Network Theory 1
Analoge Netzwerke

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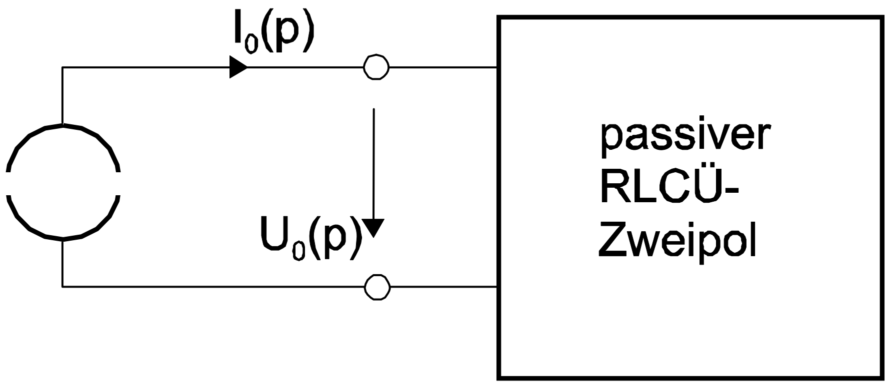
Chapter 2

Characteristics and realization of passive two-terminal network function

2.1 Passive two-terminal network functions and their properties

2.1.1 The passive RLC two-terminal network function

A passive RLCÜ two-terminal network is stimulated with the current $i_0(t)$:



$$u_0(t) \circ\!\!-\!\!\bullet\; U_0(p)$$

$$i_0(t) \circ\!\!-\!\!\bullet\; I_0(p)$$

2.1.1 The passive RLC two-terminal network function

Relationship of a number of w Ohm resistances and the currents:

$$U_w(p) = R_w \cdot I_w(p) \quad \text{with} \quad R_w > 0 \quad \text{and} \quad w = 1 \quad [1] \quad A$$

Relationship of i inductance and the currents:

$$U_i(p) = pL_i \cdot I_i(p) \quad \text{with} \quad L_i > 0 \quad \text{and} \quad i = (A+1) \quad [1] \quad B$$

Relationship of k capacitors and the currents:

$$U_k(p) = \frac{1}{pC_k} \cdot I_k(p) \quad \text{with} \quad C_k > 0 \quad \text{and} \quad k = (B+1) \quad [1] \quad D$$

Here the indices w, i and k just count all passive components.

2.1.1 The passive RLC two-terminal network function

Relation of the primary and secondary ideal transformer parts:

$$U_{\mu+1}(p) = \ddot{u}_{\mu} \cdot U_{\mu}(p) \quad \text{with} \quad \mu = (D+1) \quad [2] \quad E$$

$$I_{\mu+1}(p) = -\frac{1}{\ddot{u}_{\mu}} \cdot I_{\mu}(p) \quad \text{with} \quad \ddot{u}_{\mu} = \frac{w_{\mu+1}}{w_{\mu}} \quad \text{where} \quad \ddot{u}_{\mu}: \text{transformation constant}$$

The number of indicated branches thus is: → $Z = E + 2$

2.1.1 The passive RLC two-terminal network function

With Tellegen's theorem in total Z addends result:

$$\sum_w U_w(p)\cdot I_w^*(p) + \sum_i U_i(p)\cdot I_i^*(p) + \sum_k U_k(p)\cdot I_k^*(p)$$
$$+ \sum_\mu \left\{U_\mu(p)\cdot I_\mu^*(p) + U_{\mu+1}(p)\cdot I_{\mu+1}^*(p)\right\} - U_0(p)\cdot I_0^*(p) = 0$$

The first sum with index w covers → all branches of resistances

The second sum with index i covers → all branches of inductances

The 3rd sum with index k covers → all branches of capacitors

The 4th sum with index μ covers → all branches with ideal transformers

The minus sign of the last sum → due to the different directions of supply currents and voltages

2.1.1 The passive RLC two-terminal network function

Substitution of voltages by currents in the equations given above leads to:

$$U_0(p)\cdot I_0^*(p) = \sum_w R_w(p)\cdot I_w(p)\cdot I_w^*(p)$$

$$+ \sum_i p\cdot L_i\cdot I_i(p)\cdot I_i^*(p) + \sum_k \frac{1}{p\cdot C_k}\cdot I_k(p)\cdot I_k^*(p)$$

➡ The transformers contained in the two-terminal network and their transformer constants $\ddot{u}_\mu$ do not have influence on $U_0(p)\cdot I_o^*(p)$ and thus are not part of the sum due to:

$$U_\mu(p)\cdot I_\mu^*(p) + U_{\mu+1}(p)\cdot I_{\mu+1}^*(p) =$$

$$U_\mu(p)\cdot I_\mu^*(p) + \ddot{u}_\mu U_\mu(p)\cdot(-\frac{1}{\ddot{u}_\mu})I_\mu^*(p) = 0$$

2.1.1 The passive RLC two-terminal network function

Division on both sides of the previous equation by

$$I_0(p) \cdot I_0^*(p) = \left| \; I_0(p) \; \right|^2$$

then gives the following result:

$$\frac{U_0(p) I_0^*(p)}{\left|I_0(p)\right|^2} = \frac{U_0(p)}{I_0(p)} = Z(p)$$

$$= \sum_w R_w(p) \cdot \frac{\left|I_w(p)\right|^2}{\left|I_0(p)\right|^2} + p \cdot \sum_i L_i \cdot \frac{\left|I_i(p)\right|^2}{\left|I_0(p)\right|^2} + \frac{1}{p} \cdot \sum_k \frac{1}{C_k} \cdot \frac{\left|I_k(p)\right|^2}{\left|I_0(p)\right|^2}$$

2.1.1 The passive RLC two-terminal network function

After introduction of Brune´s pseudo energy functions F, T and V in the equation

$$Z(p) = \sum_w R_w(p) \cdot \frac{|I_w(p)|^2}{|I_0(p)|^2} + p \cdot \sum_i L_i \cdot \frac{|I_i(p)|^2}{|I_0(p)|^2} + \frac{1}{p} \cdot \sum_k \frac{1}{C_k} \cdot \frac{|I_k(p)|^2}{|I_0(p)|^2}$$

with the definitions $F_Z(p) = \sum_w R_w \cdot \frac{|I_w(p)|^2}{|I_0(p)|^2}$ and

$$V_Z(p) = \sum_k \frac{1}{C_k} \cdot \frac{|I_k(p)|^2}{|I_0(p)|^2} \qquad T_Z(p) = \sum_i L_i \cdot \frac{|I_i(p)|^2}{|I_0(p)|^2}$$

for all real and positive p it results: a) All pseudo energy functions are positive

b) $Z(p) = F_Z(p) + p \cdot T_Z(p) + \frac{1}{p} V_Z(p)$

2.1.1 The passive RLC two-terminal network function

Replacing p with $(\sigma + j\omega)$ leads to:

$$Z(p) = F_Z(p) + \sigma \cdot T_Z(p) + \frac{\sigma}{\sigma^2 + \omega^2} V_Z(p) + j\omega \cdot \left(T_Z(p) - \frac{1}{\sigma^2 + \omega^2} \cdot V_Z(p) \right)$$

or:

$$\mathrm{Re}\{Z(p)\} = F_Z(p) + \sigma \cdot T_Z(p) + \frac{\sigma}{\sigma^2 + \omega^2} V_Z(p)$$

$$\mathrm{Im}\{Z(p)\} = \omega \cdot \left(T_Z(p) - \frac{1}{\sigma^2 + \omega^2} \cdot V_Z(p) \right)$$

