Fachgebiet Nachrichtentechnische Systeme

Network Theory 2
Digital Filters

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- Detail description of two methods (impulse invariance and the the bilinear z-transform)

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Chapter 1

Introduction

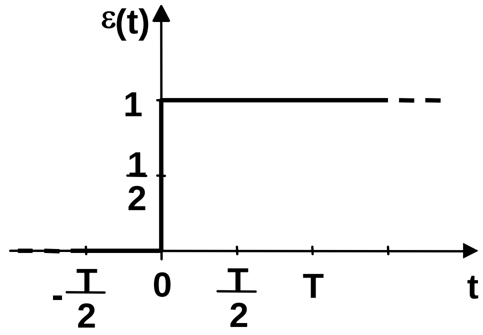
1 Introduction

- Digital Filters are discrete counterparts of analog Filters
- Digital Filters need discrete signal procesing unit and analog and analog/digital components
- Used in growing number of applications
- Found in DSP chips, mobile communication, audio/video processing
- Has properties close to analog filters but not in all details (good&bad)
- Enable to set-up high precision filtering/signal processing
- Ability for multiprocessing
- Ability for changing filter properties on the fly
- Limited frequency band
- Higher circuit effort

1.1 Basic signals

The step function

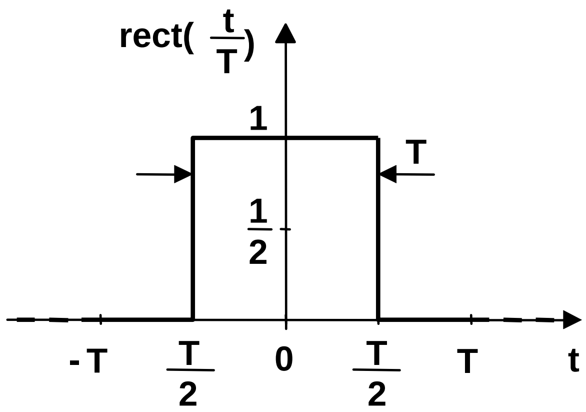
$$\varepsilon(t) = \begin{cases} 1 & \text{für } t > 0 \\ 1/2 & \text{für } t = 0 \\ 0 & \text{für } t < 0 \end{cases}$$



The rect function

$$rect\left(\frac{t}{T}\right) = \varepsilon\left(\frac{t+\frac{1}{2}T}{T}\right) - \varepsilon\left(\frac{t-\frac{1}{2}T}{T}\right)$$

$$= \varepsilon\left(t+\frac{1}{2}T\right) - \varepsilon\left(t-\frac{1}{2}T\right) = \begin{cases} 1/2 & \text{for} \quad t = \pm(1/2)T \\ 1 & \text{for} \quad -\frac{T}{2} < t < \frac{T}{2} \\ 0 & \quad \text{elsewhere} \end{cases}$$



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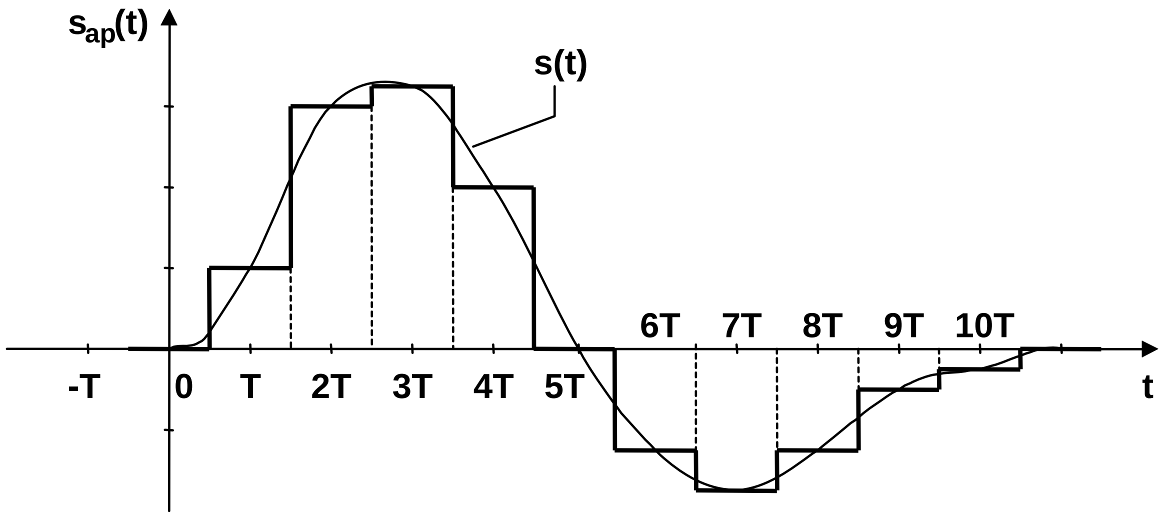
1.1 Basic signals

