Chapter 1 – Introduction and Basics

Problem 1.1

Calculate the z-transformed of the following functions, specifying the range of convergence:

1.1.1   The unit pulse $\gamma_0(k)$.

1.1.2   The unit step sequence $\gamma_{-1}(k)$.

1.1.3   The exponential sequence $e^{p_1 k T_a}$ with $p_1 = \sigma_1 + j\omega_1$.

1.1.4   The sequence $s(k - \nu)$ with $\nu = 0(1)N$. The z-transformed $S_z(z)$ of the sequence $s(k)$ shows the value $s(k) = 0 \ \ \forall \ \ k < 0$.

Problem 1.2

Given are the poles and zeros of the system-function $H_z(z)$ of a digital-filter:

$$z_{\infty 1} = +j0.5 \qquad z_{\infty 2} = -j0.5 \qquad z_{01} = +0.5$$

1.2.1   Draw the pole zero plot.

1.2.2   Determine the system-function $H_z(z)$.

1.2.3   Draw the non-canonical 'direct structure' of the described digital-filter.

1.2.4   Draw a 'canonical structure' of the described digital-filter.

1.2.5   Determine the poles and zeros of the Laplace-Transformed $H_{aL}(p)$, which are related to the system function $H_z(z)$ over $H_{aL}(p) = H_z(z = e^{pT_a})$.

Problem 1.3

Given is the following difference equation

$$g(k) = a_0 \cdot s(k) - b_1 \cdot g(k-1) \qquad \text{with} \qquad s(k) = 0 \ \forall \ k < 0$$

of a digital filter. Before excitation the system was in zero state.

1.3.1   Draw the structure of the filter.

1.3.2   Is it a recursive or non-recursive system. Justify your answer.

1.3.3   Determine the system function $H_z(z)$ of the filter.

1.3.4   Determine the poles and zeros of $H_z(z)$.

1.3.5   Which conditions a_0 and b_1 have to fulfill in order to have a stable filter?

1.3.6   At the input of the digital filter, characterised by $a_0 = 0.1$ and $b_1 = 0.9$, the sequences $s(k) = \gamma_0(k)$ (Case 1), and $s(k) = \gamma_{-1}(k)$ (Case 2) are applied. Enlist for each case in the range of $k = 0(1)5$ the values of $a_0 \cdot s(k)$, $b_1 \cdot g(k-1)$ and $g(k)$.

1.3.7   For $a_0 = 0.1$ and $b_1 = 0.9$ determine the impulse response $h(k)$ from the system function $H_z(z)$. In the range $k = 0(1)5$ calculate the values of $h(k)$ and compare them with the results of 1.3.6.

1.3.8   Determine the poles and zeros of the Laplace-transformed

$$H_{\mathrm{aL}}(p) = H_z(z = e^{pT_{\mathrm{a}}})$$

1.3.9   Determine the back-transformed $h_{\mathrm{a}}(t)$ of $H_{\mathrm{aL}}(p)$ and determine from it $H_{\mathrm{aL}}(p)$.

Problem 1.4

The given system function has no zeros but one pole at $z_{\infty 1} = 0.5$. Further on $H_z(0) = -1$.

1.4.1 Determine the system function $H_z(z)$ of the filter.

1.4.2 Determine the reaction $g(k)$ of the filter for the inputs $s(k) = \gamma_0(k)$ and $s(k) = \gamma_{-1}(k)$.

1.4.3 Determine the poles and zeros of the Laplace-transformed $H_{aL}(p) = H_z(z = e^{pT_a})$.

1.4.4 Determine the poles and zeros of the Laplace-transformed $H_L(p)$, which is related to $H_z(z)$ according to:

$$H_z(z = e^{pT_a}) = \frac{1}{T_a} \cdot \sum_{n=-\infty}^{+\infty} H_L(p - jn\omega_a) \quad \text{with} \quad \omega_a = \frac{2\pi}{T_a}$$
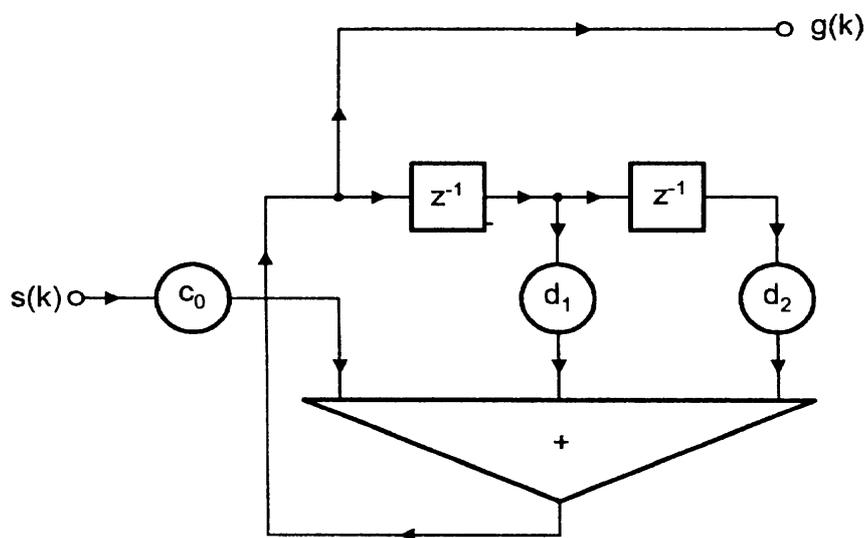
1.4.5 Determine the system function $H_L(p)$ and the Laplace back-transformed $h(t)$ for the case $H_L(0) = 1$.

