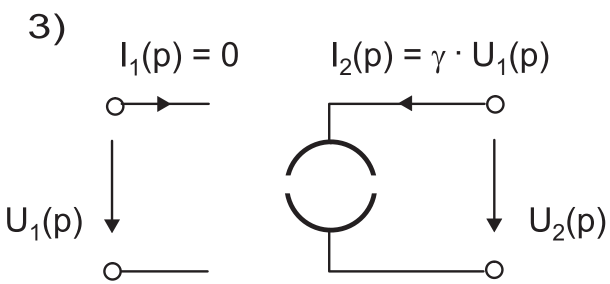
Symbols of dependent sources (controlled) and reference directions of the electricity



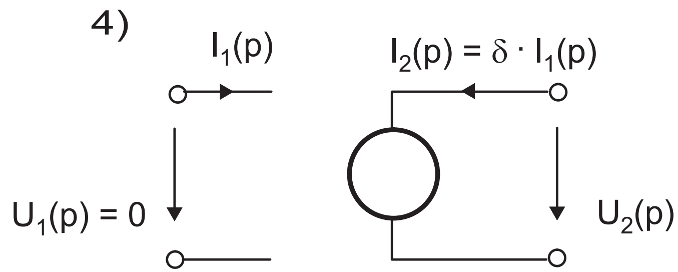
Voltage-controlled voltage supply



Current-controlled voltage supply



Voltage-controlled current source



Current-controlled current source

1.2 LTI concentrated network elements

1.2.8 Network equations:

Impedance matrix $\overleftrightarrow{Z}$

$$\begin{bmatrix} U_1(p) \\ U_2(p) \end{bmatrix} = \begin{bmatrix} Z_{11}(p) & Z_{12}(p) \\ Z_{21}(p) & Z_{22}(p) \end{bmatrix} \cdot \begin{bmatrix} I_1(p) \\ I_2(p) \end{bmatrix}$$

Admittance matrix $\overleftrightarrow{Y}$

$$\begin{bmatrix} I_1(p) \\ I_2(p) \end{bmatrix} = \begin{bmatrix} Y_{11}(p) & Y_{12}(p) \\ Y_{21}(p) & Y_{22}(p) \end{bmatrix} \cdot \begin{bmatrix} U_1(p) \\ U_2(p) \end{bmatrix}$$

Cascade matrix $\overleftrightarrow{A}$

$$\begin{bmatrix} U_1(p) \\ I_1(p) \end{bmatrix} = \begin{bmatrix} A_{11}(p) & A_{12}(p) \\ A_{21}(p) & A_{22}(p) \end{bmatrix} \cdot \begin{bmatrix} U_2(p) \\ -I_2(p) \end{bmatrix}$$

For conversion formulas see the matrix table slides.

Chapter 1

Introduction

1.3 Network Topology

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1.3 Network topology

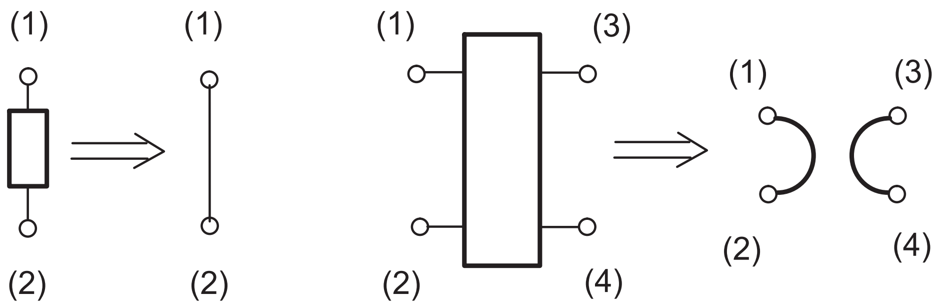
The electrical characteristics of a network depend on:

- Properties of network elements used
- **Topology** or structure of the network elements

  → Described by a graph representing the structure of the network

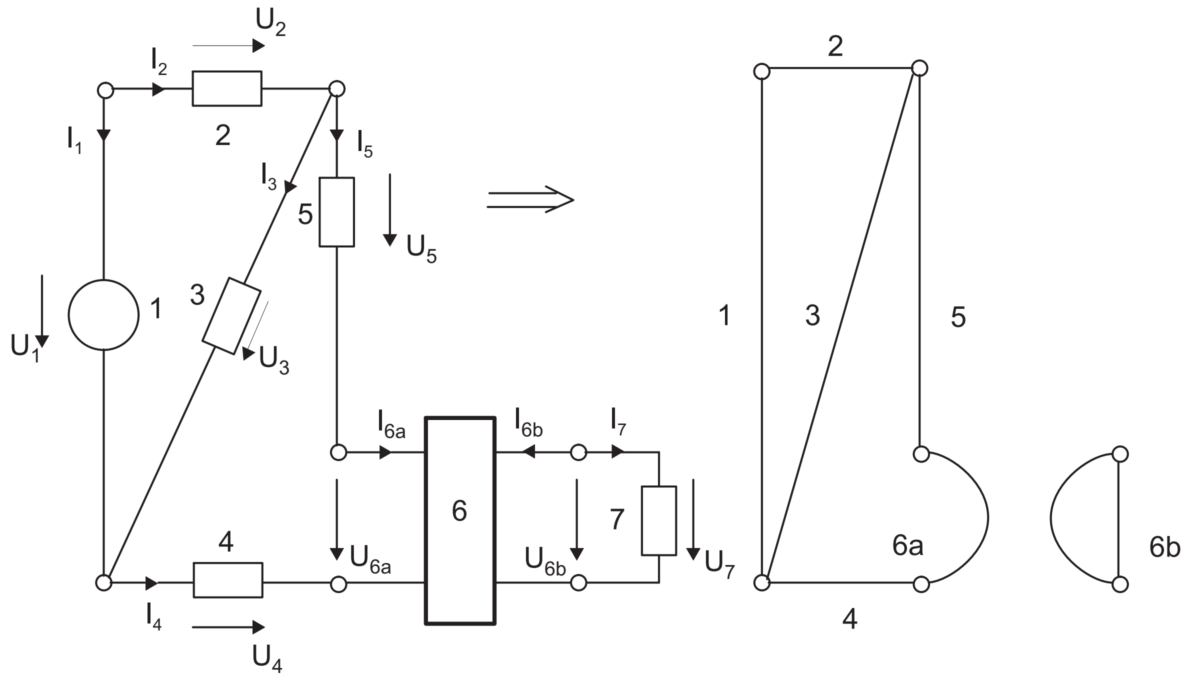
Method of drawing the network topology:

Example: Non-directional graph of a two-pole as well as a four-pole network element



1.3 Network topology

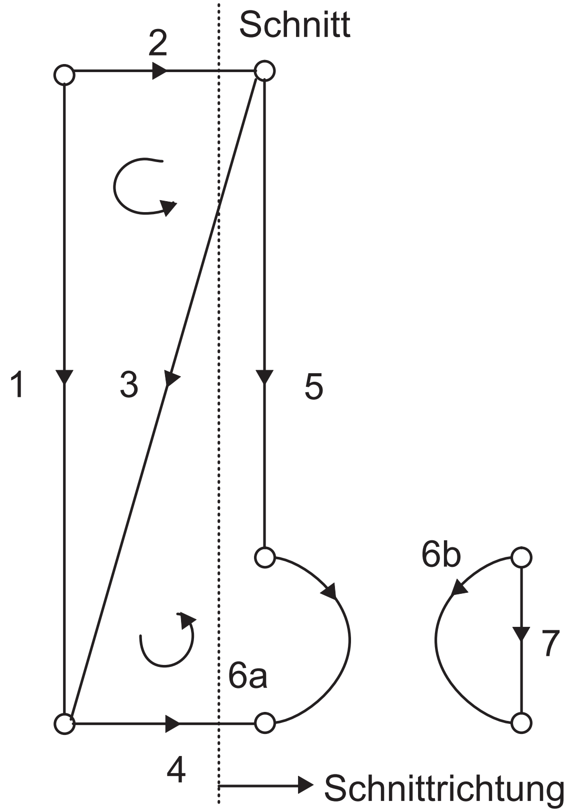
A given network and the nondirectional graph following from it



1.3 Network topology

One can cut a network in a manner that:

1. A part of the graph is completely separated from the remainder
2. This cut is not through a node



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1.3 Network topology

Applying cuts to all branches around a node leads (as charging up of a node cannot happen) to:

Kirchhoff´s current rule (summation on all branches cut):

$$\sum_{\nu} i_{\nu}(t) = 0$$

↓ Laplace transform

$$\sum_{\nu} I_{\nu}(p) = 0 \quad \text{as well as} \quad \sum_{\nu} I_{\nu}^{*}(p) = 0$$

where $I_{\nu}^{*}(p)$ is the complex conjugate of $I_{\nu}(p)$

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1.3 Network topology

Kirchhoff´s voltage rule (summation along one loop):

$$\sum_{\nu} u_{\nu}(t) = 0 \quad \forall t$$

Laplace transformation

$$\sum_{\nu} U_{\nu}(p) = 0 \quad \text{as well as} \quad \sum_{\nu} U_{\nu}^{*}(p) = 0$$

TELLEGEN's theory (summation covering all branches):

$i_{\nu}(t)$ and $u_{\nu}(t)$ are the branch current and branch voltage and it is valid:

$$\sum_{\nu=1}^{z} u_{\nu}(t) \cdot i_{\nu}(t) = 0$$

Laplace trans.

$$\sum_{\nu=1}^{z} U_{\nu}(p) \cdot I_{\nu}^{*}(p) = 0 \quad \text{and} \quad \sum_{\nu=1}^{z} U_{\nu}^{*}(p) \cdot I_{\nu}(p) = 0$$

1.3 Network topology

In a given network only a certain maximum number of branch currents arises, which

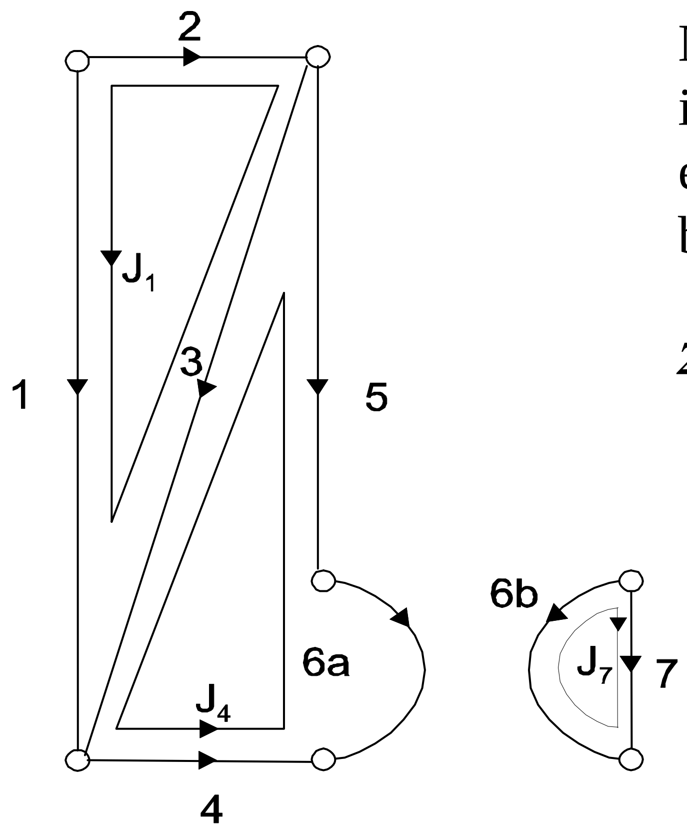
1. are independent
2. thereby specifying the remaining branch currents

<u>The independent branches of a network</u> are branches in which the independent currents flow.