2.1.1 The passive RLC two-terminal network function

Discussion of the result:

$\mathrm{Re}\{Z(p)\} > 0 \;\; \forall \mathrm{Re}\{p\} = \sigma > 0$ This is called "$Z(p)$ is positive in p"

(a positive nonzero real value of p gives a positive nonzero real value of $Z(p)$)

Note: *Z(p)* thus has no zeros in the right open *p*-plane!

$$\mathrm{Re}\{Z(j\omega)\} \geq 0 \;\; \forall \omega \;\; \text{due to} \;\; \sigma = 0$$

Examining the case of $\omega = 0$, one gets:

$$\mathrm{Re}\{Z(\sigma)\} = F_Z(\sigma) + \sigma \cdot T_Z(\sigma) + \frac{1}{\sigma} V_Z(\sigma)$$

$$\mathrm{Im}\{Z(\sigma)\} = 0$$

This is called "*Z(p)* is real in *p*" (a real *p* value gives a real *Z(p)* value).

2.1.1 The passive RLC two-terminal network function

For the admittance function of the two-terminal network

$$Y(p) = \frac{I_0(p)}{U_0(p)}$$

it also can be shown that the same properties as for the impedance functions hold.

Thus also Y(p) is positive and real in *p*.

Due to $\operatorname{Re}\{Z(p)\} > 0 \;\; \forall \operatorname{Re}\{p\} > 0$ in combination with $Y(p) = 1/Z(p)$

in the right open p-plane (defined by $\operatorname{Re}\{p\} > 0$) it holds:

Z(p) and *Y(p)* have no zeros and no poles in the right open *p*-plane!

So both impedance function *Z(p)* and the admittance function *Y(p)* of a passive RLCÜ two-terminal network are positive and real in *p*.
This is important for realizing circuits.

2.1.2 The passive LC two-terminal network function

2.1.2 The passive LC two-terminal network function

A passive lossless LC two-terminal network contains no Ohm's resistances:

$$\longrightarrow \qquad F_z(p) = 0 \quad \text{because of } R_w = 0 \; \forall w$$

Therefore the impedance function results to:

$$Z(p) = \sigma \cdot T_Z(p) + \frac{\sigma}{\sigma^2 + \omega^2} V_Z(p) + j\omega \cdot \left(T_Z(p) - \frac{1}{\sigma^2 + \omega^2} \cdot V_Z(p) \right)$$

with

$$\operatorname{Re}\{Z(p)\} = \sigma \cdot T_Z(p) + \frac{\sigma}{\sigma^2 + \omega^2} V_Z(p) \quad = 0 \text{ only for } \sigma = 0$$

$$\operatorname{Im}\{Z(p)\} = \omega \cdot \left(T_Z(p) - \frac{1}{\sigma^2 + \omega^2} \cdot V_Z(p) \right)$$

2.1.2 The passive LC two-terminal network function

Now the conditions for zeros of *Z(p)* are considered.

Note the following conditions for any zero:
Z(p) = *0* only holds if both *Re{Z(p)}* = *0* and *Im{Z(p)}* = *0* hold.
The real part *Re{Z(p)}* as shown before can only be zero on the *jω*-axis.

As specified in the previous slide in general it holds:

$$Z(p) = \sigma \cdot T_Z(p) + \frac{\sigma}{\sigma^2 + \omega^2} V_Z(p) + j\omega \cdot \left(T_Z(p) - \frac{1}{\sigma^2 + \omega^2} \cdot V_Z(p) \right)$$

Therefore, considering $\mathrm{Re}\{Z(p)\}$ it holds:

Zeros of $Z(p)$ can lie exclusively on the $j\omega$ – axis of the p-plane

2.1.2 The passive LC two-terminal network function

In addition the following can be shown:

- *Z(p)* always is imaginary for imaginary *p* values.
- As *Z(p)* is the inverse of *Y(p)* also this function *Y(p)* is imaginary for imaginary *p* values.

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2.1.2 The passive LC two-terminal network function

The partial fraction representation of *Z(p)* and/or *Y(p)*:

$$Z(p) = \frac{A_0}{p} + \sum_{\nu=1}^{N} \frac{2 \cdot A_\nu \cdot p}{p^2 + \omega_{\infty\nu}^2} + A_\infty \cdot p \quad \text{with } A_0, A_\nu, A_\infty : \text{real and positive}$$

$$Y(p) = \frac{B_0}{p} + \sum_{\mu=1}^{M} \frac{2 \cdot B_\mu \cdot p}{p^2 + \omega_{\infty\mu}^2} + B_\infty \cdot p \quad \text{with } B_0, B_\mu, B_\infty : \text{real and positive}$$

With these formulas one can determine the partial fractions development coefficients of any given impedance or admittance function.

When evaluating *Z(p)* or *Y(p)* at the poles locations it can be seen that only real and positive development coefficients ensure that these functions are positive and real in *p* , i.e. *Z(p)* and *Y(p)* are real and positive for any real and positive *p* including the pole locations.

2.1.2 The passive LC two-terminal network function

Coefficients of impedance function *Z(p)* :

$$A_0 = \lim_{p\to 0}\{p\cdot Z(p)\}$$

$$A_\nu = \frac{1}{2}\cdot \lim_{p^2\to -\omega_{\infty\,\nu}^2}\left\{\frac{Z(p)}{p}\cdot\left(p^2+\omega_{\infty\,\nu}^2\right)\right\}$$

$$A_\infty = \lim_{p\to\infty}\left\{\frac{Z(p)}{p}\right\}$$

Coefficients of admittance function *Y(p)* :

$$B_0 = \lim_{p\to 0}\{p\cdot Y(p)\}$$

$$B_\mu = \frac{1}{2}\cdot \lim_{p^2\to -\omega_{\infty\,\mu}^2}\left\{\frac{Y(p)}{p}\cdot\left(p^2+\omega_{\infty\,\mu}^2\right)\right\}$$

$$B_\infty = \lim_{p\to\infty}\left\{\frac{Y(p)}{p}\right\}$$

2.1.2 The passive LC two-terminal network function

Summary:

1. *Z(p)* and *Y(p) = l/Z(p)* of a passive LC two-terminal network are positive, and real in p.

2. Poles and zeros of *Z(p)* and *Y(p)* lie exclusively on the $j\omega$-axis of the p-plane.

3. $Z(j\omega)$ and $Y(j\omega)$ are purely imaginary for all ω

4. All coefficients of the partial fraction representation of *Z(p)* and *Y(p)* are **positive and real** (without proof).