1.4.6 Determine the absolute value $|H_{aF}(\omega)|$, the phase $\varphi(\omega)$ and the group delay $\tau_g(\omega) = -\partial\varphi(\omega)/\partial\omega$ of the transfer function

$$H_{aF}(\omega) = H_z(z = e^{j\omega T_a}) = |H_{aF}(\omega)| \cdot e^{j\varphi(\omega)}$$

as function from ω and draw the values of these three variables over ω in the range $\omega_\nu = \nu \cdot \omega_a/20$ with $\nu = 0(1)10$.

Problem 1.5

The following canonical structure of a digital filter is given.

1.5.1 Determine the difference equation of the filter.

1.5.2 Determine the system function $H_z(z)$ of the digital filter.

1.5.3 Develop a further canonical structure.

1.5.4 Determine the poles and zeros of the system function $H_z(z)$.

1.5.5 A digital filter is stable if the poles of the system function $H_z(z)$ are inside of the unit circle of the z-plane. Give the conditions d_1 and d_2 have to fulfill for a stable filter. Describe the conditions for the four cases:

   Condition 1: $d_1 > 0$ and $d_2 > 0$ Condition 2: $d_1 > 0$ and $d_2 < 0$
   Condition 3: $d_1 < 0$ and $d_2 > 0$ Condition 4: $d_1 < 0$ and $d_2 < 0$

1.5.6 Determine for $d_1 = -0.8$ and $d_2 = -0.25$ the impulse response $h(k)$ of the filter and draw $h(k)$ over k in the range $k = 0(1)15$.

1.5.7 Determine for $d_1 = -0.8$ and $d_2 = -0.25$ the magnitude of

$$|H_{aF}(\omega)| = |H_z(z = e^{j\omega T_a})|$$

of the filter and draw $|H_{aF}(\omega)|$ over ω_ν in the range $\omega_\nu = \nu \cdot \omega_a/20$ with $\nu = 0(1)10$.

Problem 1.6

A digital filter reacts to the unit pulse $\gamma_0(k)$ with the finite and causal response $g(k)$ according to the following table.

k	0	1	2	3	4	$k > 4$
$g(k)$	1	0	1/4	0	1/16	0

1.6.1 Draw $g(k)$.

1.6.2 Determine the corresponding difference equation of the digital filter.

1.6.3 Determine the system function $H_z(z)$ of the digital filter.

1.6.4 Draw the pole-zero-plot of the system function $H_z(z)$.

1.6.5 Determine the reaction $g_1(k)$ of the digital filter to the unit step sequence $\gamma_{-1}(k)$.

1.6.6 Give the z-transformed $G_{1z}(z)$ of $g_1(k)$.

1.6.7 Determine the magnitude of $|H_{aF}(\omega)| = |H_z(z = e^{j\omega T_a})|$ of the filter and draw $|H_{aF}(\omega)|$ in the range $0 \leq \omega \leq \omega_a/2$ replacing ω by $\omega_\nu = \nu \cdot \omega_a/20$ with $\nu = 0(1)10$.

1.6.8 Sketch the structure of the digital filter.

Problem 1.7

A digital filter, which originally was in the zero state reacts to the unit pulse $\gamma_0(k)$ with the infinite series

$$g(k) = \sum_{\nu=0}^{\infty} \nu \cdot \gamma_0(k - \nu)$$

1.7.1 Draw $g(k)$ over k for $k = 0(1)10$.

1.7.2 Determine the system function $H_z(z)$.

1.7.3 Draw the pole-zero-plot of the system function $H_z(z)$.

Problem 1.8

A digital filter, which originally was in the zero state reacts to the unit pulse $\gamma_0(k)$ with the infinite series

$$g(k) = \sum_{\nu=0}^{\infty} (1 + \nu) \cdot \gamma_0(k - \nu)$$

1.8.1   Draw $g(k)$ over k for $k = 0(1)10$.

1.8.2   Determine the system function $H_z(z)$.

1.8.3   Draw the pole-zero-plot of the system function $H_z(z)$.

1.8.4   Determine the difference equation of the digital filter.

1.8.5   Draw a canonical structure of the filter.

Problem 1.9

Given is the input sequence

$$s_n(k) = \sum_{\nu=0}^{\infty} (C_n)^k \cdot \gamma_0(k - \nu) \quad \text{with} \quad n = 1(1)4$$

for

$C_1 = +0.5$ (Case 1), $C_2 = -0.5$ (Case 2), $C_3 = +1.5$ (Case 3), $C_4 = -1.5$ (Case 4).

