Dear Students of the lecture Network Theory I,

this document shows the solution of the problems 3.1 and 3.3, which should have been presented on 4.2.2005. If you have questions or comments, please let me know.
On the 16.02.2005 (the 4th. date of Prof. Laws revision course) I'll present a problem of an old examination.

Yours sincerely,

Thorsten Schultze

Solution of problem 3.1

3.1.1 For the normalisation at first it is necessary to divide the numerator and the denominator of the polynomial by the normalising frequency ω_N^2.

$$H_\mathrm{L}(p) = -\frac{\dfrac{C_{1A}\cdot C_{2A}}{C_{1B}\cdot(C_{1A}+C_{2A})}\cdot\left(\dfrac{p}{\omega_\mathrm{N}}\right)^2}{\left(\dfrac{p}{\omega_\mathrm{N}}\right)^2+\dfrac{G_{1B}+G_{2B}}{C_{2B}\omega_\mathrm{N}}\cdot\dfrac{p}{\omega_\mathrm{N}}+\dfrac{G_{1B}\cdot G_{2B}}{C_{1B}\omega_\mathrm{N}\cdot C_{2B}\omega_\mathrm{N}}}$$

Afterwards the coefficients of the numerator and denominator are extended with ω_N and R_N to match the normalised resistances r (or conductances g)

$$r = \frac{R}{R_\mathrm{N}} = \frac{1}{G\cdot R_\mathrm{N}} = \frac{G_\mathrm{N}}{G} = \frac{1}{g}$$

and normalised capacitances c

$$c = \omega_\mathrm{N}\cdot R_\mathrm{N}\cdot C$$

Then the normalised frequency ρ is introduced:

$$\rho = \frac{p}{\omega_\mathrm{N}} = \frac{\sigma}{\omega_\mathrm{N}} + \mathrm{j}\frac{\omega}{\omega_\mathrm{N}} = \varsigma + \mathrm{j}\Omega$$

By the end one gets the fully normalised function $H_\mathrm{Ln}(\rho)$:

$$H_\mathrm{L}(p) = -\frac{\dfrac{c_{1A}\cdot c_{2A}}{c_{1B}\cdot(c_{1A}+c_{2A})}\cdot\rho^2}{\rho^2+\dfrac{g_{1B}+g_{2B}}{c_{2B}}\cdot\rho+\dfrac{g_{1B}\cdot g_{2B}}{c_{1B}\cdot c_{2B}}}$$

3.1.2 According to the script page 131, equations 3.3-45 to 3.3-50 one gets:

$$c_{1A} = c_{2A} = \frac{2H_0}{\frac{\omega_\mathrm{g}}{\omega_\mathrm{N}} \cdot \frac{R_\mathrm{w}}{R_\mathrm{N}}}$$

$$r_{1A} = \frac{R_\mathrm{w}}{4 \cdot b \cdot H_0 \cdot R_\mathrm{N}}$$

$$c_{1B} = \frac{1}{\frac{\omega_\mathrm{g}}{\omega_\mathrm{N}} \cdot \frac{R_\mathrm{w}}{R_\mathrm{N}}}$$

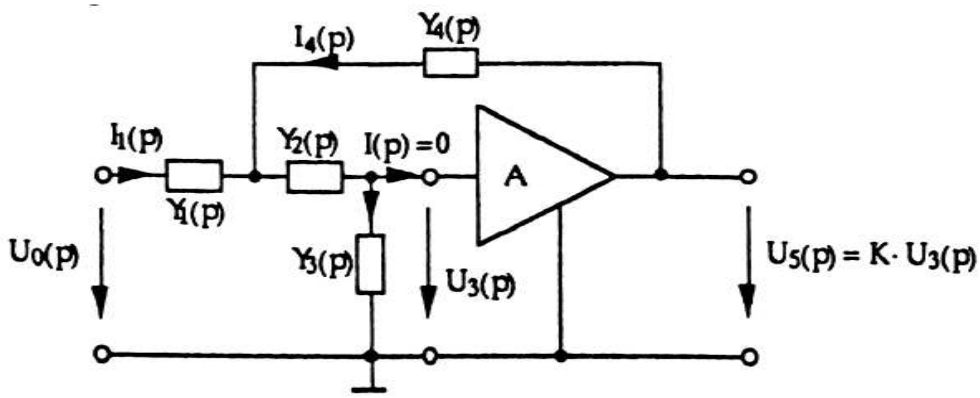
$$c_{2B} = \frac{k^2}{(b \cdot k - 1)\frac{\omega_\mathrm{g}}{\omega_\mathrm{N}} \cdot \frac{R_\mathrm{w}}{R_\mathrm{N}}}$$