2.1.3 The passive RC two-terminal network function

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2.1.3 The passive RC two-terminal network function

The passive RC two-terminal network contains no coils:

$\longrightarrow$ $T_z(p) = 0 \quad \forall w \quad$ because $L_i = 0$

Therefore the impedance function results to

$$Z(p) = F_Z(p) + \frac{1}{p} V_Z(p)$$

with

$$\mathrm{Re}\{Z(p)\} = F_Z(p) + \frac{\sigma}{\sigma^2 + \omega^2} V_Z(p)$$

$$\mathrm{Im}\{Z(p)\} = -\frac{\omega}{\sigma^2 + \omega^2} V_Z(p)$$

2.1.3 The passive RC two-terminal network function

If $p_0 = \sigma_0 + j\omega_0$ is a zero of $Z(p)$, then:

$$\mathrm{Re}\{Z(p_0)\} = F_Z(p_0) + \frac{\sigma_0}{\sigma_0^2 + \omega_0^2} V_Z(p_0) = 0$$

and
$$\mathrm{Im}\{Z(p_0)\} = -\frac{\omega_0}{\sigma_0^2 + \omega_0^2} V_Z(p_0) = 0$$

Discussion:

$$\mathrm{Im}\{Z(p_0)\} = 0 \quad \Rightarrow \quad V_z(p_0) = 0 \;\; or \;\; \omega_0 = 0$$

Property: $F_Z(p) \neq 0$ and $V_Z(p) \neq 0$ (due to RC network)

→ $\mathrm{Im}\{Z(p_0)\} = 0$ only in case of $\omega_0 = 0$

So $\omega_0 = 0$ is a necessary condition for any zero!

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2.1.3 The passive RC two-terminal network function

$$p_{0\nu} = \sigma_{0\nu} \quad \forall \nu$$

This means that all zeros can only lie on the real σ axis.

In this case for the real part of the impedance function holds:

$$\mathrm{Re}\ \{Z(\sigma_{0\nu})\} = F_Z(\sigma_{0\nu}) + \frac{1}{\sigma_{0\nu}} V_Z(\sigma_{0\nu}) = 0$$

This condition can only be satisfied for $\sigma_{0\nu} < 0$ which means that all zeros must lie in the left open p-plane!

In a similar manner it can be shown:

- All zeros of *Y(p)* must lie in the left closed p- plane.
- These zeros lie on the real axis.

2.1.3 The passive RC two-terminal network function

Summary of the characteristics of the impedance function *Z(p)*:

1. $Z(p)$ or $Y(p) = 1/Z(p)$ are positive and real in p

2. The zeros of $Z(p)$ and thus the poles of $Y(p)$ exclusively lie in the left closed half p-plane and there on the real axis.

3. It can also be shown (without proof): Poles and zeros of $Z(p)$ appear in alternating sequence on the real axis

4. $Z(p)$ can only have a single pole at $p = 0$, so $Y(p)$ can only have a single zero at $p = 0$.

5. The decomposition of $Z(p)$ or $Y(p)$ into partial fractions leads to positive real development coefficients (without proof).

2.1.3 The passive RC two-terminal network function

Partial fractions representation of such an impedance function *Z(p)* look like:

$$Z(p) = \frac{A_0}{p} + \sum_{\nu=1}^{N} \frac{A_\nu}{p - \sigma_{\infty\nu}} + A_\infty \text{ with } \sigma_{\infty\nu} < 0; \quad A_0, A_\nu, A_\infty \text{ being positive real}$$

with:

$$A_0 = \lim_{p \to 0} \{ p \cdot Z(p) \}$$

$$A_\nu = \lim_{p \to \sigma_{\infty\nu}} \{ Z(p) \cdot (p - \sigma_{\infty\nu}) \}$$

$$A_\infty = \lim_{p \to \infty} \{ Z(p) \}$$

2.1.3 The passive RC two-terminal network function

Partial fractions representation of such an admittance function *Y(p)* look like:

$$Y(p) = B_0 + \sum_{\mu=1}^{M} \frac{B_\mu \cdot p}{p - \sigma_{\infty\mu}} + B_\infty \cdot p$$

with $\sigma_{\infty\mu} < 0 \ \forall \mu$ and B_0, B_μ, B_∞ being positive real with:

$$B_0 = \lim_{p \to 0} Y(p)$$

$$B_\infty = \lim_{p \to \infty} \left\{ \frac{Y(p)}{p} \right\}$$

$$B_\mu = \lim_{p \to \sigma_{\infty\mu}} \left\{ \frac{Y(p)}{p} \cdot \left(p - \sigma_{\infty\mu} \right) \right\}$$

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2.1.4 The passive RL two-terminal network function

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2.1.4 The passive RL two-terminal network function

The passive RL two-terminal network contains no capacitors:

$\longrightarrow \; V_z(p) = 0 \quad \forall w \quad \text{because } C_k = 0$

Therefore the impedance function results to

$$Z(p) = F_Z(p) + p \cdot T_Z(p)$$

with

$$\mathrm{Re}\{Z(p)\} = F_Z(p) + \sigma \cdot T_Z(p)$$

$$\mathrm{Im}\{Z(p)\} = \omega \cdot T_Z(p)$$

2.1.4 The passive RL two-terminal network function

If $p_0 = \sigma_0 + j\omega_0$ is a zero of $Z(p)$, then the following must be true:

$$\mathrm{Re}\{Z(p_0)\} = F_Z(p_0) + \sigma_0 \cdot T_Z(p_0) = 0$$

and $$\mathrm{Im}\{Z(p_0)\} = \omega_0 \cdot T_Z(p_0) = 0$$

Discussion:

Similar to the case of the RC two-terminal networks it follows:

$$\mathrm{Im}\{Z(p_0)\} = 0 \quad \text{only in case of} \quad \omega_0 = 0$$

$\longrightarrow$ $$p_{0\nu} = \sigma_{0\nu} \quad \text{with} \quad \sigma_{0\nu} < 0 \,.$$

Additional considerations conc. *Y(p)* lead in total to the next statements:

2.1.4 The passive RL two-terminal network function

Summary:

1. *Z(p)* and *Y(p)* are positive, real and rational in *p*
2. <u>The pole and zeros of *Z(p)* and *Y(p)* only lie in the left closed p-half plane</u>
3. <u>They only lie on the real axis</u>
4. There are only single poles and zeros
5. The coefficients in the partial fraction expansion are positive

2.1.4 The passive RL two-terminal network function

Partial fraction representation of such an impedance function *Z(p)*:

$$Z(p) = A_0 + \sum_{\nu=1}^{N} \frac{p \cdot A_\nu}{p - \sigma_{\infty\nu}} + p \cdot A_\infty$$

with $\sigma_{\infty\nu} < 0$ and A_0, A_ν, A_∞ being positive real with:

$$A_0 = \lim_{p \to 0} \{Z(p)\}$$

$$A_\nu = \lim_{p \to \sigma_{\infty\nu}} \left\{ \frac{Z(p)}{p} \cdot \left(p - \sigma_{\infty\nu}\right) \right\}$$

$$A_\infty = \lim_{p \to \infty} \left\{ \frac{Z(p)}{p} \right\}$$

2.1.4 The passive RL two-terminal network function

Partial fraction representation of a corresponding admittance function *Y(p)*:

$$Y(p) = \frac{B_0}{p} + \sum_{\mu=1}^{M} \frac{B_\mu}{p - \sigma_{\infty\mu}} + B_\infty$$

with $\sigma_{\infty\mu} < 0 \ \forall \mu$ and B_0, B_μ, B_∞ being positive real with

$$B_0 = \lim_{p \to 0} \{ p \cdot Y(p) \}$$

$$B_\mu = \lim_{p \to \sigma_{\infty\mu}} \{ Y(p) \cdot (p - \sigma_{\infty\mu}) \}$$

$$B_\infty = \lim_{p \to \infty} \{ Y(p) \}$$

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Chapter 2

Characteristics and realization of passive two-terminal network function

2.2 Design of a passive two-terminal network

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2.2.1 Realization of a LC passive two-terminal network

From the partial fraction representation of the impedance function

$$Z(p) = \frac{A_0}{p} + \sum_{\nu=1}^{N} \frac{2 \cdot A_\nu \cdot p}{p^2 + \omega_{\infty\nu}^2} + A_\infty \cdot p \quad \text{with } A_0, A_\nu, A_\infty \text{ being real and positive}$$

of a lossless two-terminal network one can derive:

→ Each individual addend corresponds to a single impedance connected in series.

