Problem 2.4

Given is the wanted transfer-function of an ideal digital band-pass:

$$H_{\mathrm{wF}}(\omega) = H_{\mathrm{bFBP}}(\omega) * \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_{\mathrm{a}})$$

For the baseband-function $H_{\mathrm{bFBP}}(\omega)$ the following holds:

$$H_{\mathrm{bFBP}}(\omega) = \mathrm{rect}\left(\frac{\omega}{\Delta\omega}\right) * \left\{\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right\}$$

with

$$\omega_0 = \frac{\omega_2 + \omega_1}{2}, \quad \Delta\omega = \omega_2 - \omega_1 \quad \text{and} \quad 0 < \omega_1 < \omega_2 < \frac{\omega_a}{2}$$

A causal digital FIR band-pass shall be designed, whos magnitude characteristics $|H_{\mathrm{rF}}(\omega)|$ approximate the magnitude characteristics $|H_{\mathrm{wF}}(\omega)|$ of the given ideal digital band-pass.

2.4.1 Derive a general equation for the calculation of the values of the unweighted impulse sequence $h_{\mathrm{bk}}(k)$ of the causal digital FIR band-pass. Given are the band-pass frequencies ω_1 and ω_2, the sampling- (and clock-) frequency ω_{a} and the number of filter coefficients $N = 2N_{\mathrm{f}} + 1$.

2.4.2 Determine the general equation for the calculation of the values of the weighted impulse sequence $h_{\mathrm{rk}}(k) = h_{\mathrm{wk}}(k) \cdot h_{\mathrm{bk}}(k)$ of the causal digital FIR band-pass, if the coefficients of the weighting function $h_{\mathrm{wk}}(k)$ are equal to a Hamming-sequence $h_{\mathrm{HMk}}(k)$.

2.4.3 Determine the values of the weighted impulse sequence $h_{\mathrm{rk}}(k)$ of the causal digital FIR band-pass and plot them over k. The following data for the realisation of the filter is given:

- Cut-off-frequency $f_1 = \omega_1/2\pi = 3.4$ kHz
- Cut-off-frequency $f_2 = \omega_2/2\pi = 3.8$ kHz
- Clock-frequency $f_{\mathrm{a}} = \omega_{\mathrm{a}}/2\pi = 8$ kHz
- Number of filter coefficients $N = 11$
- Weighting of the baseband-sequence $h_{\mathrm{bk}}(k)$ with the Hamming-windowing-sequence $h_{\mathrm{HMk}}(k)$

2.4.4 How big is the minimal number N_{V} of delay-elements necessary for the realisation of the causal digital FIR band-pass?

Problem 2.5

Given is the impulse response of a digital FIR filter:

$$h(k) = \sum_{m=0}^{M} a_m \cdot \gamma_0(k-m), \quad \text{with} \quad a_m \ \text{real}$$

2.5.1 Determine the z-transformed $H_z(z)$ of the impulse response $h(k)$.

2.5.2 Express $h_1(k) = h(-k)$ as a sum of weighted unit pulses.

2.5.3 Express $h_2(k) = h(M-k)$ as a sum of weighted unit pulses and calculate the z-transformed $H_{2z}(z)$ of $h_2(k)$.

2.5.4 At the input of the digital FIR filter the signal $s(k) = h(M-k)$ is applied. Before excitation the filter was at zero-state. Determine the output sequence $g(k)$ and its z-transformed $G_z(z)$.

2.5.5 At which position k the output sequence $g(k)$ shows its maximum?

Problem 2.6

Given is the wanted transfer-function of an ideal digital high-pass:

$$H_{\mathrm{wF}}(\omega) = H_{\mathrm{bFHP}}(\omega) * \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_{\mathrm{a}})$$

For the baseband-function $H_{\mathrm{bFHP}}(\omega)$ the following holds:

$$H_{\mathrm{bFHP}}(\omega) = \mathrm{rect}\left(\frac{\omega}{\omega_{\mathrm{a}}}\right) - \mathrm{rect}\left(\frac{\omega}{2\omega_{\mathrm{H}}}\right)$$

with

$$0 < \omega_{\mathrm{H}} < \frac{\omega_a}{2}$$

A causal digital FIR high-pass shall be designed, whos magnitude characteristics $|H_{\mathrm{rF}}(\omega)|$ approximate the magnitude characteristics $|H_{\mathrm{wF}}(\omega)|$ of the given ideal digital high-pass.