1.9.1   Draw the respective sequence $s_n(k)$ over k for $k = 0(1)10$.

1.9.2   Determine for the four cases the z-transformed $S_{nz}(z)$ of $s_n(k)$.

1.9.3   Draw for the four cases the respective pole-zero-plot of $S_{nz}(z)$.

1.9.4   Draw one canonical structure of a digital filter, which reacts to the unit pulse $\gamma_0(k)$ with $s_n(k)$.

Problem 1.10

At the input of a digital FIR filter with the impulse response

$$h(k) = 3\gamma_0(k) + 4\gamma_0(k-1)$$

the sequence

$$s(k) = 2\gamma_0(k) + \gamma_0(k-1)$$

is applied. Before the filter was in zero state.

1.10.1   Determine the system function $H_z(z)$ of the digital filter, as well as the z-transformed $S_z(z)$ of the input sequence $s(k)$.

1.10.2   Determine the z-transformed $G_z(z)$ of the output sequence $g(k)$, for the case that the given sequence $s(k)$ is applied at the input.

1.10.3   Determine the output sequence $g(k)$ and draw $g(k)$ over k.

1.10.4   Determine a system function $H_{1z}(z)$ of a new digital filter, which reacts to the input of the unit step sequence $\gamma_{-1}(k)$ with the in 1.10.3 calculated output $g(k)$.

1.10.5   Give the impulse response $h_1(k)$ of the digital filter designed in 1.10.4.

1.10.6   Draw a canonical structure of the digital filter designed in 1.10.4.

Problem 1.11

A digital filter is described by the difference equation

$$g(k) = s(k) + 0.5g(k-1)$$

1.11.1   Under the assumption that the digital filter originally was in the zero state, determine the first six values of the reaction $g(k)$ of the filter to the input sequence

$$s(k) = \gamma_0(k) - 2\gamma_0(k-1) + 2\gamma_0(k-2) + \gamma_0(k-3)$$

and draw $g(k)$ for $k = 0(1)5$ as function of k.

Problem 1.12

Determine a general expression for the reaction $g(k)$ of a digital filter to the input sequence

$$s(k) = \begin{cases} 0 & \forall \quad k < 0 \\ 1 & \forall \quad k = 0, 2, 4, 6, \ldots \\ -1 & \forall \quad k = 1, 3, 5, 7, \ldots \end{cases}$$

when the filter can be described by the difference equation

$$g(k) = a_0 s(k) - b_1 g(k-1), \quad \text{with} \quad a_0, b_1 \quad \text{real.}$$

Before excitation the filter was in the zero state.

Problem 1.13

Determine the first five values of the sequences $s(k)$, whos z-transformed are equal to
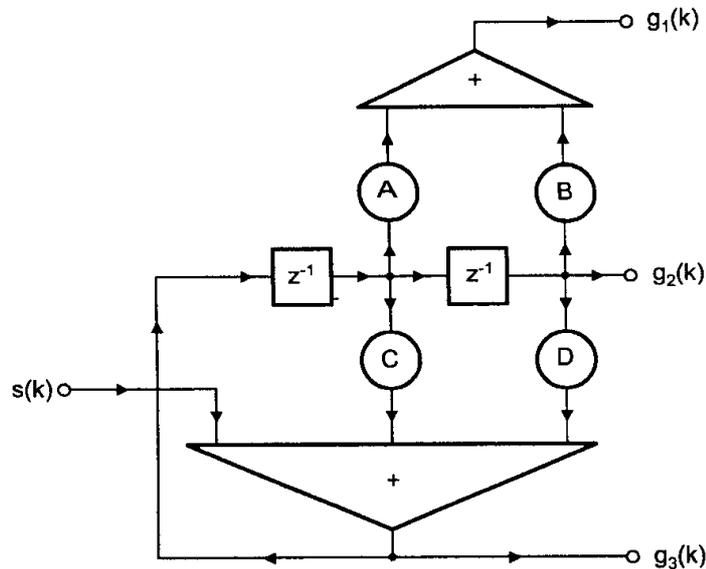
$$S_z(z) = \frac{z^{-1} + z^{-2}}{1 - z^{-2}}$$

and

$$S_z(z) = \frac{1 + 2z^{-1}}{1 - 0.5z^{-1}}$$

Therefore use the:

a) method of the division of the numerator-polynomial by the denominator-polynomial as well as the

b) expansion of $S_z(z)$ in a sum of partial fractions.