The first addend gives:

$$Z_0(p) = \frac{A_0}{p} = \frac{1}{\frac{1}{A_0} \cdot p} \xrightarrow{\text{Impedance of a capacitor}} C_0 = \frac{1}{A_0}$$

2.2.1 Realization of a LC passive two-terminal network

The further addends of the fractions give:

$$Z_\nu(p) = \frac{2 \cdot A_\nu \cdot p}{p^2 + \omega_{\infty\nu}^2} = \frac{1}{\dfrac{1}{2 \cdot A_\nu} \cdot p + \dfrac{\omega_{\infty\nu}^2}{2 \cdot A_\nu} \cdot \dfrac{1}{p}}$$

They correspond in each case to the **parallel connection of**

• a capacitor and • a coil

$$C_\nu = \frac{1}{2 \cdot A_\nu} \qquad L_\nu = \frac{2 \cdot A_\nu}{\omega_{\infty\nu}^2}$$

2.2.1 Realization of a LC passive two-terminal network

The last addend corresponds to the impedance of a coil:

$$L_\infty = A_\infty$$

This leads directly to the 1st FOSTER form (canonical circuit):

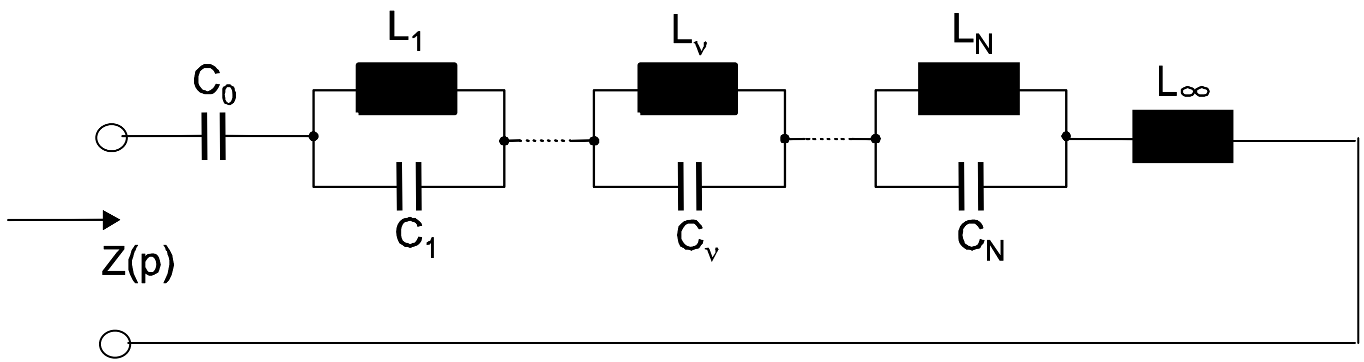

Fig.: Realization of the LC two-terminal impedance *Z(p)* as partial fraction circuit

2.2.1 Realization of a LC passive two-terminal network

From the partial fraction representation of the admittance function

$$Y(p) = \frac{B_0}{p} + \sum_{\mu=1}^{M} \frac{2 \cdot B_\mu \cdot p}{p^2 + \omega_{\infty\mu}^2} + B_\infty \cdot p \quad \text{with } B_0, B_\mu, B_\infty \text{ being real and positive}$$

one can derive:

→ Each individual addend of the partial fractions represents a single admittance which is connected in parallel to the others.

The first addend gives:

$$\frac{B_0}{p} = \frac{1}{\frac{1}{B_0} \cdot p} \quad \xrightarrow{\text{Admittance of an inductance}} \quad L_0 = \frac{1}{B_0}$$

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2.2.1 Realization of a LC passive two-terminal network

The further addends of the form:

$$Y_\mu(p) = \frac{2 \cdot B_\mu \cdot p}{p^2 + \omega_{\infty\nu}^2} = \frac{1}{\frac{1}{2 \cdot B_\mu} \cdot p + \frac{\omega_{\infty\nu}^2}{2 \cdot B_\mu} \cdot \frac{1}{p}}$$

correspond in each case to the **serial connection of**

- a coil: and
- a capacitor:

$$L_\mu = \frac{1}{2 \cdot B_\mu}$$

$$C_\mu = \frac{2 \cdot B_\mu}{\omega_{\infty\mu}^2}$$

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2.2.1 Realization of a LC passive two-terminal network

The last addend corresponds to the admittance of a capacitor:

$$C_\infty = B_\infty$$

Thus the 2nd FOSTER form is obtained (canonical circuit):

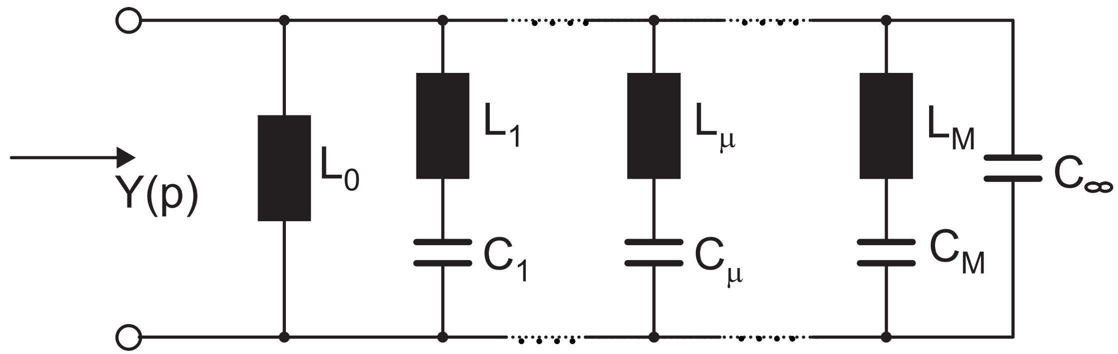


Fig.: Realization of the LC two-terminal admittance *Y(p)* as a partial fraction circuit

2.2.1 Realization of a LC passive two-terminal network

The first CAUER form procedure is a method to set up two-terminal networks from a continued fraction:

Step 1: Pick up from $Z(p)$ the last addend $A_\infty \cdot p$, so that the remainder is:

$$Z_1(p) = Z(p) - A_\infty \cdot p$$

Step 2: The remainder admittance will then be: → $Y_1(p) = \dfrac{1}{Z_1(p)}$

Step 3: Take off from the remainder admittance $Y_1(p)$ the term $B_{\infty 1} \cdot p$, so that:

$$Y_2(p) = Y_1(p) - B_{\infty 1} \cdot p$$

Step 4: Inverting $Y_2(p)$ gives the impedance of single LC two-pole network:

$$Z_2(p) = \frac{1}{Y_2(p)}$$

2.2.1 Realization of a LC passive two-terminal network

A continued application leads to a fraction representation of the first CAUER form:

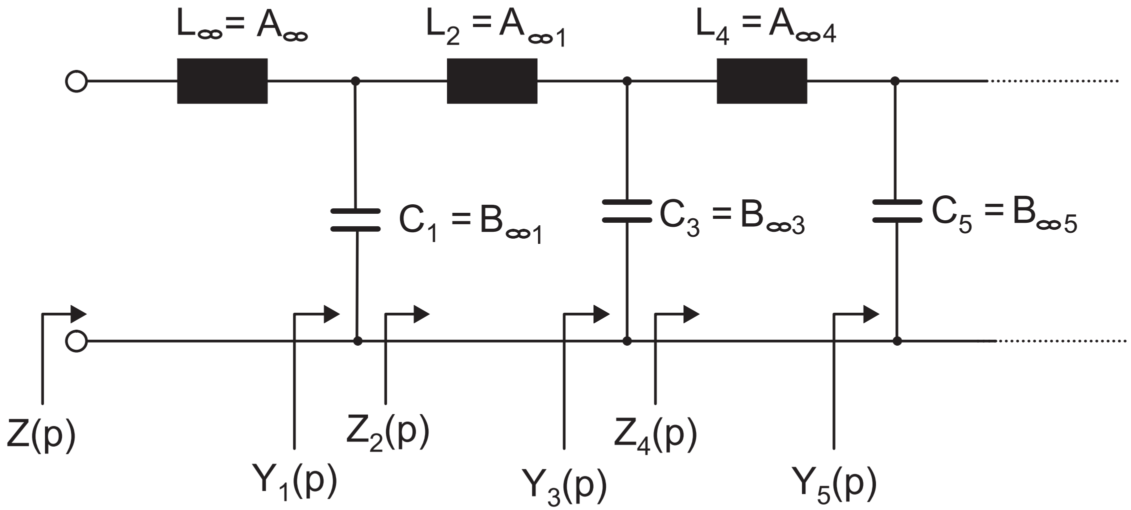
$$Z(p) = A_{\infty} \cdot p + \cfrac{1}{B_{\infty 1} \cdot p + \cfrac{1}{A_{\infty 2} \cdot p + \cfrac{1}{B_{\infty 3} \cdot p + \cfrac{1}{A_{\infty 4} \cdot p + \cdots}}}}$$

The procedure ends if none of the following expressions remains: $A_{\infty \nu} \cdot p$ or $\mathrm{B}_{\infty \mu} \cdot p$

Note: Both are poles located at *p* going to infinite!

2.2.1 Realization of a LC passive two-terminal network

The appropriate continued fraction - or also branch circuit looks as follows:



Realization of the LC two-terminal impedance as a continued fraction circuit (First CAUER form)

In the realized circuit, inductances are in the long branches and capacitors are in the transverse branches.

2.2.1 Realization of a LC passive two-terminal network

<u>Background of the method</u>

- $Z_1(p) = Z(p) - A_\infty p$ exhibits a zero for $p \to \infty$ due to C_1
- Thus $Y_1(p) = 1/Z_1(p)$ has a pole for $p \to \infty$

  and a development coefficient $B_{\infty 1}$

- $Y_2(p) = Y_1(p) - B_{\infty 1} p$ has a zero at $p \to \infty$ due to L_2
- Thus $Z_2(p) = 1/Y_1(p)$ has a pole for $p \to \infty$

  and a development coefficient $A_{\infty 2}$

- $Z_3(p) = Z_2(p) - A_{\infty 2} p$ has a zero at $p \to \infty$ due to C_3

  ... and so on.

So this method always takes off the pole at infinitely large p !

2.2.1 Realization of a LC passive two-terminal network

<u>Other comments</u>

- It is also possible to write an admittance function *Y(p)* as a continued fraction
- Such a procedure in general follows the 4 steps given above
- The difference is: The procedure starts with step 3!
- Thus the circuits show the same structure as in the previous figure
- The difference here: Coils and capacitors are exchanged

2.2.1 Realization of a LC passive two-terminal network

Second CAUER form procedure (with taking off poles at $p = 0$):

Step 1: From the given $Z(p)$, the term A_0 / p is taken off, thus the remainder is:

$$Z_1(p) = Z(p) - \frac{A_0}{p}$$

Step 2: The corresponding admittance is formed by: → $Y_1(p) = \frac{1}{Z_1(p)}$

Step 3: The term B_{01} / p is taken off from the remainder of $Y_1(p)$:

$$Y_2(p) = Y_1(p) - \frac{B_{01}}{p}$$

Step 4: The impedance is the defined by the inverse of the remainder:

$$Z_2(p) = \frac{1}{Y_2(p)}$$

2.2.1 Realization of a LC passive two-terminal network

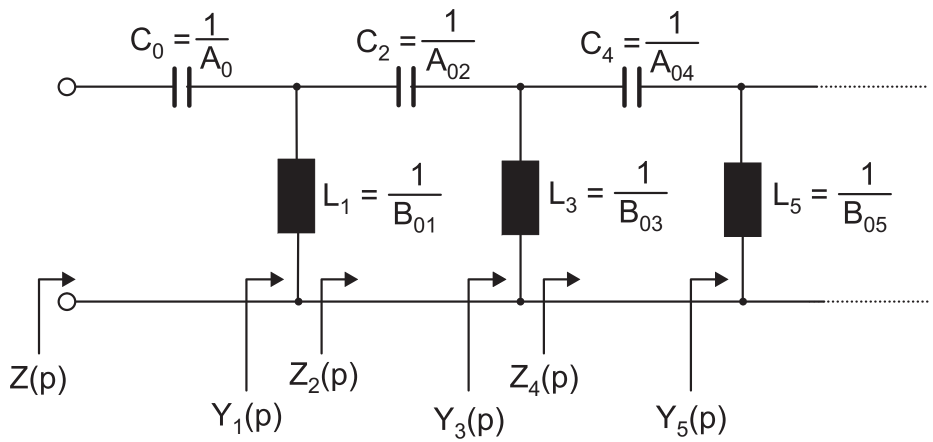
Thus one obtains the following LC-impedance function *Z(p)*:

$$Z(p) = \frac{A_0}{p} + \cfrac{1}{\cfrac{B_{01}}{p} + \cfrac{1}{\cfrac{A_{02}}{p} + \cfrac{1}{\cfrac{B_{03}}{p} + \cfrac{1}{\cfrac{A_{04}}{p} + \cdots}}}}$$

➡ The procedure described above ends as soon as no more expressions can be taken off!