2.6.1 Derive a general equation for the calculation of the values of the unweighted impulse sequence $h_{\mathrm{bk}}(k)$ of the causal digital FIR high-pass. Given are the high-pass frequency ω_{H}, the sampling- (and clock-) frequency ω_{a} and the number of filter coefficients $N = 2N_{\mathrm{f}} + 1$.

2.6.2 Determine the general equation for the calculation of the values of the weighted impulse sequence $h_{\mathrm{rk}}(k) = h_{\mathrm{wk}}(k) \cdot h_{\mathrm{bk}}(k)$ of the causal digital FIR high-pass, if the coefficients of the weighting function $h_{\mathrm{wk}}(k)$ are equal to a Blackman-sequence $h_{\mathrm{Bk}}(k)$.

2.6.3 Determine the values of the weighted impulse sequence $h_{\mathrm{rk}}(k)$ of the causal digital FIR high-pass and plot them over k. The following data for the realisation of the filter is given:

- Cut-off-frequency $f_{\mathrm{H}} = \omega_{\mathrm{H}}/2\pi = 20$ kHz
- Clock-frequency $f_{\mathrm{a}} = \omega_{\mathrm{a}}/2\pi = 100$ kHz
- Number of filter coefficients $N = 9$
- Weighting of the baseband-sequence $h_{\mathrm{bk}}(k)$ with the Blackman-windowing-sequence $h_{\mathrm{Bk}}(k)$

2.6.4 How big is the minimal number N_{V} of delay-elements necessary for the realisation of the causal digital FIR high-pass?

Problem 2.7

Given is the wanted transfer-function of an ideal digital bandstop:

$$H_{\mathrm{wF}}(\omega) = H_{\mathrm{bFBS}}(\omega) * \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_{\mathrm{a}})$$

For the baseband-function $H_{\mathrm{bFBS}}(\omega)$ the following holds:

$$H_{\mathrm{bFBS}}(\omega) = \mathrm{rect}\Big(\frac{\omega}{\omega_{\mathrm{a}}}\Big) - \Big[\mathrm{rect}\Big(\frac{\omega}{\Delta\omega}\Big) * \Big\{\delta(\omega-\omega_0) + \delta(\omega+\omega_0)\Big\}\Big]$$

with

$$\omega_0 = \frac{\omega_2 + \omega_1}{2}, \quad \Delta\omega = \omega_2 - \omega_1 \quad \text{and} \quad 0 < \omega_1 < \omega_2 < \frac{\omega_a}{2}$$

A causal digital FIR bandstop shall be designed, whos magnitude characteristics $|H_{\mathrm{rF}}(\omega)|$ approximate the magnitude characteristics $|H_{\mathrm{wF}}(\omega)|$ of the given ideal digital bandstop.

2.7.1 Derive a general equation for the calculation of the values of the unweighted impulse sequence $h_{\mathrm{bk}}(k)$ of the causal digital FIR bandstop. Given are the bandstop frequencies ω_1 and ω_2, the sampling- (and clock-) frequency ω_{a} and the number of filter coefficients $N = 2N_{\mathrm{f}} + 1$.

2.7.2 Determine the general equation for the calculation of the values of the weighted impulse sequence $h_{\mathrm{rk}}(k) = h_{\mathrm{wk}}(k) \cdot h_{\mathrm{bk}}(k)$ of the causal digital FIR bandstop, if the coefficients of the weighting function $h_{\mathrm{wk}}(k)$ are equal to a Kaiser-sequence $h_{\mathrm{Kk}}(k)$.

2.7.3 Determine the values of the weighted impulse sequence $h_{\mathrm{rk}}(k)$ of the causal digital FIR bandstop and plot them over k. The following data for the realisation of the filter is given:

- Cut-off-frequency $f_1 = \omega_1/2\pi = 13$ kHz
- Cut-off-frequency $f_2 = \omega_2/2\pi = 9$ kHz
- Clock-frequency $f_{\mathrm{a}} = \omega_{\mathrm{a}}/2\pi = 40$ kHz
- Number of filter coefficients $N = 11$
- Weighting of the baseband-sequence $h_{\mathrm{bk}}(k)$ with the Kaiser-windowing-sequence $h_{\mathrm{Kk}}(k)$ with $\alpha = 5$

2.7.4 How big is the minimal number N_{V} of delay-elements necessary for the realisation of the causal digital FIR bandstop?