The independent branches of a network are determined by first designing a “complete tree” .

A complete tree of a connected graph is a partial graph which contains all of its nodes and some of its branches, but contains no loops.

1.3 Network topology



Number z_u of linearly independent loop equations for in total z branches and k nodes is:

$$z_u = z - k + 1$$

The figure shows branches of a complete tree (2, 3, 5, 6a, 6b) and the independent branches (1, 4, 7) of the graph as well as the appropriate loop current s.

J_μ

1.3 Network topology

Method of branch current determination

$$I_\nu(p) = \sum_\mu m_{\nu\mu} \cdot J_\mu(p)$$ where $J_\mu(p)$ are the independent loop currents

Independant branches produce a loop when adding it to a complete tree.

In each of these loops the corresponding „loop current“ runs through an independant branch which is not contained in the other loops!

This can be expressed more generally in matrix form:

$$\vec{I}(p) = \overleftrightarrow{M} \cdot \vec{J}(p)$$

"Branch current vector" - *(z x 1)*

Incidence matrix

"Loop current vector" $(z_u \ x \ 1)$

1.3 Network topology

Incidence matrix coefficients:

$$m_{\nu\mu} = \begin{cases} 1 & \text{if branch } \nu \text{ belongs to loop } \mu\text{; branch and loop directions agree} \\ -1 & \text{if branch } \nu \text{ belongs to loop } \mu\text{; branch and loop directions do not agree} \\ 0 & \text{if branch } \nu \text{ does not belong to loop } \mu \end{cases}$$

Kirchhoff's voltage rule can then be formulated as

$$\sum_{\nu=1}^{z} m_{\nu\mu} \cdot U_\nu(p) = 0 \quad \text{with } U_\nu(p) \text{ as the voltage of the branch } \nu$$

1.3 Network topology

All branch voltages can be combined into the branch voltage vector:

$$\vec{U}(p) = \begin{pmatrix} \vec{U}_1(p) \\ \vdots \\ \vec{U}_\nu(p) \\ \vdots \\ \vec{U}_z(p) \end{pmatrix}$$

Kirchhoff's voltage rule can then be represented as follows:

$$\overleftrightarrow{M}^T \cdot \vec{U}(p) = \vec{0}$$

Replacing branch voltages with voltages at all components gives (and under the condition that the network does not contain uncontrolled power sources, controlled sources, transformers and gyrators):

$$\vec{U}(p) = \overleftrightarrow{Z}_{BI} \cdot \vec{I}(p) + \vec{U}_s(p)$$

Branch impedance matrix

Voltage source vector

1.3 Network topology

<u>Branch impedance matrix</u>: (purely diagonally with z x z elements)

$$\overleftrightarrow{Z}_{BI} = \begin{bmatrix} Z_1(p) & 0 & 0 \\ 0 & Z_\nu(p) & 0 \\ 0 & 0 & Z_z(p) \end{bmatrix}$$

1.3 Network topology

Now the following equations will be combined:

$$\vec{U}(p) = \overleftrightarrow{Z}_{BI} \cdot \vec{I}(p) + \vec{U}_s(p) \qquad \vec{I}(p) = \overleftrightarrow{M} \cdot \vec{J}(p)$$

They give: $\vec{U}(p) = \overleftrightarrow{Z}_{BI} \cdot \overleftrightarrow{M} \cdot J(p) + \vec{U}_s(p)$

Another relation results from using: $\overleftrightarrow{M}^T \cdot \vec{U}(p) = \vec{0}$

$$\overleftrightarrow{M}^T \cdot \vec{U}(p) = \overleftrightarrow{M}^T \cdot \overleftrightarrow{Z}_{BI} \cdot \overleftrightarrow{M} \cdot \vec{J}(p) + \overleftrightarrow{M}^T \cdot \vec{U}_s(p) = \vec{0}$$

Thus the <u>loop impedance matrix</u> *Z(p)* appears: $(z_u \; x \; z_u)$

$$\overleftrightarrow{Z}(p) = \overleftrightarrow{M}^T \cdot \overleftrightarrow{Z}_{BI} \cdot \overleftrightarrow{M} = \begin{pmatrix} Z_{11} & \cdots & Z_{12} & \cdots & Z_{1z_u} \\ Z_{21} & \cdots & & \cdots & Z_{2z_u} \\ \vdots & & \vdots & & \vdots \\ Z_{z_u 1} & \cdots & & \cdots & Z_{z_u z_u} \end{pmatrix}$$

1.3 Network topology

Thus the previous equation can be simplified to:

$$\overleftrightarrow{Z}(p) \cdot \vec{J}(p) = -\overleftrightarrow{M}^T \cdot \vec{U}_s(p)$$

It is a system of z_u linearly independent equations!

Thus the following procedure results based on:

a) a given loop impedance matrix

b) a given vector $-\overleftrightarrow{M}^T \cdot \vec{U}_s(p)$

1) Determination of the current loop vector $J(p)$
2) From this the current branch vector is determined by $\vec{I}(p) = \overleftrightarrow{M} \cdot \vec{J}(p)$
3) Finally the branch impedance vector is determined using

$$\vec{U}(p) = \overleftrightarrow{Z}_{BI} \cdot \vec{I}(p) + \vec{U}_s(p)$$

1.3 Network topology

For RLC-networks with independent voltage sources the loop impedance matrix can be set up as follows:

$$Z_{mm} = \sum_{\nu} Z_{\nu} \quad \text{for all branches } \nu, \text{ which belong to loop } \mu_m$$

Here Z_{mm} is the sum of all impedances in a loop.

$$Z_{nm} = \sum_{\nu} a_{\nu} Z_{\nu} \quad \text{for all branches } \nu, \text{ which belong to loop } \mu_m \text{ and loop } \mu_n$$

$$a_{\nu} = \begin{cases} +1 & \text{if the direction of loop } \mu_m \text{ and } \mu_n \text{ at the common branch are equal} \\ -1 & \text{If the direction of loop } \mu_m \text{ and } \mu_n \text{ at the common branch are not equal} \end{cases}$$

Chapter 1

Introduction

1.4 Network Functions

1.4 Network functions

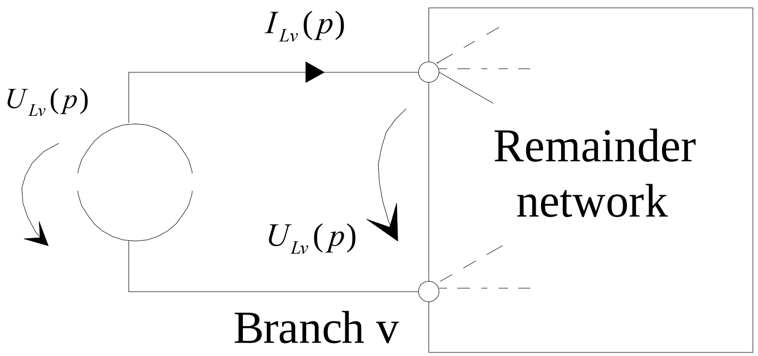
The network function $N_L(p)$ is defined as the ratio of the Laplace transform of the response signal and the Laplace transform of the input signal under the conditions that:

1. All network elements are linear and time invariant (in short LTI or LZI)
2. All network elements are in the energyless initial condition (zero state).

If the response signal and the input signal are at the same branch, the network function is called two-terminal function or impedance function or admittance function.

1.4 Network functions

An example of an impedance function:



Two-terminal network

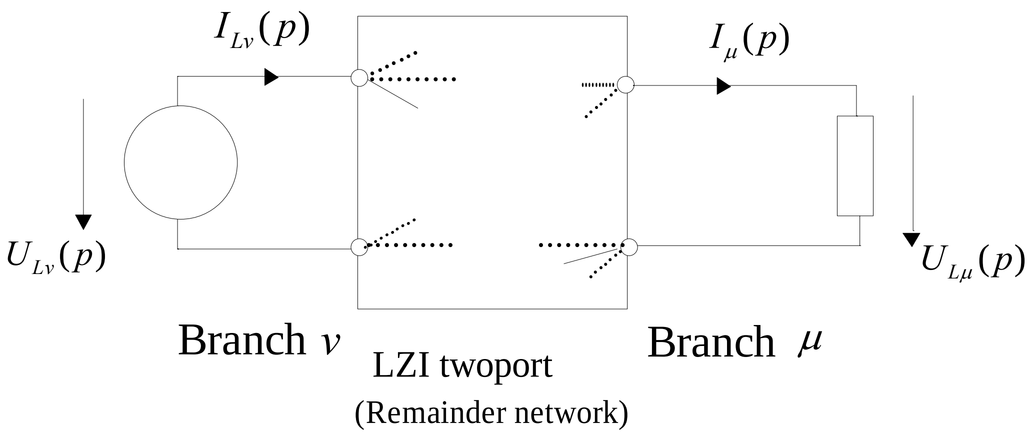
Input $I_{L\nu}(p)$ and response signal $U_{L\nu}(p)$ of a two-port LTI system gives the network impedance function:

$$Z_L(p) = \frac{U_{L\nu}(p)}{I_{L\nu}(p)}$$

1.4 Network functions

If the excitation signal and the response are located at different branches of the network or at different ports, then one calls the appropriate network function an effective function $H_L(p)$

An example of an effective function:

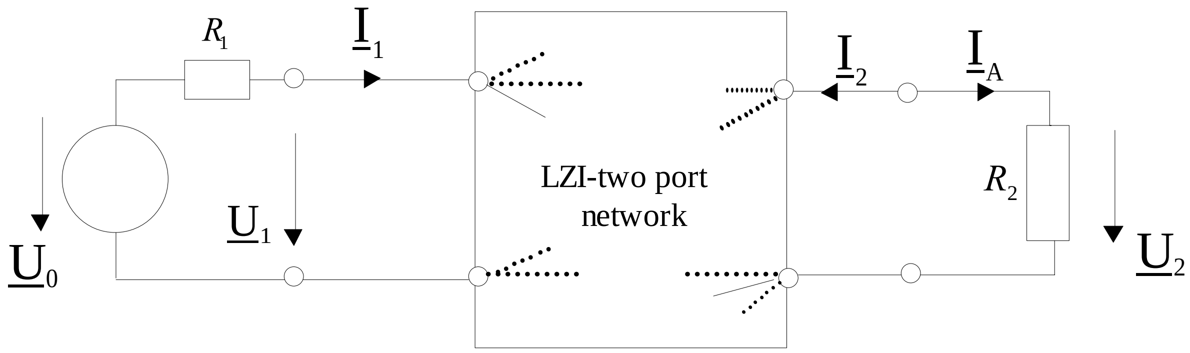


Input $U_{L\nu}(p)$ and the response signal $U_{L\mu}(p)$ of a two-port LTI network:

$$H_L(p) = \frac{U_{L\mu}(p)}{U_{L\nu}(p)}$$

1.4 Network functions

Modern network theory uses a description of the transmission characteristics of networks in a representation which follows partly that of classical network theory:



The classical representation is in time domain instead of p domain:

$$\frac{1}{\sqrt{2}} u(t) = U_{eff} \cdot \cos\ (\omega t + \varphi_u)$$

and

$$\frac{1}{\sqrt{2}} i(t) = I_{eff} \cdot \cos\ (\omega t + \varphi_u)$$

1.4 Network functions

For passive networks it is reasonable to look at both voltage transmission and current transmission through the network.

This is done for a typical case with <u>impedance matching</u> at the input.