Approximation of arbitrary analog signals:

$$s_{ap}(t) = \sum_{n=-\infty}^{+\infty} s(nT) \cdot rect\left(\frac{t-nT}{T}\right) \approx s(t)$$



Signal *s(t)* (thin line) and approximating signal $s_{ap}(t)$

Error over time: $s_{\Delta}(t) = s(t) - s_{ap}(t)$ improves for smaller T

1.1 Basic signals

The mean square error F within the approximated interval $t_u \le t \le t_o$:

$$F(s(t),T) = \int_{t_u}^{t_o} s_\Delta{}^2(t)\,dt = \int_{t_u}^{t_o} \left(s(t) - \sum_{n=-\infty}^{+\infty} s(nT) \cdot rect\left(\frac{t-nT}{T} \right) \right)^2 dt$$

It is possible to determine a certain width T_0 such that a certain error F_0 is reached:

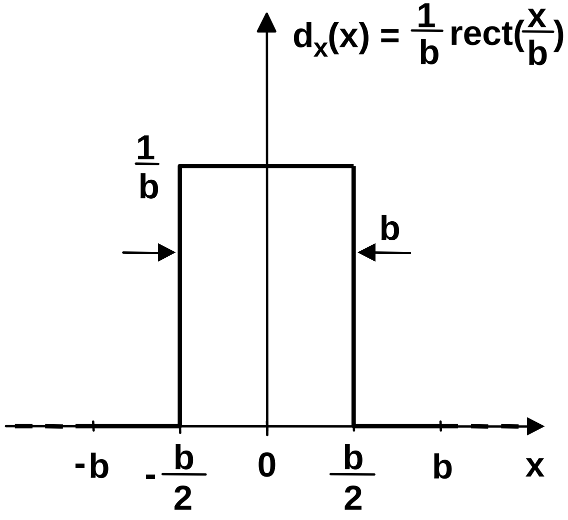
$$F(s(t),T_0) = \int_{t_u}^{t_o} \left(s(t) - \sum_{n=-\infty}^{+\infty} s(nT_0) \cdot rect\left(\frac{t-nT_o}{T_0} \right) \right)^2 dt = F_0$$

<u>The Dirac-distribution</u>

Starting point for the interpretation of the unit pulse by means of the equation:

$$d_x(x) = \frac{1}{b} \cdot rect\left(\frac{x}{b} \right)$$ with b being positive and real

1.1 Basic signals



It applies:

$$\int_{-\infty}^{+\infty} d_x(x)\, dx = \int_{-\infty}^{+\infty} \frac{1}{b} \cdot rect(\frac{x}{b})\, dx = A = 1$$

(For all positive and real b)

Rect function

If one multiplies this function by a given function a windowing of this given function is realised.

In a limiting process with finally infitely small *b* the Dirac function is obtained.