2.2.1 Realization of a LC passive two-terminal network

The appropriate continued fraction or branch circuit then looks as follows:



Realization of the LC two-terminal impedance *Z(p)* in the 2nd CAUER form

→ One obtains a network which contains capacitors in the long branches and inductances in the transverse branches.

2.2.2 Realization of a RC passive two-terminal network

Based on the partial fraction representation of the impedance function

$$Z(p) = \frac{A_0}{p} + \sum_{\nu=1}^{N} \frac{A_\nu}{p - \sigma_{\infty\nu}} + A_\infty \quad \text{with } \sigma_{\infty\nu} < 0; \quad A_0, A_\nu, A_\infty \ \text{:positive real}$$

one can interpret it as follows:

The last addend A_∞ of the summation

$R_\infty = A_\infty$ ⟶ is the impedance of a resistance

2.2.2 Realization of a RC passive two-terminal network

The further addends of the form

$$Z_\nu(p) = \frac{A_\nu}{p - \sigma_{\infty\nu}} = \frac{1}{\frac{1}{A_\nu} \cdot p + \frac{|\sigma_{\infty\nu}|}{A_\nu}} \text{ due to } \sigma_{\infty\nu} < 0 \ \forall \nu$$

correspond in each case to the **parallel connection** of

- a capacitor and
- a resistor

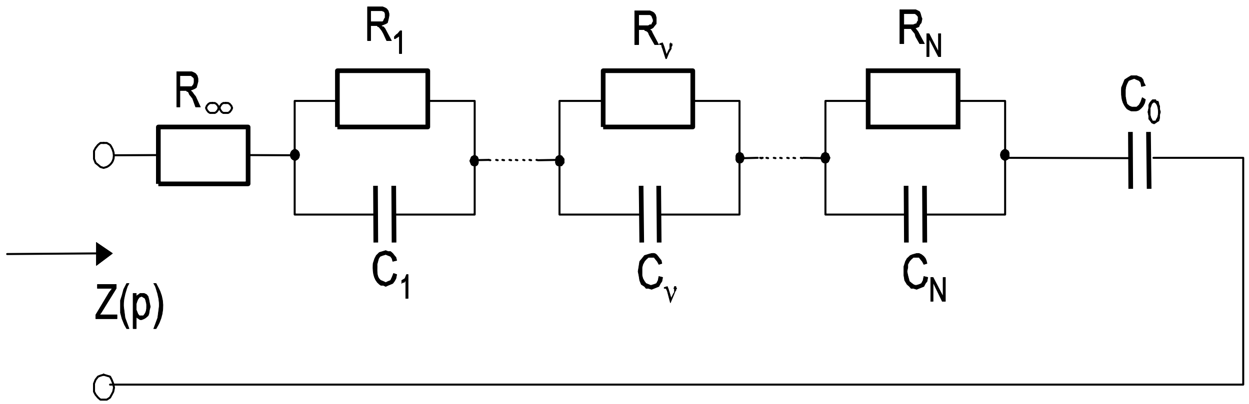
$$C_\nu = \frac{1}{A_\nu} \qquad R_\nu = \frac{A_\nu}{|\sigma_{\infty\nu}|}$$

2.2.2 Realization of a RC passive two-terminal network

The first addend of the summation is A_0/p .

This corresponds to the impedance of the capacity: $C_0 = \frac{1}{A_0}$

<u>Thus it results the partial fraction circuit or 1st FOSTER form of an RC circuit:</u>



Realization of the RC two-terminal partial fraction circuit

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2.2.2 Realization of a RC passive two-terminal network

In case of the partial fractions representation of the admittance function one obtains:

$$Y(p) = B_0 + \sum_{\mu=1}^{M} \frac{B_\mu \cdot p}{p - \sigma_{\infty\mu}} + B_\infty \cdot p$$

with $\sigma_{\infty\mu} < 0$ and B_0, B_μ, B_∞ being positive real

This means that the first addend

$$R_0 = \frac{1}{B_0}$$ → corresponds to a resistor

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2.2.2 Realization of a RC passive two-terminal network

The further addends of the form

$$Y_\mu(p) = \frac{B_\mu p}{p - \sigma_{\infty\mu}} = \frac{1}{\frac{1}{B_\mu} + \frac{|\sigma_{\infty\mu}|}{B_\mu} \cdot \frac{1}{p}} \quad \text{due to } \sigma_{\infty\mu} < 0 \ \forall \mu$$

correspond in each case to the **series connection** of:

- a resistor

  $$R_\mu = \frac{1}{B_\mu}$$

and

- a capacitor

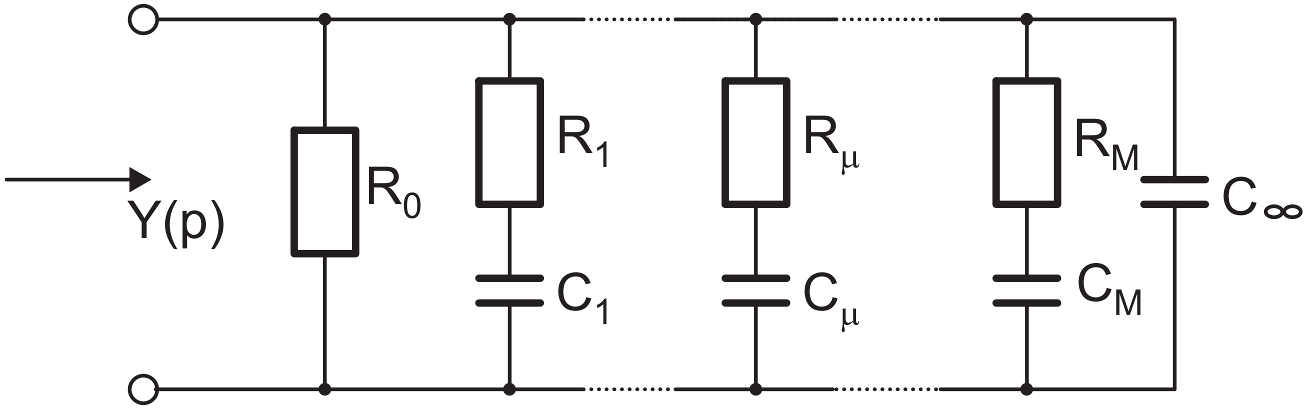
  $$C_\mu = \frac{B_\mu}{|\sigma_{\infty\mu}|}$$

2.2.2 Realization of a RC passive two-terminal network

The last addend $B_\infty \cdot p$ corresponds to the admittance of a capacitor:

$$C_\infty = B_\infty$$

Thus the corresponding partial fractions circuit looks as follows:



Realization of the RC two-terminal admittance *Y(p)* as partial fraction circuit (Second FOSTER form)

2.2.2 Realization of a RC passive two-terminal network

Just as LC two-terminal networks, the impedance function *Z(p)* and the admittance function *Y(p)* of a passive RC network can also be written as a continued fraction with a corresponding (branch) circuit, either by:

- The 4-step method of first CAUER form by taking off poles with $p = \infty$ in which:

→ Resistors are in the long branches and the capacitors are in the transverse branches of the branch circuit or

- The 4-step method of second CAUER form by taking off poles with $p = 0$ in which:

→ Capacitors are in the long branches and resistors are in the transverse branches of the branch circuit