Voltage amplification then is: $$A_U = \left.\frac{\underline{U}_2}{\underline{U}_1}\right|_{R_1=R_L} = \frac{\underline{U}_2}{\underline{U}_0/2} \text{ with } R_L = \frac{\underline{U}_1}{\underline{I}_1}$$

Current amplification then is: $$A_I = \left.\frac{-\underline{I}_2}{\underline{I}_1}\right|_{R_1=R_L} = \frac{-\underline{I}_2}{\underline{U}_0/2R_1} = \frac{\underline{I}_A}{\underline{U}_0/2R_1}$$

An operation transmission factor then can be defined:

$$A_B = \sqrt{A_U A_I} = \sqrt{\frac{\underline{U}_2}{\underline{U}_0/2} \cdot \frac{-\underline{I}_2}{\underline{U}_0/2R_1}} = \sqrt{\frac{\underline{U}_2}{(\underline{U}_0/2)^2} \cdot \frac{\underline{U}_2/R_2}{1/R_1}} = \frac{\underline{U}_2}{\underline{U}_0/2}\sqrt{\frac{R_1}{R_2}} = H_{LB}(p)$$

The inverse of this factor leads then to the insertion loss function.

1.4 Network functions

Definition of the insertion loss function:

$$H_{BD}(\omega) = \frac{1}{A_B}\Big|_{p\to j\omega} = \sqrt{\frac{\dfrac{\underline{U}_0^2}{4R_1}}{\dfrac{\underline{U}_2^2}{R_2}}} = \sqrt{\frac{R_2}{R_1}} \cdot \frac{\frac{1}{2}\cdot\underline{U}_0}{\underline{U}_2}$$

This definition follows the idea that the magnitude of the insertion loss function can be expressed using 2 effective powers (related to the input and to the load):

$$P_{0\,\text{max.}} = \frac{|\underline{U}_0|^2}{4\cdot R_1} \qquad P_2 = \frac{|\underline{U}_2|^2}{R_2}$$ Here voltages represent effective values

Therefore here the magnitude of the insertion loss function is considered and can be rewritten using the two effective powers.

1.4 Network functions

Thus we obtain:

$$\left|H_{BD}(\omega)\right| = \sqrt{\frac{P_{o\,\max}}{P_2}} = \sqrt{\frac{R_2}{R_1}} \cdot \frac{\frac{1}{2}\cdot\left|\underline{U}_0\right|}{\left|\underline{U}_2\right|}$$

Here the two effective powers are represented by:

$P_{0\,\max}$: The maximum effective power passed to an external load resistance R_L attached to a voltage supply (or the input power)

(The maximum is given in case of $R_L = R_1$)

P_2 The effective power at the load R_2

This definition of the insertion loss function is due to the fact that in early days of communications technology the transmission of telephone/telegraph signals over lossy long lines was quite an important problem.

1.4 Network functions

Long lines (compared to wavelength) have to be operated with suitable loads for avoiding reflections at input and output ports. At all interfaces loads have to be equal to the wave impedance of the line, e.g.: $Z_w = 600\Omega = R_1 = R_2$
The same is true for two-ports for RF signals!

Definition of the effective transmission factor:

$$g_B(\omega) = a_B(\omega) + j \cdot b_B(\omega)$$

Attenuation constant (Phase) Wavelength constant

Relation with insertion loss function:

$$H_{BD}(\omega) = |H_{BD}(\omega)| \cdot e^{j \cdot \sphericalangle H_{BD}(\omega)}$$

$$= e^{g_B(\omega)} = e^{a_B(\omega) + j \cdot b_B(\omega)}$$

Using the natural logarithm it holds:

$$\ln\{H_{BD}(\omega)\} = \ln\left\{ |H_{BD}(\omega)| e^{j \sphericalangle H_{BD}(\omega)} \right\} =$$

$$\ln\left\{ |H_{BD}(\omega)| \right\} + j \cdot \sphericalangle H_{BD}(\omega) = a_B(\omega) + j \cdot b_B(\omega)$$

1.4 Network functions

Thus we arrive at the following relation with the insertion loss function due to:

$$H_{BD}(\omega) = \sqrt{\frac{R_2}{R_1}} \cdot \frac{\frac{1}{2}\cdot \underline{U}_0}{\underline{U}_2}$$

$$a_B(\omega) = \ln\left\{\left|H_{BD}(\omega)\right|\right\} = \ln\left\{\sqrt{\frac{R_2}{R_1}} \cdot \frac{\frac{1}{2}\cdot\left|\underline{U}_0\right|}{\left|\underline{U}_2\right|}\right\}$$

$$b_B(\omega) = \arctan\left(\frac{\operatorname{Im}\left\{\frac{\frac{1}{2}\cdot\left|\underline{U}_0\right|}{\left|\underline{U}_2\right|}\right\}}{\operatorname{Re}\left\{\frac{\frac{1}{2}\cdot\left|\underline{U}_0\right|}{\left|\underline{U}_2\right|}\right\}}\right)$$

If impedances (instead of resistors) are used, also these values go into the determination of the angle!

1.4 Network functions

Nowadays the usual method of the description of the transmission characteristics of two-port networks is the LAPLACE transform of the voltage signals *u(t)* or of the current signals *i(t)*.

Four operational cases conc. the effective functions are distinguished.

The first operational case "Two-port network without input&output load":

1) A two-port network fed by voltage source $U_0(p)$ and negligible internal resistance $R_1 = 0$, two-port load $R_2 = \infty$ (zero-load):

Voltage effective funktion (expressed by two-port matrix elements):

$$\frac{U_2(p)}{U_0(p)} = \frac{Z_{21}(p)}{Z_{11}(p)} = \frac{-Y_{21}(p)}{Y_{22}(p)} = \frac{1}{A_{11}(p)}$$

1.4 Network functions

2) A two-port network fed by voltage supply source $U_0(p)$ and internal resistance $R_1 = 0$, two-port load $R2 = 0$ and $U_2(p) = 0$ (short-circuit at output)

<u>Transmission admittance function:</u>

$$\frac{I_A(p)}{U_0(p)} = \frac{Z_{21}(p)}{\det \overleftrightarrow{Z}} = -Y_{21}(p) = \frac{1}{A_{12}(p)}$$

3) A two-port network fed by power supply source with $I_0(p)$ and internal conductance $Y_1 = (1/R_1) = 0$ with load $R_2 = \infty$ (open circuit).

<u>Transmission impedance function:</u>

$$\frac{U_2(p)}{I_0(p)} = Z_{21}(p) = -\frac{Y_{21}(p)}{\det \overleftrightarrow{Y}} = \frac{1}{A_{21}(p)}$$

1.4 Network functions

4) A two-port network fed by power supply source $I_0(p)$ and internal conductance $Y_1 = (1/R_1) = 0$ and $R_2 = 0$ (closed circuit at output).

Current effective funktion:

$$\frac{I_A(p)}{I_0(p)} = \frac{Z_{21}(p)}{Z_{22}(p)} = -\frac{Y_{21}(p)}{Y_{11}(p)} = \frac{1}{A_{22}(p)}$$

1.4 Network functions

The second operational case „Two-port with resistance at input"

1) A two-port fed by voltage supply source $U_0(p)$ and internal resistance with two-port load $R_2 = \infty$ (zero-load).

$$H_{Leu}(p) = \frac{U_2(p)}{U_0(p)} = \frac{U_2(p)}{R_1 \cdot I_0(p)} = \frac{Z_{21}(p)}{Z_{11}(p) + R_1}$$

$$= -\frac{Y_{21}(p)}{Y_{22}(p) + R_1 \cdot \det \overleftrightarrow{Y}} = \frac{1}{A_{11}(p) + A_{21}(p) \cdot R_1}$$

2) A two-port fed by power supply source $I_0(p)$ and internal conductance $Y_1 = (1/R_1) = 0$ with load $R_2 = 0$.

$$H_{Lei}(p) = \frac{I_A(p)}{I_0(p)} = \frac{R_1 \cdot I_A(p)}{U_0(p)} = \frac{Z_{21}(p) \cdot R_1}{\det \overleftrightarrow{Z} + Z_{22}(p) \cdot R_1}$$

$$= -\frac{Y_{21}(p) \cdot R_1}{1 + Y_{11}(p) \cdot R_1} = \frac{R_1}{A_{21}(p) + A_{22}(p) \cdot R_1}$$

1.4 Network functions

The third operational case "two-port with output load"

(load resistance is finite and nonzero)

1. A two-port fed by voltage supply source $U_0(p)$ and internal resistance $R_1 = 0$ with two-port termination R_2 gives voltage effective function:

$$H_{Lau}(p) = \frac{U_2(p)}{U_0(p)} = \frac{R_2 \cdot I_A(p)}{U_0(p)} = \frac{Z_{21}(p)\cdot R_2}{\det \overleftrightarrow{Z} + Z_{11}(p)\cdot R_2}$$

$$= -\frac{Y_{21}(p)\cdot R_2}{1 + Y_{22}(p)\cdot R_2} = \frac{R_2}{A_{12}(p) + A_{11}(p)\cdot R_2} \qquad with \quad I_A(p) = -I_2(p)$$

2. A two-port fed by power supply source with $I_0(p)$ and internal conductance $Y_1 = (1/R_1) = 0$ and load R_2 gives current effective function:

$$H_{Lai}(p) = \frac{I_A(p)}{I_0(p)} = \frac{U_2(p)}{R_2 \cdot I_0(p)} = \frac{Z_{21}(p)}{Z_{22}(p) + R_2}$$

$$= -\frac{Y_{21}(p)}{Y_{11}(p) + R_2 \cdot \det \overleftrightarrow{Y}} = \frac{1}{A_{22}(p) + A_{21}(p)\cdot R_2}$$

1.4 Network functions

The fourth operational case "two-port with input and output load"

(characterized by finite resistance values) with $I_A(p) = -I_2(p)$

Voltage effective function:

$$H_{Lu}(p) = A_U = \frac{U_2(p)}{\frac{1}{2}\cdot U_0(p)} = \frac{2\cdot Z_{21}(p)}{\frac{\det \overleftrightarrow{Z}}{R_2} + Z_{11}(p) + \frac{R_2}{R_1}\cdot Z_{22}(p) + R_1}$$

$$= \frac{2}{A_{11}(p) + \frac{A_{12}(p)}{R_2} + R_1\cdot A_{21}(p) + \frac{R_1}{R_2}\cdot A_{22}(p)}$$

Current effective function:

$$H_{Li}(p) = A_I = \frac{I_A(p)}{\frac{1}{2}\cdot I_0(p)} = \frac{R_1}{R_2}\cdot H_{Lu}(p)$$

1.4 Network functions

Combination of corresponding current and voltage effective functions give the operation effective function:

$$H_{LB}(p) = A_B = \sqrt{H_{Lu}(p) \cdot H_{Li}(p)}$$

$$= \frac{2 \cdot Z_{21}(p)}{\frac{\det \overleftrightarrow{Z}}{\sqrt{R_1 \cdot R_2}} + \sqrt{\frac{R_2}{R_1}} \cdot Z_{11}(p) + \sqrt{\frac{R_1}{R_2}} \cdot Z_{22}(p) + \sqrt{R_1 \cdot R_2}}$$

$$= \frac{2}{\sqrt{\frac{R_2}{R_1}} \cdot A_{11}(p) + \frac{A_{11}(p)}{\sqrt{R_1 \cdot R_2}} + \sqrt{R_1 \cdot R_2} \cdot A_{21}(p) + \sqrt{\frac{R_1}{R_2}} \cdot A_{22}(p)}$$

The determination of the formula makes use of the the definition of A_B , the relations for voltages at input and output and the two-port cascade matrix equations.