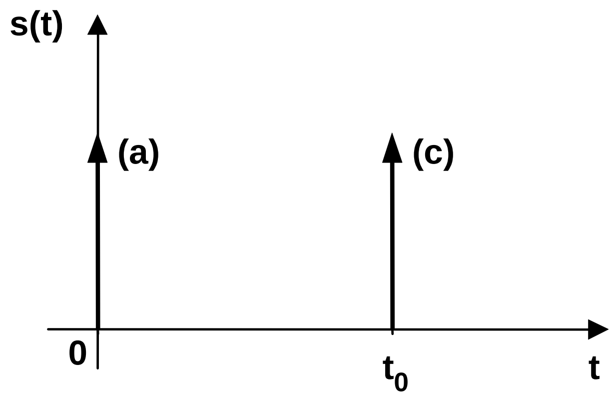
1.1 Basic signals

It holds:

$$\int_{-\infty}^{+\infty} \delta(t)\, dt = 1$$

$$\int_{-\infty}^{+\infty} \delta(t-t_0)\cdot y(t)\, dt = y(t_0) \quad \textit{for all } t_0 \textit{ with } -\infty < t_0 < +\infty$$

Example:



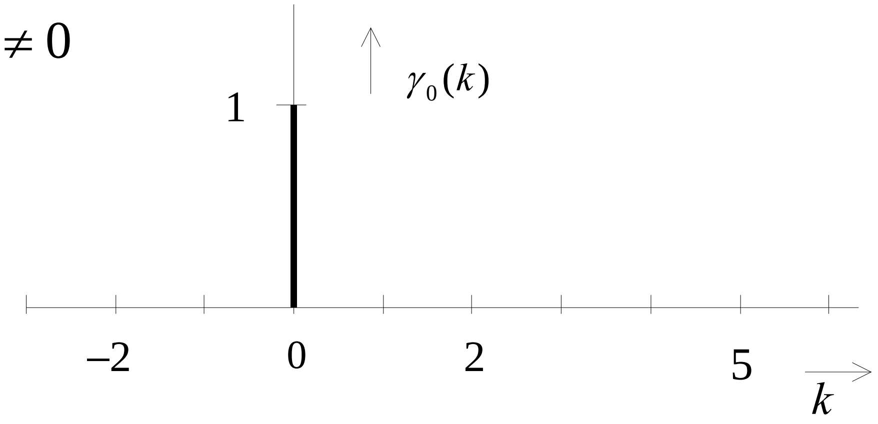
$$s(t) = a\cdot\delta(t) + c\cdot\delta(t-t_0)$$

1.1 Basic Signals

Additional basic sequences (basic discrete signals)

The unit-impulse:

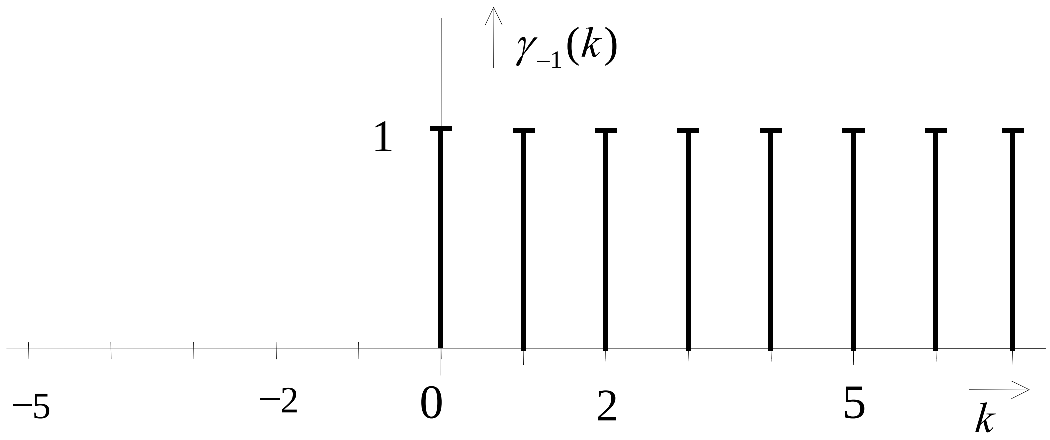
$$\{s(k)\} = \gamma_0(k) = \begin{cases} 1 & \text{for} \quad k = 0 \\ 0 & \text{for} \quad k \neq 0 \end{cases}$$