Problem 1.14

Given is the canonical structure of a digital filter build up by two delay-elements (clock-period T_a) and two adder.

1.14.1　Determine the describing difference equations and system functions $H_{\nu z}(z)$ of the three digital sub-systems, defined by $s(k) \to g_1(k)$, $\quad s(k) \to g_2(k)$ and $s(k) \to g_3(k)$.

1.14.2　The three filter sub-systems are of which order?

1.14.3　Determine for $A = 0$, $B = 1$, $C = -2$ and $D = -0.19$ the impulse responses of the three digital sub-systems and draw them for $k = 0(1)8$ over k.

1.14.4　Determine for $A = 0$, $B = 1$, $C = -2$ and $D = -0.19$ the output-sequences $g_1(k)$, $g_2(k)$ and $g_3(k)$ of the three digital sub-systems for the case that at the input $s(k)$ the unit step sequence $\gamma_{-1}(k)$ has been applied. Before excitation the filter was at zero state.

Problem 1.15

Given are the following recursive difference equations, usually used for the numerical integration:

Trapezium-Law: $g_\mathrm{T}(k) = g_\mathrm{T}(k-1) + \frac{1}{2}s(k) + \frac{1}{2}s(k-1)$

Simpson-Law: $g_\mathrm{S} = g_\mathrm{S}(k-2) + \frac{1}{3}s(k) + \frac{4}{3}s(k-1) + \frac{1}{3}s(k-2)$

Newton-3/8-Law: $g_\mathrm{N} = g_\mathrm{N}(k-3) + \frac{3}{8}s(k) + \frac{9}{8}s(k-1) + \frac{9}{8}s(k-2) + \frac{3}{8}s(k-3)$

Each of these difference equations should be realised by a respective digital IIR filter, which works with the sampling (clock-) frequency $\omega_\mathrm{a} = 2\pi/T_\mathrm{a}$.

1.15.1   Determine the system functions $H_\mathrm{Tz}(z)$, $H_\mathrm{Sz}(z)$ and $H_\mathrm{Nz}(z)$ and draw the respective pole-zero-plots.

1.15.2   Determine the transfer function $H_\mathrm{aF}(\omega) = H_\mathrm{z}(z = \mathrm{e}^{\mathrm{j}\omega T_\mathrm{a}})$ of each filter, express $H_\mathrm{aF}(\omega)$ by sin- and cos-functions and draw $|H_\mathrm{aF}(\omega)|$ in the range $0 \leq \omega \leq \omega_a/2$ replacing ω by $\omega_\nu = \nu \cdot \omega_\mathrm{a}/20$ with $\nu = 0(1)10$.

1.15.3   Determine the impulse responses $h_\mathrm{T}(k)$, $h_\mathrm{S}(k)$ and $h_\mathrm{N}(k)$ of the causal digital IIR filter and draw the impulse responses in the range $k = 0(1)6$.

1.15.4   Draw a canonical structure for each digital IIR filter and place the values of the multiplying coefficients in the drawing.

Problem 1.16

A causal digital filter for the mean calculation of $M + 1$ consecutive values of a causal signal $s(k)$ (a so called digital averager) is described by the following difference equation (with $M > 1$ and integer):

$$g(k) = \frac{1}{M + 1} \cdot \sum_{\nu = k - M}^{k} s(\nu)$$

The clock frequency is given by $\omega_\mathrm{a} = 2\pi/T\mathrm{a}$.

1.16.1 Is the system recursive or not? Justify your answer.

1.16.2 Show that the upper difference equation can be transformed into

$$g(k) = \frac{1}{M + 1} \cdot \sum_{m = 0}^{M} s(k - m)$$

1.16.3 Determine the impulse response $h(k)$ and the related system function $H_\mathrm{z}(z)$ of the digital system.

1.16.4 Determine the transfer function $H_\mathrm{aF}(\omega) = H_\mathrm{z}(z = \mathrm{e}^{\mathrm{j}\omega T_\mathrm{a}})$ and transform $H_\mathrm{aF}(\omega)$ in such way that $|H_\mathrm{aF}(\omega)|$ can be expressed by by sin- and cos-functions.

1.16.5 Draw the impulse response $h(k)$ in the range $k = 0(1)5$ over k, with $M = 4$.

1.16.6 Draw for $M = 4$ the magnitude of $|H_\mathrm{aF}(\omega)|$ in the range $0 \leq \omega \leq \omega_a/2$ replacing ω by $\omega_\nu = \nu \cdot \omega_\mathrm{a}/20$ with $\nu = 0(1)10$.