1.4 Network functions

Each of these effective functions follows the general definition:

$$H_L(p) = \frac{L\{\text{ Zerostate Response Signal }\}}{L\{\text{ Input Signal }\}}$$

This is often also called “system function” or “system-driving function”.

An alternative representation is made using rational real fractions in p:

$$H_L(p) = \frac{P(p)}{Q(p)} = \frac{\sum_{m=0}^{M} a_m \cdot p^m}{\sum_{n=0}^{N} b_n \cdot p^n} = A \cdot \frac{\prod_{i=1}^{M}(p - p_{0\,i})}{\prod_{k=1}^{N}(p - p_{\infty\,k})} = |H_L(p)| \cdot e^{j\sphericalangle H_L(p)}$$

with

$p_{0i}\big|_{i=1..M}$ zeros of the numerator *P(p)*

$p_{\infty k}\big|_{k=1..N}$ zeros of the denominator *Q(p)*

1.4 Network functions

One can write

$$(p - p_{0\,i}) = \left| p - p_{0\,i} \right| \cdot e^{j\alpha_i(p)} \qquad \text{and} \qquad (p - p_{\infty\,k}) = \left| p - p_{\infty\,k} \right| \cdot e^{j\beta_k(p)}$$

so that

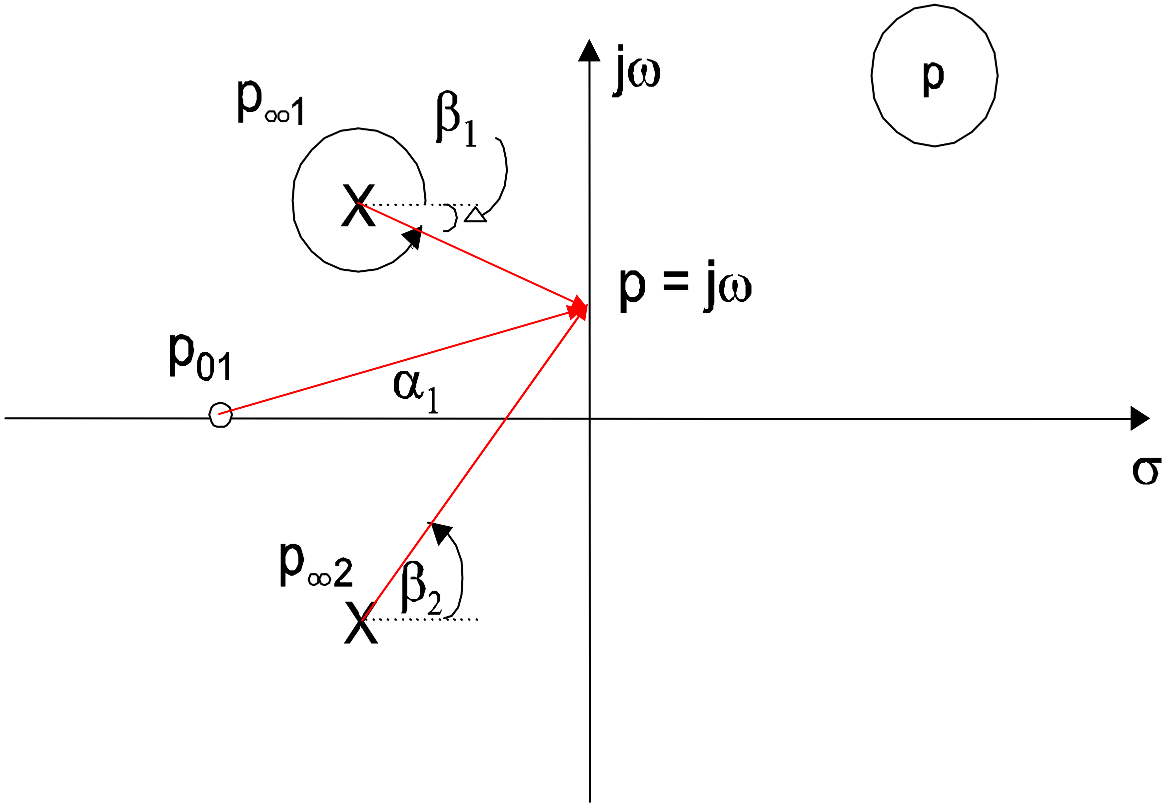
$$\left| H_L(p) \right| = \left| A \right| \cdot \frac{\left| p - p_{0\,1} \right| \cdot \left| p - p_{0\,2} \right| \cdot \cdots \cdot \left| p - p_{0\,M} \right|}{\left| p - p_{\infty\,1} \right| \cdot \left| p - p_{\infty\,2} \right| \cdot \cdots \cdot \left| p - p_{\infty\,N} \right|}$$

and

$$\begin{aligned} \varphi(p) &= \sphericalangle H_L(p) \\ &= \alpha_1(p) + \alpha_2(p) + \cdots + \alpha_M(p) \\ &\qquad - \beta_1(p) - \beta_2(p) - \cdots - \beta_N(p) \\ &\qquad + \gamma_A(p) \end{aligned}$$

with $\gamma_A(p) = \sphericalangle(A)$

1.4 Network functions



Example of a pole-zero diagram

1.4 Network functions

In practice, one often looks at the corresponding transfer function:

$$H_F(\omega) = H_L(p = j\omega)$$

Condition: Poles of $H_L(p)$ must lie in left half p plane ⟹ $\mathrm{Re}\{p_{\infty k}\} < 0$

In this case the following the equation holds:

$$H_F(\omega) = H_L(j\omega) = A \cdot \frac{(j\omega - p_{0\,1})\cdot(j\omega - p_{0\,2}) \cdot \cdots \cdot (j\omega - p_{0\,M})}{(j\omega - p_{\infty\,1})\cdot(j\omega - p_{\infty\,2}) \cdot \cdots \cdot (j\omega - p_{\infty\,N})}$$

⟹ Fourier transform gives the impulse response *h(t)* of the system!

$$H_F(\omega) = H_L(j\omega) \quad \overset{F}{\bullet\!\!-\!\!\circ} \quad h(t) \quad \text{with} \quad \mathrm{Re}\{p_{\infty k}\} < 0$$

Note: This also is possible from the system function using inverse Laplace transform!

1.4 Network functions

Stability of the two-port network

$$M < N \Rightarrow |\mathrm{H_L(p)}| \to 0 \quad for \quad |p| \to \infty$$

is a condition for subsequent considerations!

(with *M* being the order of the numerator, *N* being the order of the denominator)

The partial fraction method gives for N_ν-fold poles at location $p_{\infty\nu}$:

$$H_L(p) = \sum_\nu \left(\sum_{\mu=1}^{N_\nu} \frac{A_{\nu\mu}}{(p - p_{\infty\nu})^\mu} \right)$$

with the pole factors:

$$A_{\nu\,\mu} = \frac{1}{(N_\nu - \mu)!} \cdot \left[\frac{d^{(N_\nu-\mu)}}{dp^{(N_\nu-\mu)}} \left\{ H_L(p) \cdot (p - p_{\infty\,\nu})^{N_\nu} \right\} \right]_{p=p_{\infty\,\nu}}$$

1.4 Network functions

With the correspondence

$$\frac{1}{(p-p_{\infty\nu})^{\mu}} \;\bullet\!\!-\!\!\circ^{L}\; \begin{cases} \dfrac{t^{\mu-1}}{(\mu-1)!}\cdot e^{p_{\infty\nu}\cdot t} & \text{für } t>0 \\ 0 & \text{sonst} \end{cases}$$

the impulse response *h(t)* thus can be determined as:

$$H_L(p) \;\bullet\!\!-\!\!\circ^{L}\; h(t) = \begin{cases} \sum\limits_{\nu}\left(\sum\limits_{\mu=1}^{N} A_{\nu\mu}\cdot\dfrac{t^{\mu-1}}{(\mu-1)!}\cdot e^{p_{\infty\nu}\cdot t}\right) & \text{für } t>0 \\ 0 & \text{sonst} \end{cases}$$

$$\Rightarrow \quad h_{\nu\mu}(t) = A_{\nu\mu}\cdot\frac{t^{\mu-1}}{(\mu-1)!}\cdot e^{p_{\infty\nu}\cdot t} = A_{\nu\mu}\cdot\frac{t^{\mu-1}}{(\mu-1)!}\cdot e^{\sigma_{\infty\nu}\cdot t}\cdot e^{j\omega_{\infty\nu}\cdot t}$$

1.4 Network functions

Thus the impulse response h(t) will satisfy the following condition

$$|h(t)| < W < \infty \quad \text{for all } t > 0$$ (W is a finite value)

only under the condition that:

$$\operatorname{Re}\{p_{\infty\nu}\} = \sigma_{\infty\nu} < 0 \quad \text{for} \quad N_\nu \leq 1 \qquad \text{and}$$

$$\max.\operatorname{Re}\{p_{\infty\nu}\} = \max.\sigma_{\infty\nu} = 0 \quad \text{for} \quad N_\nu = 1^{1)}$$

Amplitude delimitation of h(t) corresponds
to the stability definition for two-port systems.

1) In this case a special stability situation is given characterized by constant oszillations!

1.4 Network functions

Stability of the system with the HURWITZ criterion:

Hurwitz – rational polynomial:

- All coefficients are real numbers
- All zeros are located in the left half of the complex *p* plane

The stability criterion of a two-port system thus can be expressed:

A two-port with the characteristic function or system function $H_L(p) = P(p)/Q(p)$ is stable if the denominator polynomial Q(p) is a **Hurwitz - polynomial**.

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1.4 Network functions

Modified Hurwitz polynomial:

- All coefficients are real numbers
- There may be simple or multiple zeros in the left half p-plane
- There may be simple poles on the $j\omega$-axis but no multiple ones!

A two-port network with the system function $H_L(p) = P(p)/Q(p)$ is stable if the denominator polynomial Q(p) is a **modified Hurwitz - polynomial**.

Chapter 1

Introduction

1.5 Two-terminal networks with special effective functions (System functions)

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1.5.1 The all-pass network

The all-pass behaviour is defined by the system function:

$$H_L(p) = A \cdot \frac{Q(-p)}{Q(p)}$$ with A being real and $Q(p)$ being a Hurwitz - polynomial

Then in addition to conjugated complex poles and zeros it holds:

$p_{0i} = -p_{\infty k}$ for corresponding i and k and the system function will be:

$$H_L(p) = A \cdot \frac{Q(-p)}{Q(p)} = A \cdot \frac{b_0 - b_1 \cdot p + b_2 \cdot p^2 - \cdots + b_N \cdot p^N}{b_0 + b_1 \cdot p + b_2 \cdot p^2 + \cdots + b_N \cdot p^N}$$

$$= A' \cdot \frac{(p + p_{\infty 1}) \cdot (p + p_{\infty 2}) \cdot \cdots \cdot (p + p_{\infty N})}{(p - p_{\infty 1}) \cdot (p - p_{\infty 2}) \cdot \cdots \cdot (p - p_{\infty N})}$$

gives square-symmetrical pole zero configuration typical for an all-pass network

1.5.1 The all-pass network

For $p = j\omega$ the corresponding transfer function gives:

$$\left|H_L(j\omega)\right| = A' \cdot \frac{\left|(j\omega + p_{\infty 1})\right| \cdot \left|(j\omega + p_{\infty 2})\right| \cdot \ \cdots \ \cdot \left|(j\omega + p_{\infty N})\right|}{\left|(j\omega - p_{\infty 1})\right| \cdot \left|(j\omega - p_{\infty 2})\right| \cdot \ \cdots \ \cdot \left|(j\omega - p_{\infty N})\right|} = A'$$

Constant

with $H_L(j\omega) = \left|H_F(\omega)\right| \cdot e^{j \cdot \sphericalangle H_F(\omega)}$ and for complex conjugated poles:

$$\left|j\omega + p_{\infty 2}\right| = \left|j\omega + \sigma_{\infty 1} - j\omega_{\infty 1}\right| \text{ and } \left|j\omega - p_{\infty 2}\right| = \left|j\omega - \sigma_{\infty 1} + j\omega_{\infty 1}\right|$$

so that here $p_{\infty 1,2}$ has the same distance to $j\omega$ as $p_{01,2} = -p^*_{\infty 1,2}$!