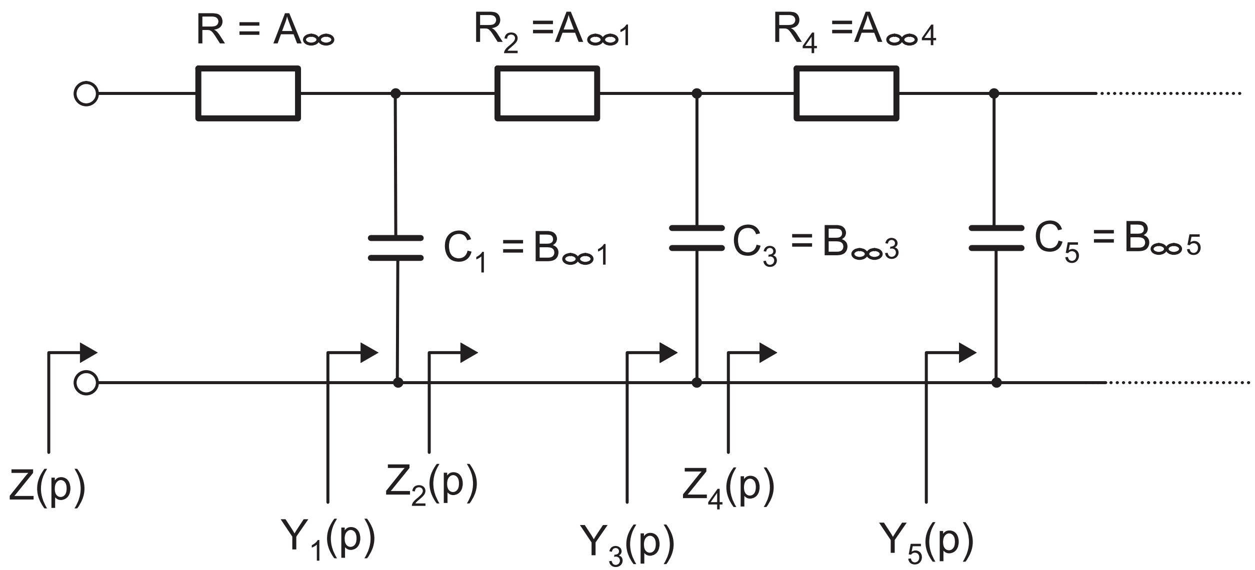
2.2.2 Realization of a RC passive two-terminal network

The assigned continued fraction representation (First CAUER Form):

$$Z(p) = A_{\infty} + \cfrac{1}{B_{\infty 1} \cdot p + \cfrac{1}{A_{\infty 2} + \cfrac{1}{B_{\infty 3} \cdot p + \cfrac{1}{A_{\infty 4} \; \cdot \; \cdots}}}}$$

2.2.2 Realization of a RC passive two-terminal network

The appropriate continued fraction or branch circuit looks as follows:



Realization of the RC two-terminal network *Z(p)* as continued fraction branch circuit (1st CAUER form)

2.2.2 Realization of a RC passive two-terminal network

The assigned continued fraction representation (Second CAUER form):

$$Z(p) = \frac{A_0}{p} + \cfrac{1}{B_{01} + \cfrac{1}{\cfrac{A_{02}}{p} + \cfrac{1}{B_{03} + \cfrac{1}{\cfrac{A_{04}}{p} + \cdots}}}}$$

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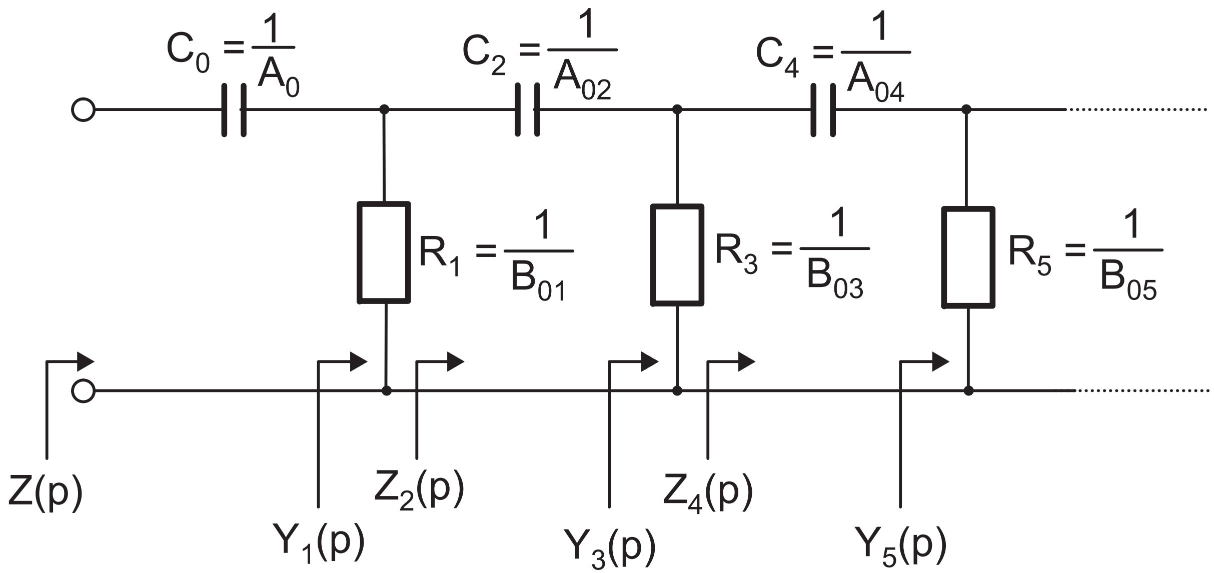
2.2.2 Realization of a RC passive two-terminal network

The appropriate continued fraction or branch circuit:



Realization of the RC two-terminal network *Z(p)* as continued fraction Branch circuit (Second CAUER form)

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2.2.3 Realization of a RL passive two-terminal network

The partial fractions representation of the impedance function *Z(p)* gives:

$$Z(p) = A_0 + \sum_{\nu=1}^{N} \frac{p \cdot A_\nu}{p + \sigma_{\infty\nu}} + p \cdot A_\infty \quad \text{with } \sigma_{\infty\nu} < 0; \quad A_0, A_\nu, A_\infty \text{ are positive real}$$

One can interpret again each individual addend as a single impedance and translate the sum of partial fractions into a serial connection of single impedances.