$$r_{1B} = \frac{R_\mathrm{w}}{k \cdot R_\mathrm{N}}$$

$$r_{2B} = \frac{(b \cdot k - 1)R_\mathrm{w}}{k \cdot R_\mathrm{N}}$$

Solution of problem 3.3

3.3.1 The currents I_1 to I_4 are set as follows:



3.3.2 From the schematics the following equations can be extracted:

$$\frac{1}{Y_1}I_1 + \frac{1}{Y_2}I_3 + \frac{1}{Y_3}I_3 = U_0 \tag{1}$$

$$\frac{1}{Y_1}I_1 - \frac{1}{Y_4}I_4 = U_0 - U_5 \tag{2}$$

$$I_1 + I_4 = I_3 = Y_3 \cdot U_3 = \frac{Y_3}{K} \cdot U_5 \tag{3}$$

Inserting equation (3) in equation (1) one gets:

$$I_1 = U_0 Y_1 - \frac{Y_1}{KY_2}(Y_2 + Y_3) \cdot U_5 \tag{4}$$

Inserting equation (4) in equation (2) one gets:

$$U_0 - U_5 = U_0 - \frac{U_5}{K} \cdot \frac{Y_2 + Y_3}{Y_2} - \frac{I_4}{Y_4}$$

$$I_4 = \frac{Y_4}{KY_2} \cdot (KY_2 - Y_2 - Y_3) \cdot U_5 \tag{5}$$

Now, by inserting I_1 from equation (4) and I_4 from equation (5) in equation (3) one finally gets:

$$\frac{U_5}{U_0} = H_\mathrm{L}(p) = \frac{K \cdot Y_1 \cdot Y_2}{Y_1(Y_2 + Y_3) + Y_3(Y_2 + Y_4) + Y_2 Y_4 (1 - K)} \tag{6}$$

3.3.3 Inserting $Y_1 = 1/R_1 = G_1$, $Y_2 = 1/R_2 = G_2$, $Y_3 = pC_3$ and $Y_4 = pC_4$ in equation (6):

$$
\begin{aligned}
H_\mathrm{L}(p) &= \frac{K \cdot G_1 \cdot G_2}{G_1(G_2 + pC_3) + pC_3(G_2 + pC_4) + G_2 \cdot pC_4(1 - K)} \\
&= \frac{K \cdot \dfrac{G_1 G_2}{C_3 C_4}}{p^2 + \left(\dfrac{G_1 + G_2}{C_4} + \dfrac{G_2(1-K)}{C_3} \right) \cdot p + \dfrac{G_1 G_2}{C_3 C_4}}
\end{aligned}
$$

3.3.4 The function $H_\mathrm{L}(p)$ describes a 2nd order low-pass.

3.3.5 By dividing equation (4) by U_0 one gets:

$$
\begin{aligned}
\frac{I_1}{U_0} = \frac{1}{Z_\mathrm{e}} &= Y_1 - \frac{Y_1}{KY_2}(Y_2 + Y_3) \cdot \frac{U_5}{U_0} \\
&= Y_1 - \frac{Y_1}{KY_2}(Y_2 + Y_3) \cdot H_\mathrm{L}(p)
\end{aligned}
$$

After some transformations the final result can be written as follows:

$$
Z_\mathrm{e} = \frac{1}{Y_1} + \frac{Y_2 + Y_3}{Y_3(Y_2 + Y_4) + Y_2 \cdot Y_4 \cdot (1 - K)}
$$