A pole with the index k contributes with:

$$-\sphericalangle(j\omega - p_{\infty k}) = -\sphericalangle(j\omega - \sigma_{\infty k} - j\omega_{\infty k}) = -\arctan\left(\frac{\omega - \omega_{\infty k}}{-\sigma_{\infty k}}\right) = \arctan\left(\frac{\omega - \omega_{\infty k}}{\sigma_{\infty k}}\right)$$

A corresponding zero contributes with:

$$\sphericalangle(j\omega - p_{ok}) = \sphericalangle(j\omega + \sigma_{\infty k} - j\omega_{\infty k}) = \arctan\left(\frac{\omega - \omega_{\infty k}}{\sigma_{\infty k}}\right)$$

In total it results:

$$\sphericalangle H_F(\omega) = 2 \cdot \sum_{k=1}^{N} \arctan\left(\frac{\omega - \omega_{\infty k}}{\sigma_{\infty k}}\right) \quad , \quad \sigma_{\infty k} < 0$$

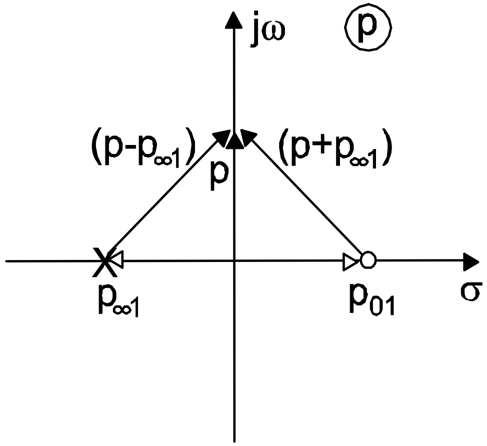
1.5.1 The all-pass network

The corresponding all-pass group delay gives:

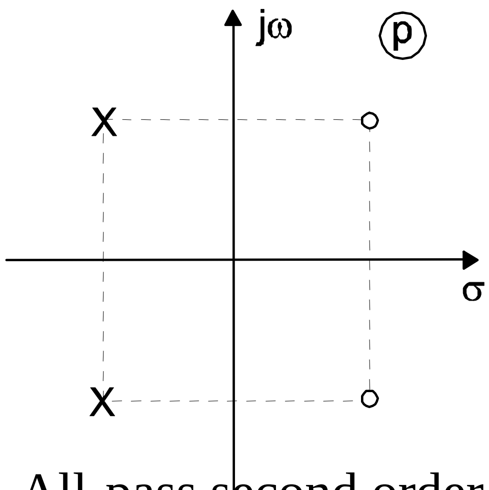
$$\tau_g(\omega) = \frac{db(\omega)}{d(\omega)} = -2\sum_{k=1}^{N} \frac{1}{1+\left(\frac{\omega-\omega_{\infty k}}{\sigma_{\infty k}}\right)^2} \qquad \text{with } b(\omega) = -\sphericalangle H_F(\omega)$$

$$= -2\cdot\sum_{k=1}^{N} \arctan\left(\frac{\sigma_{\infty k}{}^2}{\sigma_{\infty k}^2 + (\omega-\omega_{\infty k})^2}\right)$$

Examples of quadrant-symmetrical pole zero configuration for all-passes:



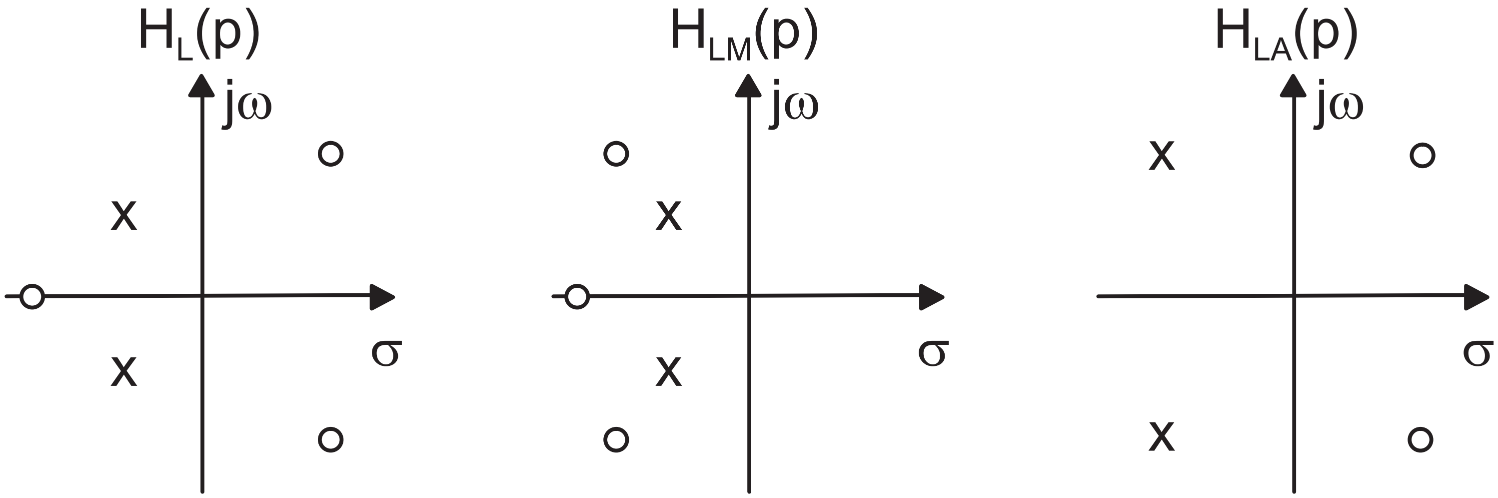
All-pass first order network



All-pass second order network

1.5.1 The all-pass network

Minimum phase two-port network:



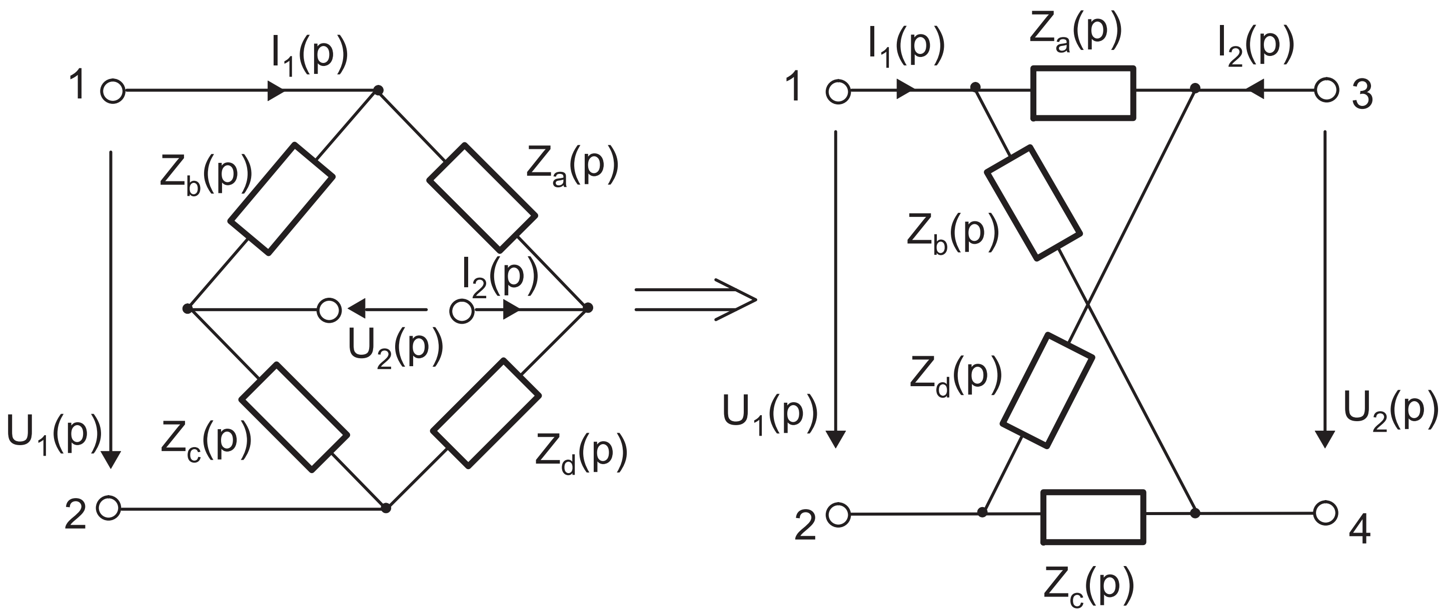
Two-port contains all-pass | Minimum phase system | Pure all-pass

$$H_L(p) = H_{LM} \cdot H_{LA}$$

Two-ports containing all-passes can be divided into a minimum phase two-port and an all-pass two-port network.

1.5.2 Bridge circuits

A special two-port network with interesting properties is represented by:



Bridge connection representation

Cross connection representation

Bridge network advantages: Easy synthesis relations (under conditions)

1.5.2 Bridge circuits

The Z-matrix of the asymmetrical bridge two-port of previous figure:

$$\overleftrightarrow{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}$$

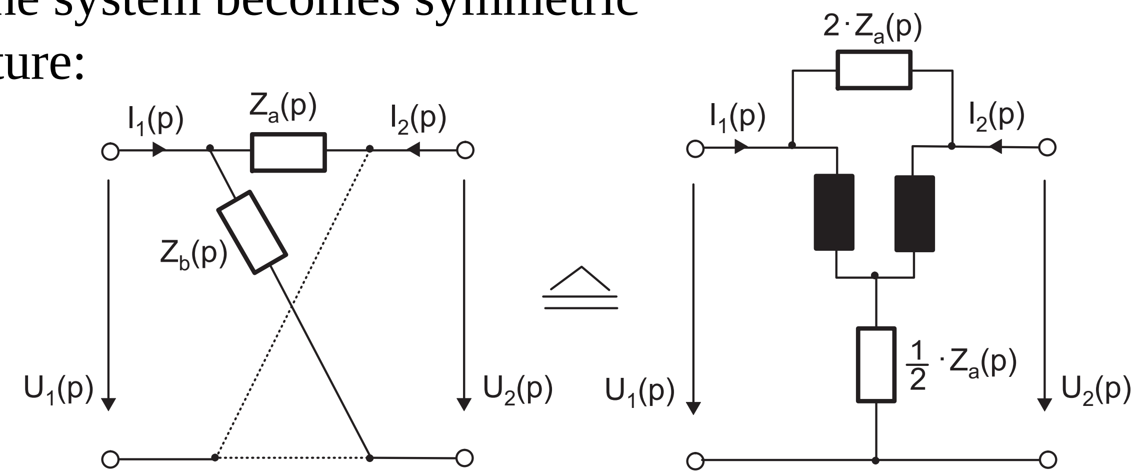
$$= \frac{1}{Z_a + Z_b + Z_c + Z_d} \cdot \begin{bmatrix} (Z_a + Z_d)(Z_b + Z_c) & Z_b Z_d - Z_a Z_c \\ Z_b Z_d - Z_a Z_c & (Z_a + Z_b)(Z_c + Z_d) \end{bmatrix}$$