Problem 2.8

A periodic sequence $s_{\mathrm{p}}(k)$ with the period $(M+1)$ can be described as follows:

$$
\begin{aligned}
s_{\mathrm{p}}(k) &= 0 \quad \forall k < 0 \\
s_{\mathrm{p}}(k) &= s_{\mathrm{p}}(k + n \cdot \{M+1\}) \quad \forall k \geq 0 \ \text{ and } \ n = 0(1)\infty
\end{aligned}
$$

This sequence can be realised by adding to the impulse response $h(k)$ of a digital FIR filter all sequences $h(k - n \cdot \{M+1\})$ with $n = 1(1)\infty$. Further on

$$
h(k) = \sum_{m=0}^{M} a_m \cdot \gamma_0(k-m), \quad \text{with} \ \ a_m \ \ \text{real}
$$

2.8.1 Express $s_{\mathrm{p}}(k)$ as a sum using the impulse response $h(k)$.

2.8.2 Express $s_{\mathrm{p}}(k)$ as a sum of unit pulses $\gamma_0(k)$.

2.8.3 Determine the z-transformed $S_{\mathrm{pz}}(z)$ of $s_{\mathrm{p}}(k)$.

Problem 2.9

Given is a causal, linear, time-invariant analog system, with a system-function $H_\mathrm{L}(p)$ – the Laplace-transformed of the system impulse response $h(t)$ – which has only poles in the left open p-plane.

2.9.1 How the transfer-function $H_\mathrm{F}(\omega)$ of the analog system – which is the Fourier-transformed of the system-impulse response $h(t)$ – can be calculated from $H_\mathrm{L}(p)$? Give the correspondence between $H_\mathrm{F}(\omega)$ and $H_\mathrm{L}(p)$.

2.9.2 The analog system shall now be 'transformed' in a digital system. In the range $-\omega_\mathrm{a}/2 < \omega < +\omega_\mathrm{a}/2$ the transfer-function $H_\mathrm{wF}(\omega) = H_\mathrm{z}(z = \mathrm{e}^{\mathrm{j}\omega T_\mathrm{a}})$ of the digital system should correspond to the transfer-function $H_\mathrm{F}(\omega)$ of the analog system. The clock- and sampling-frequency of the digital system is equal to $\omega_\mathrm{a} = 2\pi/T_\mathrm{a}$.
Give an equation for $H_\mathrm{wF}(\omega)$, where the relation between $H_\mathrm{wF}(\omega)$ and $H_\mathrm{F}(\omega)$ is expressed as a convolution.

2.9.3 Calculate the Fourier-back-transformed $h_\mathrm{w}(t)$ of $H_\mathrm{wF}(\omega)$ for the following two cases:

2.9.3.1 Case 1: the wanted transfer-function $H_\mathrm{wF}(\omega)$ of a digital differentiator is in the range $-\omega_\mathrm{a}/2 < \omega < +\omega_\mathrm{a}/2$ equal to the transfer-function $H_\mathrm{DF}(\omega) = \mathrm{j}\omega/\omega_\mathrm{N}$ of an ideal analog differentiator. ω_N is the normalising frequency.

2.9.3.2 Case 2: the wanted transfer-function $H_\mathrm{wF}(\omega)$ of a digital Hilbert-transformer is in the range $-\omega_\mathrm{a}/2 < \omega < +\omega_\mathrm{a}/2$ equal to the transfer-function $H_\mathrm{HTF}(\omega) = -\mathrm{j} \cdot \mathrm{sign}(\omega/\omega_\mathrm{N})$ of an ideal analog Hilbert-transformer. ω_N is the normalising frequency.

2.9.4 Determine on basis of the results of 2.9.3, $\omega_\mathrm{N} = \omega_\mathrm{a}/2$ and $k = 0(1)2 \cdot N_\mathrm{f}$ with $N_\mathrm{f} = 5$ the values of the unweighted causal impulse responses $h_\mathrm{rkD}(k)$ and $h_\mathrm{rkHT}(k)$ of the approximated, causal digital FIR differentiator and Hilbert-transformer.