The first addend is considered as:

$$R_0 = A_0 \longrightarrow \text{impedance of a resistance}$$

2.2.3 Realization of a RL passive two-terminal network

The further addends of the form

$$Z_\nu(p) = \frac{A_\nu \cdot p}{p - \sigma_{\infty\nu}} = \frac{1}{\frac{1}{A_\nu} + \frac{|\sigma_{\infty\nu}|}{A_\nu} \cdot \frac{1}{p}}$$

correspond in each case to the **parallel connection** of

- a resistor and
- a coil

$$R_\nu = A_\nu$$

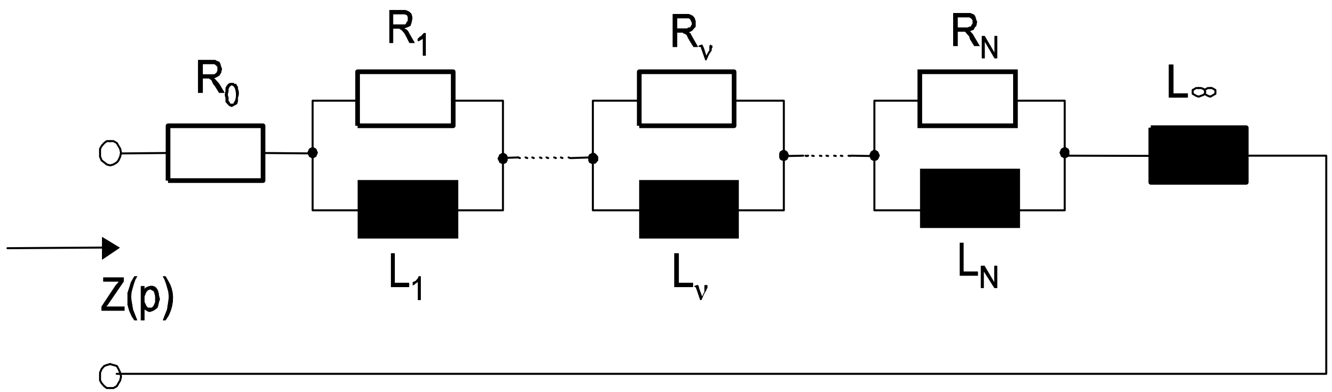
$$L_\nu = \frac{A_\nu}{|\sigma_{\infty\nu}|}$$

2.2.3 Realization of a RL passive two-terminal network

The last addend $A_\infty p$ corresponds to the impedance of a coil:

$$L_\infty = A_\infty$$

Then the assigned partial fractions circuit of *Z(p)* - (in first FOSTER form) results:



Realization of the RC two-terminal impedance function *Z(p)* as partial fractions circuit (First FOSTER form)

2.2.3 Realization of a RL passive two-terminal network

The partial fraction representation of the admittance function *Y(p)* has the form:

$$Y(p) = \frac{B_0}{p} + \sum_{\mu=1}^{M} \frac{B_\mu}{p - \sigma_{\infty\mu}} + B_\infty$$

with $\sigma_{\infty\mu} < 0$ and B_0, B_μ, B_∞ being positive real

The last term arising in the sum

→ corresponds to the admittance of a resistor:

$$R_\infty = \frac{1}{B_\infty}$$

2.2.3 Realization of a RL passive two-terminal network

The further addends

$$Y_\mu(p) = \frac{B_\mu}{p - \sigma_{\infty\mu}} = \frac{1}{\frac{p}{B_\mu} + \frac{|\sigma_{\infty\mu}|}{B_\mu}}$$

become in each case a series connection of

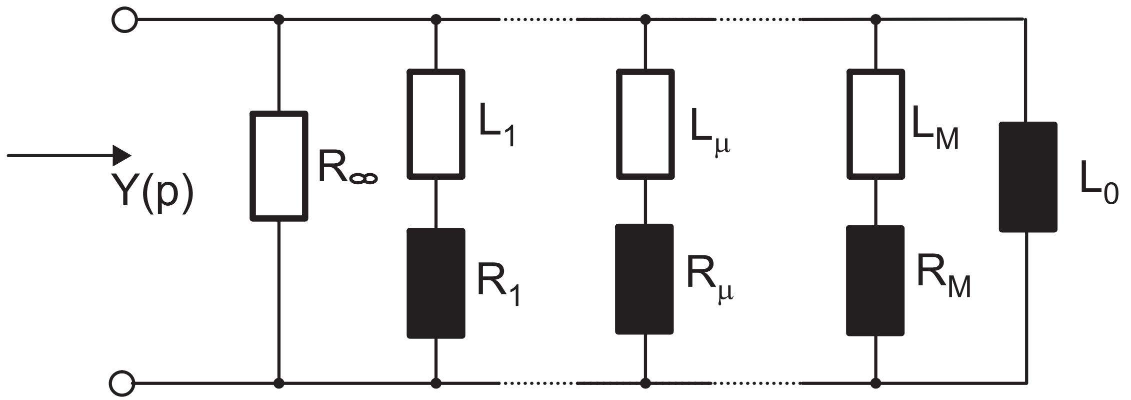
- a coil: $L_\mu = \frac{1}{B_\mu}$
- and a resistor $R_\mu = \frac{|\sigma_{\infty\mu}|}{B_\mu}$

2.2.3 Realization of a RL passive two-terminal network

The first term of the form

$$L_0 = \frac{1}{B_0} \longrightarrow$$ corresponds to the admittance of a coil.

The resulting partial fraction circuit looks as follows:



Realization of the RL two-terminal admittance function *Y(p)* as partial fractions circuit (Second FOSTER form)

Summary – Partial fractions method

For LC networks:

$$Z(p) = \frac{A_0}{p} + \sum_{\nu=1}^{N} \frac{2 \cdot A_\nu \cdot p}{p^2 + \omega_{\infty\nu}^2} + A_\infty \cdot p \quad \text{with } A_0, A_\nu, A_\infty : \text{real and positive}$$

$$Y(p) = \frac{B_0}{p} + \sum_{\mu=1}^{M} \frac{2 \cdot B_\mu \cdot p}{p^2 + \omega_{\infty\mu}^2} + B_\infty \cdot p \quad \text{with } \mathrm{B}_0, B_\mu, B_\infty \text{being real and positive}$$

For RC networks:

$$Z(p) = \frac{A_0}{p} + \sum_{\nu=1}^{N} \frac{A_\nu}{p - \sigma_{\infty\nu}} + A_\infty \text{ with } \sigma_{\infty\nu} < 0$$

$$Y(p) = B_0 + \sum_{\mu=1}^{M} \frac{B_\mu \cdot p}{p - \sigma_{\infty\mu}} + B_\infty \cdot p \text{ with } \sigma_{\infty\mu} < 0 \text{ and } B_0, B_\mu, B_\infty \text{being positive real}$$

For RL networks:

$$Z(p) = A_0 + \sum_{\nu=1}^{N} \frac{p \cdot A_\nu}{p - \sigma_{\infty\nu}} + p \cdot A_\infty \quad \text{with } \sigma_{\infty\nu} < 0$$

$$Y(p) = \frac{B_0}{p} + \sum_{\mu=1}^{M} \frac{B_\mu}{p - \sigma_{\infty\mu}} + B_\infty \text{ with } \sigma_{\infty\mu} < 0$$

Summary – Chain of fractions method

LC networks: $$Z(p) = \frac{A_0}{p} + \cfrac{1}{\cfrac{B_{01}}{p} + \cfrac{1}{\cfrac{A_{02}}{p} + \cfrac{1}{\cfrac{B_{03}}{p} + \cfrac{1}{\cfrac{A_{04}}{p} + \cdots}}}}$$

RC networks: $$Z(p) = \frac{A_0}{p} + \cfrac{1}{B_{01} + \cfrac{1}{\cfrac{A_{02}}{p} + \cfrac{1}{B_{03} + \cfrac{1}{\cfrac{A_{04}}{p} + \cdots}}}}$$