Under the condition of $Z_a = Z_c$ and $Z_b = Z_d$ it results:

$$\Rightarrow \quad \overleftrightarrow{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(Z_a + Z_b) & \frac{1}{2}(Z_b - Z_a) \\ \frac{1}{2}(Z_b - Z_a) & \frac{1}{2}(Z_b + Z_a) \end{bmatrix}$$

1.5.2 Bridge circuits

…and the system becomes symmetric in structure:



A symmetrical bridge → Its equivalent circuit with an ideal transformer

Two conditions sufficiently and necessary for the symmetry of a two-port (exchange of the ports has no effect) <u>are fulfilled by this circuit</u>:

$$Z_{12} = Z_{21}$$

$$Z_{11} = Z_{22}$$

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1.5.2 Bridge circuits

The matrix $\overleftrightarrow{Y}$ of this symmetrical bridge two-port applies with

$$Y_a = \frac{1}{Z_a} \qquad \text{and} \qquad Y_b = \frac{1}{Z_b}$$

gives the result:

$$\overleftrightarrow{Y} = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(Y_a + Y_b) & \frac{1}{2}(Y_b - Y_a) \\ \frac{1}{2}(Y_b - Y_a) & \frac{1}{2}(Y_b + Y_a) \end{bmatrix}$$

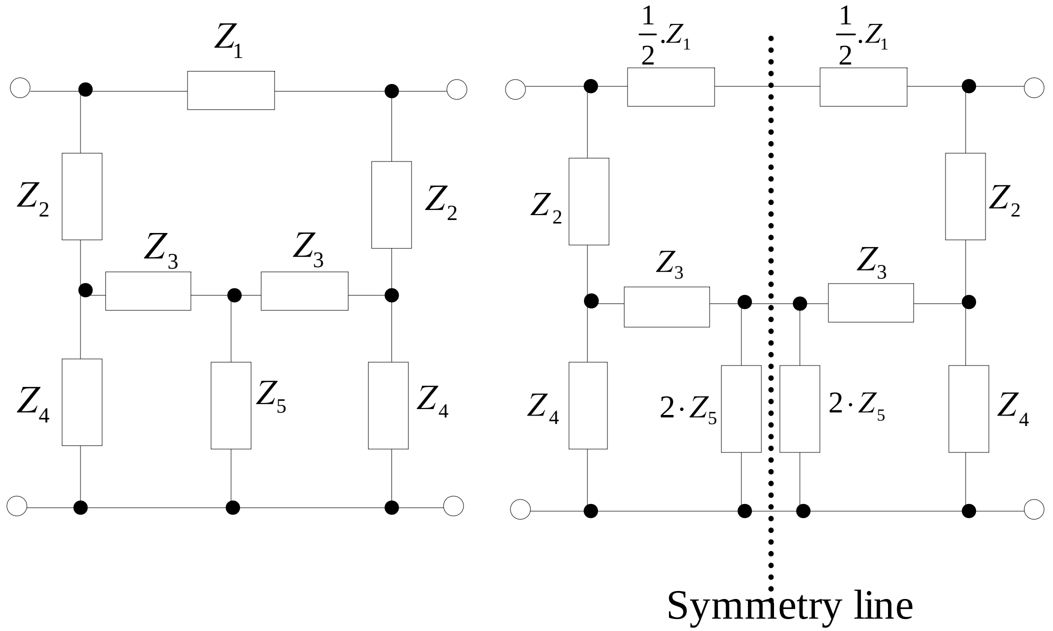
<u>Longitudinal branch impedance</u> Z_a and <u>transverse branch impedance</u> Z_b

are always positive real functions of p for a passive

symmetrical four-pole network!

1.5.2 Bridge circuits

Condition of a two-port network with symmetrical structure:



The properties of the two-port remains unchanged when exchanging the ports.

Symmetric two-ports can be transformed into equivalent circuits using the symmetry rule of Bartlett.

1.5.2 Bridge circuits

Due to S. 73 a special operation condition of a two-port with the same currents at both ports can be used to determine the longitudinal impedance:

$I_1 = I_2$ gives $U_1 = U_2$ due to symmetrical two-port and in addition to:

$$U_1 = Z_{11} I_1 + Z_{12} I_1 = Z_b I_1 \Rightarrow Z_b = \frac{U_1}{I_1}$$

Due to the symmetrical two-port all node potentials φ are equal at corresponding nodes of both halfes of the two-port. Thus no current flows through branches crossing the symmetry line. Consequently, these branches could be extracted from the network without changes to currents or voltages.

Considering the two-port with extracted branches gives an easier overview and corresponds to an open circuit operation of seperated halfes of the original network.

So Z_b can also be determined by looking at one separated half of the two-port in open circuit operation.

1.5.2 Bridge circuits

Again due to S. 73 a second special operation of a two-port with the opposite current values at port 1 and port 2 can be used to determine the needed transverse impedance Z_a:

$I_1 = -I_2$ gives $U_1 = -U_2$ due to symmetrical two-port and in addition to:

$$U_1 = Z_{11}I_1 - Z_{12}I_1 = Z_a I_1 \Rightarrow Z_a = \frac{U_1}{I_1}$$

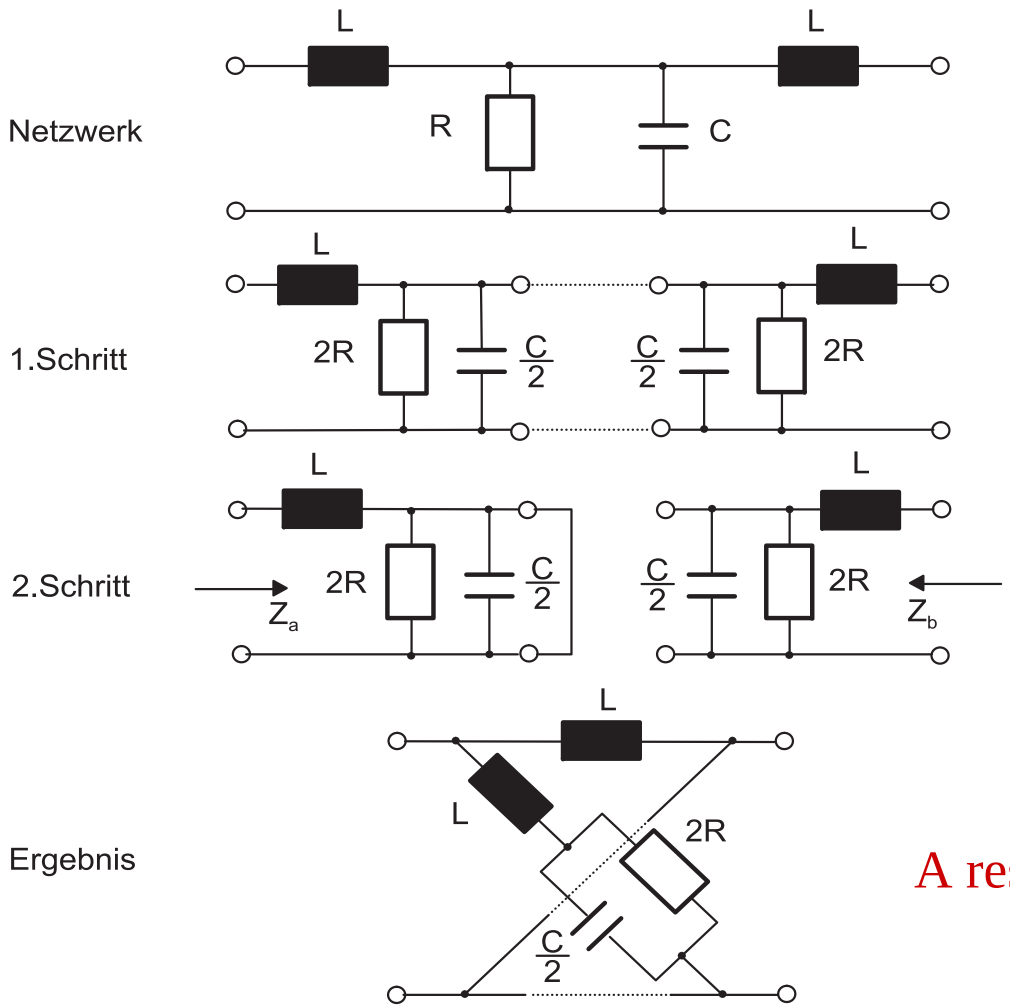
Due to the symmetrical two-port and considering superposition of the cases when only the first and then only the second source is attached, all node potentials φ at the branches crossing the symmetry line now have the same value. Thus all of theses points can be connected to each other without changes to currents or voltages.

Consequently, the two halfes of the symmetrical network can be separated without changing currents or voltages. This corresponds to a short circuit operation of the seperated halves of the original network.

Considering the short circuited separated half of the two-port gives an easier overview and thus Z_a can be determined by looking at this simplified network.

1.5.2 Bridge circuits

An example of the BARTLETT' symmetry rule:



A given two-port with symmetrical structure

↓

Symmetrical allocation of the components

↓

Determination of the longitudinal and transverse branch impedance

↓

A resulting symmetrical bridge two-port

1.5.2 Bridge circuits

Note:

- Now a network with loads is considered for determination of relations with the operation effective function
- Method: Insertion of impedance matrix elements for a bridge network into formula for $H_{LB}(p)$
- Details are shown in the next slide.

1.5.2 Bridge circuits

$$H_{LB}(p) = \sqrt{H_{Lu}(p)\ \cdot\ H_{Li}(p)}\ = \frac{2\cdot Z_{21}}{\dfrac{\det \overleftrightarrow{Z}}{\sqrt{R_1\cdot R_2}} + \sqrt{\dfrac{R_2}{R_1}}\cdot Z_{11} + \sqrt{\dfrac{R_1}{R_2}}\cdot Z_{22} + \sqrt{R_1\cdot R_2}}$$

$$= \frac{\sqrt{R_1\cdot R_2}\cdot 2\cdot Z_{21}}{\det \overleftrightarrow{Z} + R_2\cdot Z_{11} + R_1\cdot Z_{22} + R_1\cdot R_2} \quad \text{(due to S.57)}$$

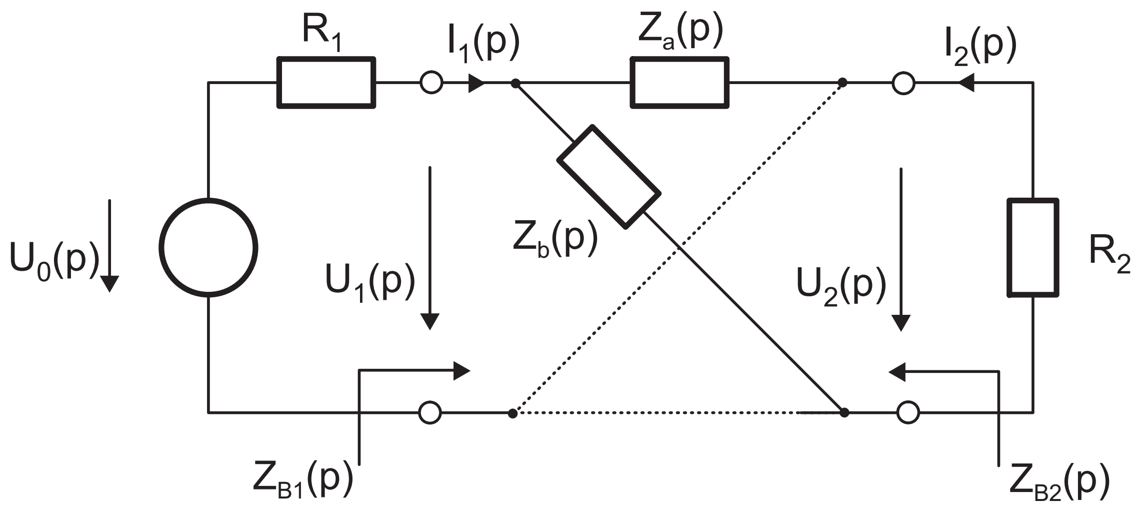
with $\overleftrightarrow{Z} = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(Z_a+Z_b) & \frac{1}{2}(Z_b-Z_a) \\ \frac{1}{2}(Z_b-Z_a) & \frac{1}{2}(Z_b+Z_a) \end{bmatrix}$

and $\det \overleftrightarrow{Z} = \frac{1}{4}((Z_a+Z_b)^2 - (Z_b-Z_a)^2 = \frac{1}{4}4Z_aZ_b$

$$\Rightarrow H_{LB}(p) = \frac{\sqrt{R_1\cdot R_2}\cdot\left(Z_b(p)-Z_a(p)\right)}{Z_a(p)\cdot Z_b(p) + \frac{1}{2}(R_1+R_2)\cdot\left(Z_a(p)+Z_b(p)\right) + R_1\cdot R_2}$$

1.5.2 Bridge circuits

Reciprocally wired symmetrical bridge two-port



and the condition

So for this circuit the effective transmission factor results as shown in last slide:

$$H_{LB}(p) = \frac{\sqrt{R_1 \cdot R_2} \cdot \left(Z_b(p) - Z_a(p)\right)}{Z_a(p) \cdot Z_b(p) \; + \; \frac{1}{2}(R_1 + R_2) \cdot \left(Z_a(p) + Z_b(p)\right) \; + \; R_1 \cdot R_2}$$

1.5.2 Bridge circuits

After solving for $Z_b(p)$ an easily usable relation results (for a given operation effective function and given loads):

$$Z_b(p) = Z_a(p) \cdot \frac{1+\left\{\dfrac{\sqrt{R_1 \cdot R_2}}{Z_a(p)} + \dfrac{R_1+R_2}{2\cdot\sqrt{R_1 \cdot R_2}}\right\}\cdot H_{LB}(p)}{1-\left\{\dfrac{Z_a(p)}{\sqrt{R_1 \cdot R_2}} + \dfrac{R_1+R_2}{2\cdot\sqrt{R_1 \cdot R_2}}\right\}\cdot H_{LB}(p)}$$

The operation impedances (input and output impedances) can be determined as:

For the input:

$$Z_{B1}(p) = \frac{U_1(p)}{I_1(p)} = \frac{2\cdot Z_a(p)\cdot Z_b(p) \ + \ R_2\cdot\{Z_a(p)+Z_b(p)\}}{2\cdot R_2 \ + \ Z_a(p) \ + \ Z_b(p)}$$

For the output:

$$Z_{B2}(p) = \left.\frac{U_2(p)}{I_2(p)}\right|_{U_0(p)=0} = \frac{2\cdot Z_a(p)\cdot Z_b(p) \ + \ R_1\cdot\{Z_a(p)+Z_b(p)\}}{2\cdot R_1 \ + \ Z_a(p) \ + \ Z_b(p)}$$

1.5.2 Bridge circuits

A further simplification results with the condition of $R_1 = R_2$

In this often wanted case it holds:

$$H_{LB}(p) = \frac{R_1 \cdot \left(Z_b(p) - Z_a(p)\right)}{R_1^2 + R_1(Z_a(p) + Z_b(p)) + Z_a(p) Z_b(p)} = \frac{R_1 \cdot \left(Z_b(p) - Z_a(p)\right)}{\left(R_1 + Z_a(p)\right) \cdot \left(R_1 + Z_b(p)\right)}$$

$$Z_b(p) = Z_a(p) \cdot \frac{1 + \left\{\dfrac{R_1}{Z_a(p)} + 1\right\} \cdot H_{LB}(p)}{1 - \left\{\dfrac{Z_a(p)}{R_1} + 1\right\} \cdot H_{LB}(p)}$$

$$Z_{B1}(p) = Z_{B2}(p) = \frac{2 \cdot Z_a(p) \cdot Z_b(p) \ + \ R_1 \cdot \left\{Z_a(p) + Z_b(p)\right\}}{2 \cdot R_1 \ + \ Z_a(p) \ + \ Z_b(p)}$$

with $Z_{B1,2}(p) = \dfrac{U_{1,2}(p)}{I_{1,2}(p)}$

1.5.2 Bridge circuits

Another simplification uses the additional condition $Z_a(p) \cdot Z_b(p) = R_1^2$ and gives:

$$H_{LB}(p) = \frac{R_1 - Z_a(p)}{R_1 + Z_a(p)} \quad \text{or} \quad H_{LB}(p) = \frac{Z_b(p) - R_1}{Z_b(p) + R_1}$$

⇕ dissolving ⇕

$$Z_a(p) = R_1 \cdot \frac{1 - H_{LB}(p)}{1 + H_{LB}(p)} \qquad Z_b(p) = R_1 \cdot \frac{1 + H_{LB}(p)}{1 - H_{LB}(p)}$$

$Z_b(p)$ as well as $Z_a(p)$ depend only on the internal resistance and the Operation effective function!

1.5.2 Bridge circuits

To observe:

Not all effective transmission factors lead to impedances realizable just by passive RLCÜ elements!

The conditions of realizing rational two-poles just with passive RLCÜ elements are:

$$\mathrm{Im}\{Z(p)\} = 0 \quad \text{for all } p \text{ with } \; p = \sigma \text{ (DC-case due to } \omega = 0)$$

$$\mathrm{Re}\{Z(p)\} \geq 0 \quad \text{for all } p \text{ with } \; \mathrm{Re}\{p\} = \sigma \geq 0$$

The last equation means that the two-pole cannot deliver energy (true for all passive two-poles)! (see also S.11 of chapter 2)

These conditions are now considered with respect to the two impedances of the symmetric bridge circuit.

1.5.2 Bridge circuits

Conditions for realizing the bridge circuit with the two impedances

$$Z_a(p) = R_1 \cdot \frac{1 - H_{LB}(p)}{1 + H_{LB}(p)} \quad \text{and} \quad Z_b(p) = R_1 \cdot \frac{1 + H_{LB}(p)}{1 - H_{LB}(p)}$$

The operation effective function is now split up into real and imaginary parts:

$$H_{LB}(p) = H_r(p) + jH_i(p)$$

This leads to:

$$Z_a(p) = R_1 \cdot \frac{1 - H_r(p) - jH_i(p)}{1 + H_r(p) + jH_i(p)} \qquad Z_b(p) = R_1 \cdot \frac{1 - H_r(p) - jH_i(p)}{1 + H_r(p) + jH_i(p)}$$

Positive real parts of these impedances are given for $\left|H_{LB}(p)\right| \leq 1$ for all $\mathrm{Re}\{p\} \geq 0$ (without proof)

This means just no amplification of currents/voltages!

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1.5.2 Bridge circuits

It can also be shown that positive real parts of the bridge impedances relies on the additional conditions:

a) The operation effective function has no poles in closed positive p plane

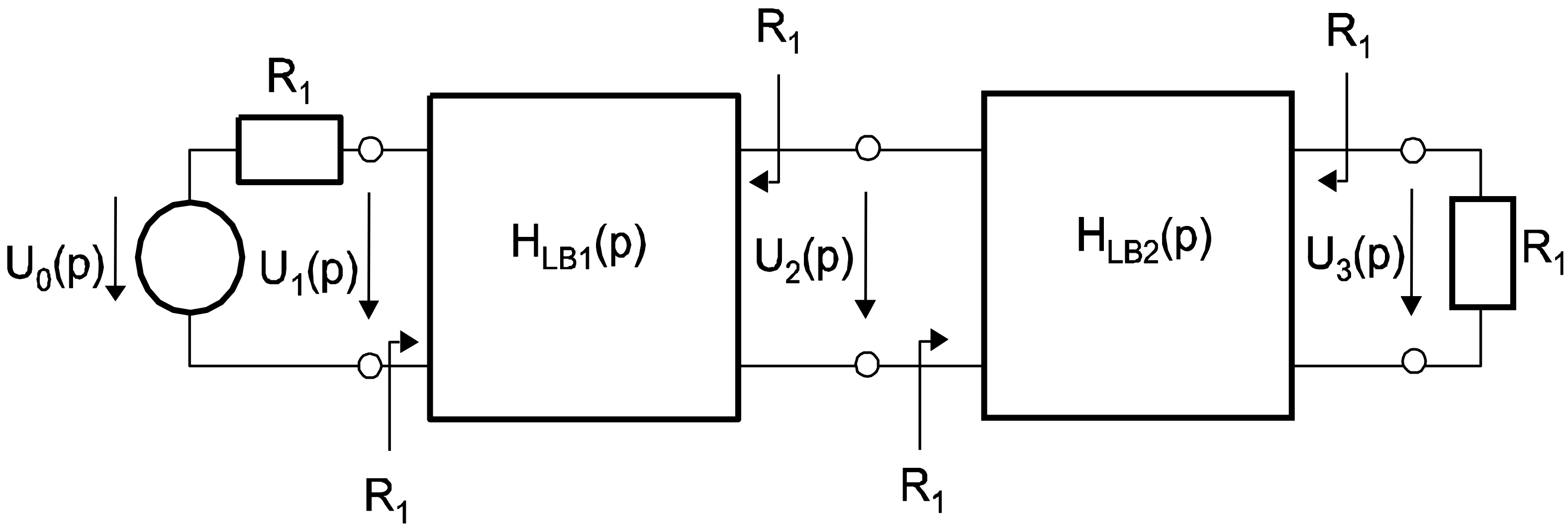
b) It holds $Z_{B1}(p) = Z_{B2}(p) = R_1$

The last relation is also a condition for being able to set up a chain of bridge circuits with in some sense "effectless" or "non-reactive" connections (i.e. no reflections, no effect of a second network on $H_{LB}(p)$ of a first network).

Or: The load of one element in the chain thus has no negative effect on the transmission properties of the previous chain elements.

Reason:
The loads of all inputs and outputs are equal to: $R_1 = R_2$

1.5.2 Bridge circuits



Effectless chain network of two symmetrical bridge two-ports with dual branch impedances $Z_b(p)$ and $Z_a(p)$

For Z symmetric bridge circuits (connected in a chain) it holds:

$$H_{LB}(p) = \frac{U_{Z+1}(p)}{\frac{1}{2} \cdot U_0(p)} = \prod_{v=1}^{Z} H_{LBv}(p)$$

1.5.2 Bridge circuits

Thus a given operation effective function can be realized using:

- A number of Z chained symmetric bridge circuits
- Symmetrical loads
- Identical operation input and output impedances
- For simplicity: Chain elements of the order of 2 (in addition to one with the order of 1)

Chapter 1

Introduction

1.6 Normalization Procedures

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1.6.1 Normalized representation of network functions

Normalization procedures help to keep the overview in a big network where usally it is hard to compare values.

Normalisations are transforms of variables using reference constants. They can be applied to frequencies and component values.

Typical normalization procedures applied to a network function *N(p)* might be:

- An impedance-function *Z(p)* is normalized
- Its argument, the complex frequency is normalized $p = \sigma + j\omega$
- An operation effective function is normalized

1.6.1 Normalized representation of network functions

The standardizing auxiliary variable used during the frequency normalization is the normalized complex frequency:

$$\rho = \frac{p}{\omega_N} = \frac{\sigma}{\omega_N} + j \cdot \frac{\omega}{\omega_N} = \varsigma + j \cdot \Omega = P$$

$$N(p) \xrightarrow{\text{Frequency normalization}} N_{nf}(P)$$

$$N_{nf}(P) = N(P \cdot \omega_N)$$

or

$$N(p) = N_{nf}\left(\frac{p}{\omega_N}\right)$$

1.6.1 Normalized representation of network functions

Example of a frequency normalization:

Network function *N(p)* is given in the form of *U(p)* , a voltage at a series connection of a resistance *R*, an inductance *L* and a capacity *C*.
I(p) is the current of the series connection.

$$U(p) = \left(R + pL + \frac{1}{pC} \right) \cdot I(p)$$

Frequency normalisation extends *p* as follows:

$$U(p) = \left(R + \frac{p}{\omega_N} \cdot \omega_N \cdot L + \frac{1}{\frac{p}{\omega_N} \cdot \omega_N \cdot C} \right) \cdot I\left(\frac{p}{\omega_N} \cdot \omega_N\right)$$

1.6.1 Normalized representation of network functions

Thus:
$$U_{nf}(P) = \left(R + P \cdot \omega_N \cdot L + \frac{1}{P \cdot \omega_N \cdot C} \right) \cdot I_{nf}(P)$$

and
$$I\left(\frac{p}{\omega_N} \cdot \omega_N\right) = I(P \cdot \omega_N) = I_{nf}(P)$$

Example of the resistance normalization:

This normalisation is applied to all impedance functions of the network function:

$$U(p) = \left(R + pL + \frac{1}{pC} \right) \cdot I(p) = Z(p) \cdot I(p)$$

Impedance functions

1.6.1 Normalized representation of network functions

Thus the extension with R_N starts with:

$$U(p) = \left(\frac{R}{R_N} \cdot R_N + \frac{pL}{R_N} \cdot R_N + \frac{R_N}{p \cdot C \cdot R_N} \right) \cdot I(p)$$

After extraction of the normalization resistance R_N from the parenthesis it results:

$$U(p) = R_N \cdot Z_{nw}(p) \cdot I(p)$$

with $$Z_{nw} = \frac{Z(p)}{R_N} = \left(\frac{R}{R_N} + \frac{pL}{R_N} + \frac{1}{p \cdot C \cdot R_N} \right) \cdot I(p)$$

→ unitless (normalized) impedance

1.6.1 Normalized representation of network functions

Use of auxiliary variables in network normalization

Such variables can be applied both to normalized voltages/currents or for Laplace transforms.

Example:

$$\frac{U(p)}{U_N} = U_{nu}(p) = Z(p)\cdot\frac{I(p)}{U_N}$$

It is possible to normalize all other elements in formulas describing network functions/elements by applying such a normalization individually!

This is called a complete normalization.

1.6.1 Normalized representation of network functions

An example of the complete normalization:

$$\frac{U(p)}{U_N}\cdot U_N = \left(\frac{R}{R_N}\cdot R_N + \frac{p\cdot\omega_N\cdot L\cdot R_N}{\omega_N\cdot R_N} + \frac{R_N}{\frac{p\cdot\omega_N\cdot R_N\cdot C}{\omega_N}} \right)\cdot I(\frac{p}{\omega_N}\cdot\omega_N)$$

From this arises:

1. $U_{nu}(p) = \frac{U(p)}{U_N}$ Voltage normalized concerning its value

2. $r = \frac{R}{R_N} = RG_N$ Normalized resistance

1.6.1 Normalized representation of network functions

3. $l = \frac{\omega_N \cdot L}{R_N}$ Normalized inductance

4. $c = \omega_N \cdot R_N \cdot C$ Normalized capacity

5. $P = \frac{p}{\omega_N}$ Normalized complex frequency

6. $I_{nf}(P) = I\left(\frac{p}{\omega_N} \cdot \omega_N\right) = I(P \cdot \omega_N)$ Frequency normalized current

1.6.1 Normalized representation of network functions

The equation

$$U_{nu}(p) = \frac{R_N}{U_N} \cdot \left(r + Pl + \frac{1}{Pc} \right) \cdot I_{nf}(P) = U_{nuf}(P)$$

using $$I_N = \frac{U_N}{R_N}$$

leads to: $$U_{nuf}(P) = \left(r + Pl + \frac{1}{Pc} \right) \cdot \frac{I_{nf}(P)}{I_N}$$

Now $I_{nf}(P)$ is substituted: $$I_{nfi}(P) = \frac{I_{nf}(P)}{I_N}$$

1.6.1 Normalized representation of network functions

Then the completely normalized network function results:

$$U_{nuf}(P) = \left(r + Pl + \frac{1}{Pc} \right) \cdot I_{nfi}(P)$$

1.6.1 Normalized representation of network functions

Resistances R and conductances G appear in a network function after a complete standardization of *N(p)* in the diagrams as pure numerical values.

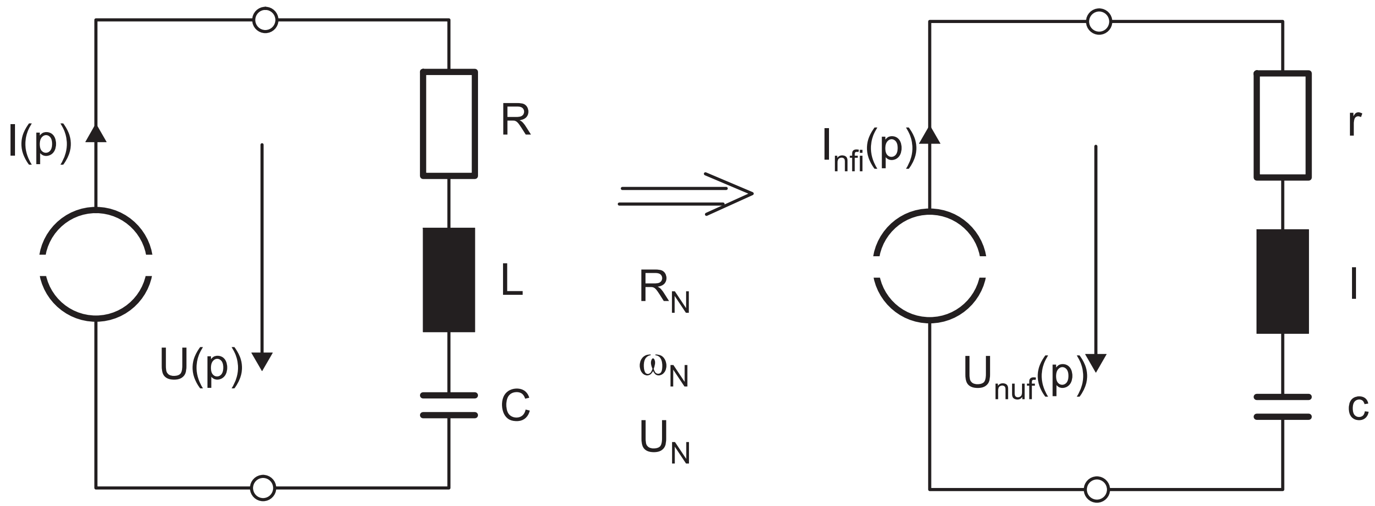

Diagram and designation of the sizes before (left) and after the normalization (right) of the network function

1.6.2 Normalization in the time and frequency domain

The description of signals in the time domain has two common representation types:

1. The representation with units
2. The completely normalized representation

The mathematical description of a signal *x(t)* in the time domain looks like this:

$$x(t) = A_0 \cdot s(t)$$

A_0 is a constant with a certain unit, for example Volt if *x(t)* is a voltage signal

s(t) is a unitless function and

t is its argument with the unit s .

1.6.2 Normalization in the time and frequency domain

The transform of the signal representation with units into the completely standardized representation of the signal happens with the help of two standardization operations.

The first normalising operation leads to the dimensionless signal:

$$x_n(t) = \frac{x(t)}{A_N} = \frac{A_0}{A_N} \cdot s(t)$$

The second normalising operation leads to dimensionless time:

$$\tau = \frac{t}{T_N}$$

1.6.2 Normalization in the time and frequency domain

Then one obtains: $$x_n(\tau \cdot T_N) = \frac{x(\tau \cdot T_N)}{A_N} = \frac{A_0}{A_N} \cdot s(\tau \cdot T_N)$$

By means of the relation $x_n(\tau \cdot T_N) = s_n(\tau)$ with $s_n(\tau) = \frac{A_0}{A_n} s(\tau)$

a completely normalized signal results.

Application

Under the condition of $s_n(\tau) = 0 \quad , \quad \forall \tau < 0$

the Laplace transform can be applied to a normalized signal as follows:

$$S_{L\,n}(P) = \int_0^\infty s_n(\tau) \cdot e^{-P \cdot \tau} d\tau$$

1.6.2 Normalization in the time and frequency domain

The inverse Laplace transformation is defined as:

$$s_n(\tau) = \lim_{\Omega_0 \to \infty} \left\{ \frac{1}{2\pi \cdot j} \int_{\xi - j\Omega_0}^{\xi + j\Omega_0} S_{L\,n}(P) \cdot e^{j\,P \cdot \tau} dP \right\}$$

In English literature usually instead of *P* the dimensionless variable *s* is used.

1.6.2 Normalization in the time and frequency domain

Using normalized variables the usual Laplace transform can be written as follows:

$$S_L(p) = \int_{\tau=0}^{\infty} s(\tau \cdot T_N) \cdot e^{-p \cdot \tau \cdot T_N} d(\tau \cdot T_N)$$

The relations $s_n(\tau) = s(\tau \cdot T_N)$ $\quad T_N = \dfrac{1}{\omega_N}$ $\quad d(\tau T_N) = T_N \cdot d\tau$ give:

$$S_L(p) = T_N \cdot \int_{\tau=0}^{\infty} s_n(\tau) \cdot e^{-\frac{p}{\omega_N} \cdot \tau} d\tau$$

Thus it follows:

$$S_L(p) = T_N \cdot \int_{\tau=0}^{\infty} s_n(\tau) \cdot e^{-P \cdot \tau} d\tau \overset{!}{=} T_N \cdot S_{Ln}(P)$$

1.6.2 Normalization in the time and frequency domain

Thus one obtains: $S_{Ln}(P) = \int\limits_{\tau=0}^{\infty} s_n(\tau) \cdot e^{-P \cdot \tau} d\tau$

In a similar way by setting $p = j\omega$ or $P = j\Omega$ the normalised version of the Fourier transform is obtained:

$$S_L(p) \xrightarrow{p = j\omega} S_F(\omega)$$

$$S_{Ln}(P) \longrightarrow S_{Fn}(\Omega)$$

1.6.2 Normalization in the time and frequency domain

Thus it follows under certain restrictions:

$$S_F(\omega) = S_L(j\omega) \longrightarrow S_{Fn}(\Omega) = S_{Ln}(j\Omega)$$

Here the following relationships hold:

$$S_F(\omega) = T_N \cdot S_{Fn}(\Omega) \quad \text{with} \qquad S_{Fn}(\Omega) = \int_{-\infty}^{+\infty} s_n(\tau) \cdot e^{-j\Omega\tau} d\tau$$

and of course:

$$S_F(\omega) = \int_{-\infty}^{+\infty} s(t) \cdot e^{-j\omega t} dt$$

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