1.1 Basic Signals

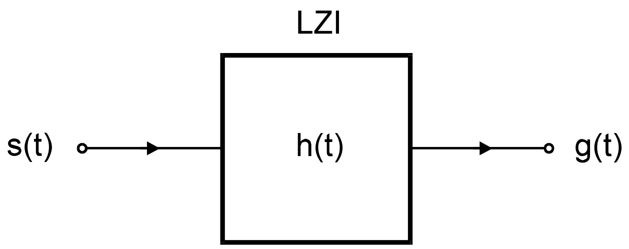
The unit-step sequence:

$$\gamma_{-1}(k) = \begin{cases} 0 & \text{for} \quad k < 0 \\ 1 & \text{for} \quad k \geq 0 \end{cases}$$



1.2 Signals at analog LTI systems

Description of the system properties in the time domain:



$$g(t) = s(t) * h(t) = \int_{-\infty}^{+\infty} s(\tau) \cdot h(t-\tau) d\tau$$

$$= h(t) * s(t) = \int_{-\infty}^{+\infty} h(\tau) \cdot s(t-\tau) d\tau$$

with $h(t)$ as the impulse response

Further possibility for determination of the output signal is given by means of solving a linear inhomogeneous differential equation:

$$\sum_{n=0}^{N} b_n \cdot \frac{d^{(n)} g(t)}{dt^n} = \sum_{m=0}^{M} a_m \cdot \frac{d^{(m)} s(t)}{dt^m}$$

But: Description, computation and analysis of signals and LTI systems is often easier in the frequency range, i.e. using **the Laplace** or **the Fourier transform**.

1.2 Signals at analog LTI systems

Laplace transform and the system function $H_L(p)$

A basis for the description of signals within the p-domain is:

$$S_L(p) = \int_0^{+\infty} s(t) \cdot e^{-pt}\, dt, \text{ where } s(t) = 0 \text{ for } t < 0 \text{ (causality condition)}$$

The inverse Laplace transform is determined by:

$$s(t) = \frac{1}{2\pi j} \cdot \oint_C S_L(p) \cdot e^{pt}\, dp$$

With the help of the Laplace transform, the **system function** can be defined:

$$\begin{array}{ccccc} g(t) & = & s(t) & * & h(t) \\ \updownarrow \text{L} & & \updownarrow \text{L} & & \updownarrow \text{L} \\ G_L(p) & = & S_L(p) & \cdot & H_L(p) \end{array}$$

$$H_L(p) = \frac{G_L(p)}{S_L(p)}$$

1.2 Signals at analog LTI systems

The Fourier transform and the transfer function $H_F(\omega)$:

The Fourier transform of the signal *s(t)* and its inverse transform:

$$S_F(\omega) = \int_{-\infty}^{+\infty} s(t) \cdot e^{-j\omega t} dt \qquad s(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_F(\omega) \cdot e^{j\omega t} d\omega$$

Some basic properties of the transforms:

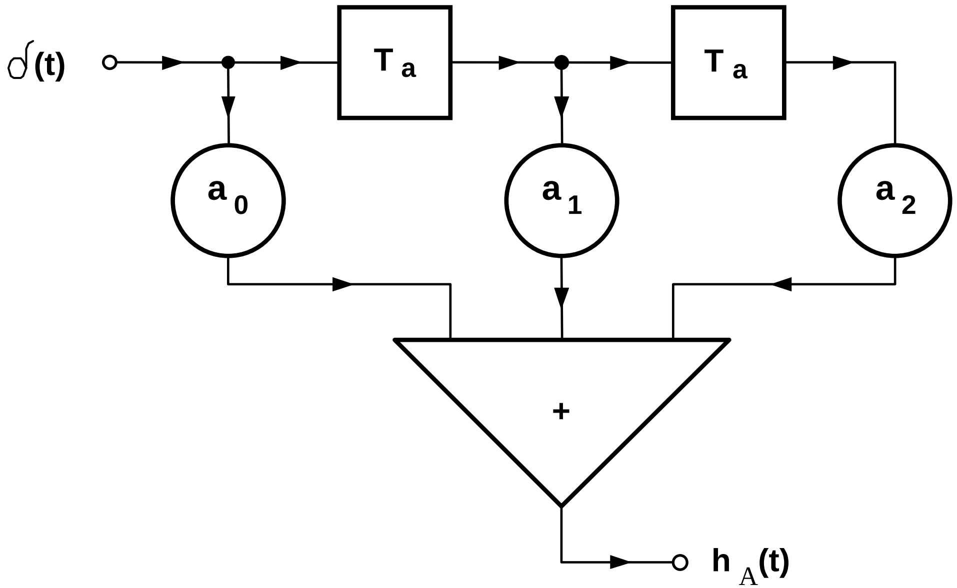
1. Stable systems have all the poles of the system function $H_L(p)$ in the left open p-half plane
2. Transfer function in this case is also given by: $H_F(\omega) = H_L(j\omega)$
3. Under the condition of a causal impulse response with poles only in left open p-plane, one can write:

$$H_L(p) = H_F(\omega)\big|_{j\omega \to p}$$

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

For discrete systems the z-transform is used. The following shows why by means of an example.

Example: An analog system (transverse filters) is given as follows:



1.3 The Z-transform and its relationship with the Laplace and Fourier transform

The impulse response can be determined as follows:

$$h_a(t) = a_0 \cdot \delta(t) + a_1 \cdot \delta(t - T_a) + a_2 \cdot \delta(t - 2T_a)$$

A Laplace transformation gives:

$$\delta(t - kT_a) \;\circ\!\!\stackrel{L}{\longrightarrow}\!\!\bullet\; e^{-pkT_a}$$

Thus the system function (a non-rational function in p) gives:

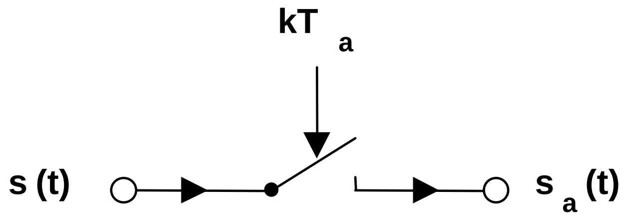
$$H_{aL}(p) = a_0 + a_1 \cdot e^{-pT_a} + a_2 \cdot e^{-2pT_a}$$

A substitution with $z = e^{pT_a}$ gives a rational fractioned system function in z .

$$H_z(z) = a_0 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2} = \frac{a_0 \cdot z^2 + a_1 \cdot z + a_2}{z^2}$$

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

Additional relations of z and Laplace transform are shown using an "**Ideal sampler**":



Assuming that the input signal *s(t)* is a causal and real signal, one gets:

$$s_a(t) = s(t) \cdot \sum_{k=-\infty}^{+\infty} \delta(t - kT_a) = \sum_{k=-\infty}^{+\infty} s(kT_a) \cdot \delta(t - kT_a)$$

$$= \sum_{k=0}^{+\infty} s(kT_a) \cdot \delta(t - kT_a) \text{ (because of the causality of } s(t)\text{)}$$

If the Laplace transform is applied, it is obtained:

$$S_{aL}(p) = \sum_{k=0}^{+\infty} s(kT_a) \cdot e^{-pkT_a}$$

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

This equation can be rewritten, by implementing the following steps:

Step 1: Abbreviation $s(kT_a) \rightarrow s(k)$ gives a new discrete function

Step 2: $z = e^{pT_a}$ with the complex frequency $p = \sigma + j\omega$

Step 3: Combination of step 1+2, one gets:

$$S_{aL}(p) = \sum_{k=0}^{+\infty} s(k) \cdot z^{-k} = S_z(z)$$

giving the z-transform of $s(k)$

For example:

$$s(k) = a_1 \cdot s_1(k) + a_2 \cdot s_2(k)$$

$\circ\!\!-\!\!\bullet$ z

$$S_z(z) = a_1 \cdot S_{1z}(z) + a_2 \cdot S_{2z}(z)$$

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

The Laplace transform can always be recovered from the z-transform:

$$S_{aL}(p) \;=\; S_z(e^{pT_a})$$

The inverse z-transform is defined as follows:

$$s(k) \;=\; \frac{1}{2\pi j} \cdot \oint_C S_z(\mathrm{z}) \cdot z^{k-1}\,\mathrm{dz}$$

Examples: Two basic functions and their z-transform:

The unit impulse

$$s(k) \;=\; \gamma_0(k)$$

$$\circ\!\!-\!\!\bullet\ \mathrm{z} \qquad\qquad \circ\!\!-\!\!\bullet\ \mathrm{z}$$

$$S_z(z) \;=\; 1$$

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

The unit step sequence

$$s(k) \quad = \gamma_{-1}(k) \qquad \longleftrightarrow \qquad \gamma_{-1}(k) = \sum_{\nu=0}^{\infty} \gamma_0(k-\nu)$$

$\circ\!\!-\!\!\bullet$ z $\qquad$ $\circ\!\!-\!\!\bullet$ z

$$S_z(z) \quad = \quad \frac{z}{z-1}$$

Under the condition $\qquad \mathrm{Re}\left\{ p_{\infty\upsilon} \right\} \; = \; \sigma_{\infty\upsilon} \; < \; 0$

of a Laplace transform of a signal *s(t)* (i.e. all poles of the transform are in the left open p-plane), the following is valid:

$$S_{aL}(p) \; = \; S_z(e^{pT_a})$$

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1.3 The Z-transform and its relationship with the Laplace and Fourier transform

Then it is also valid: $S_{aF}(\omega) = S_{aL}(j\omega) = S_z(e^{j\omega T_a})$

and thus the following relations:

Magnitude spectrum: $\left|S_{aF}(\omega)\right| = \left|S_{aL}(j\omega)\right| = \left|S_z(e^{j\omega T_a})\right|$

Phase spectrum: $\varphi_{aF}(\omega) = \sphericalangle S_{aF}(\omega) = \arctan\left(\frac{\mathrm{Im}\left\{S_z(e^{j\omega T_a})\right\}}{\mathrm{Re}\left\{S_z(e^{j\omega T_a})\right\}}\right)$

Due to $z = e^{pT_a}$ important mappings for a simplified description of discrete LTI-analog system can be observed:

- Property 1: The $j\omega$-axis of the p-plane $\leftrightarrow$ the unit circle of the z-plane
- Property 2: The left open p-plane $\leftrightarrow$ area inside of the unit circle of z-plane
- Property 3: The right open p-plane $\leftrightarrow$ areas outside of the unit circle of z-plane

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

Property 1 can be further examined by setting $p = j\omega$.

This gives: $z = |z| \cdot e^{j\sphericalangle z} = e^{j\omega T_a}$ ⇨ **Pointer z is always on the unit circle**

with $|z| = 1$, $\sphericalangle z = \sphericalangle e^{pT_a} = \omega T_a$

and $\omega_a = 2\pi f_a = \frac{2\pi}{T_a} \Rightarrow T_a = \frac{2\pi}{\omega_a}$

Example: The pointer p now moves on the $j\omega$-axes from the point

$$p_{bu} = -j\frac{\omega_a}{2} = -j\frac{\pi}{T_a} \quad \text{to} \quad p_{bo} = +j\frac{\omega_a}{2} = +j\frac{\pi}{T_a}$$

Thus z moves on the unit circle starting at $z = -1$ on the unit circle counter clockwise again to the point $z = -1$.

Reason: $e^{p_{bu,0}T_a} = e^{\mp j\pi} = -1$

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

Application: **Determination of the frequency response**

For the frequency response within the base band range $-\omega_a/2 \le \omega \le +\omega_a/2$ holds:

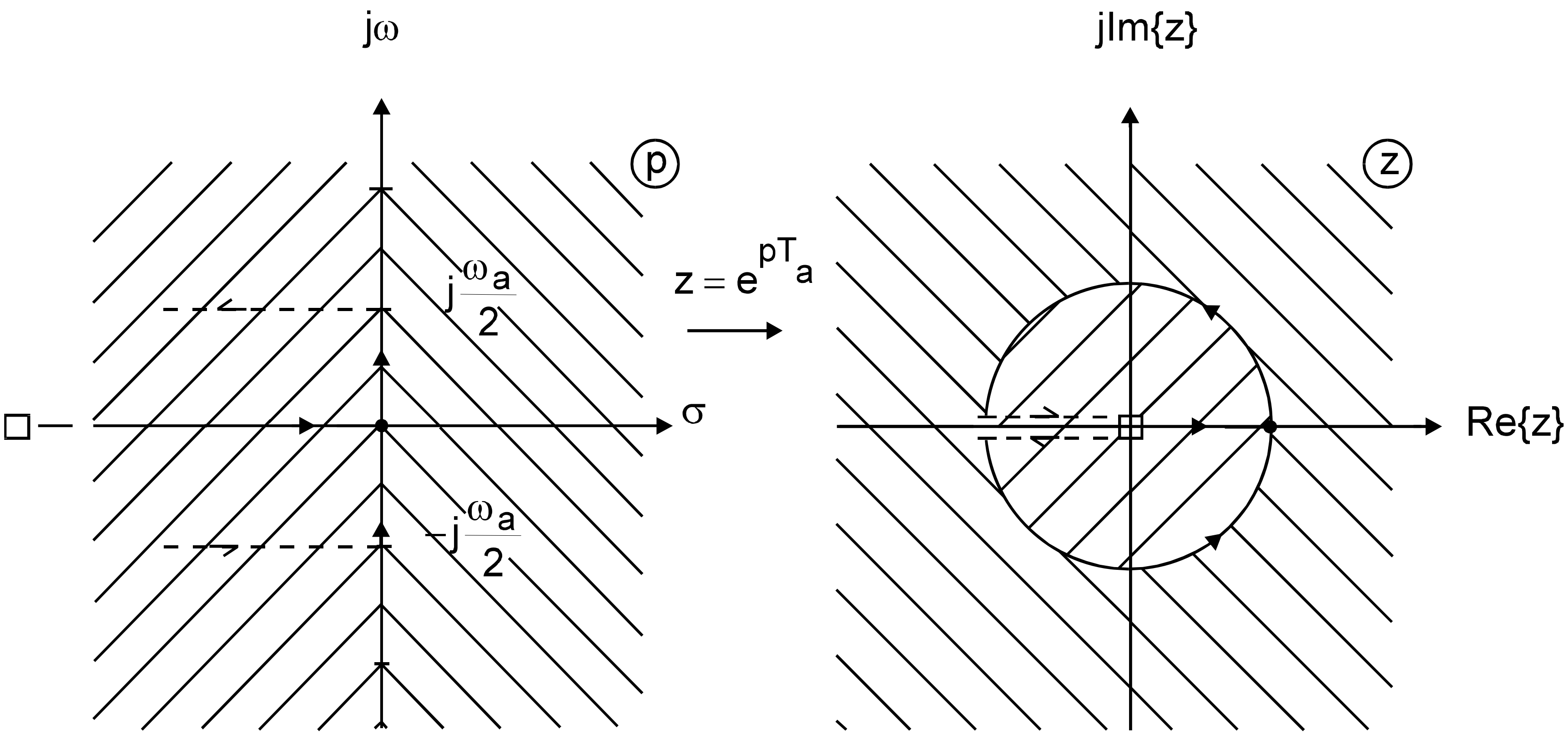
$$S_{bF}(\omega) = rect\left(\frac{\omega}{\omega_a}\right) \cdot S_{aF}(\omega)$$

This is a function which follows the z-values once around the unit circle.

A repetition of the base band frequency response gives the whole periodic frequency response:

$$S_{aF}(\omega) = S_{bF}(\omega) * \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_a)$$

1.3 The Z-transform and its relationship with the Laplace and Fourier transform



Mapping of the p-plane into the z-plane by the function $z = e^{pT_a}$

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

Property 2: With p being on the left half plane of the pole zero diagram gives:

$$\mathrm{Re}\{p\} = \sigma = -|\sigma| < 0$$

Thus it results: $$z = e^{pT_a} = e^{(-|\sigma| + j\omega)T_a} = e^{-|\sigma| T_a} \cdot e^{j\omega T_a}$$

From the above equation it is clear that:

$$|z| = e^{-|\sigma| T_a} < 1 \text{ for all } |\sigma| < 0$$

z is always inside of the unit circle

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

Property 3: With the condition that p is on the right half plane of pole zero diagram, the following is valid:

$$\mathrm{Re}\{p\} = \mathrm{Re}\{\sigma + j\omega\} = \sigma > 0 = |\sigma|$$

It results:

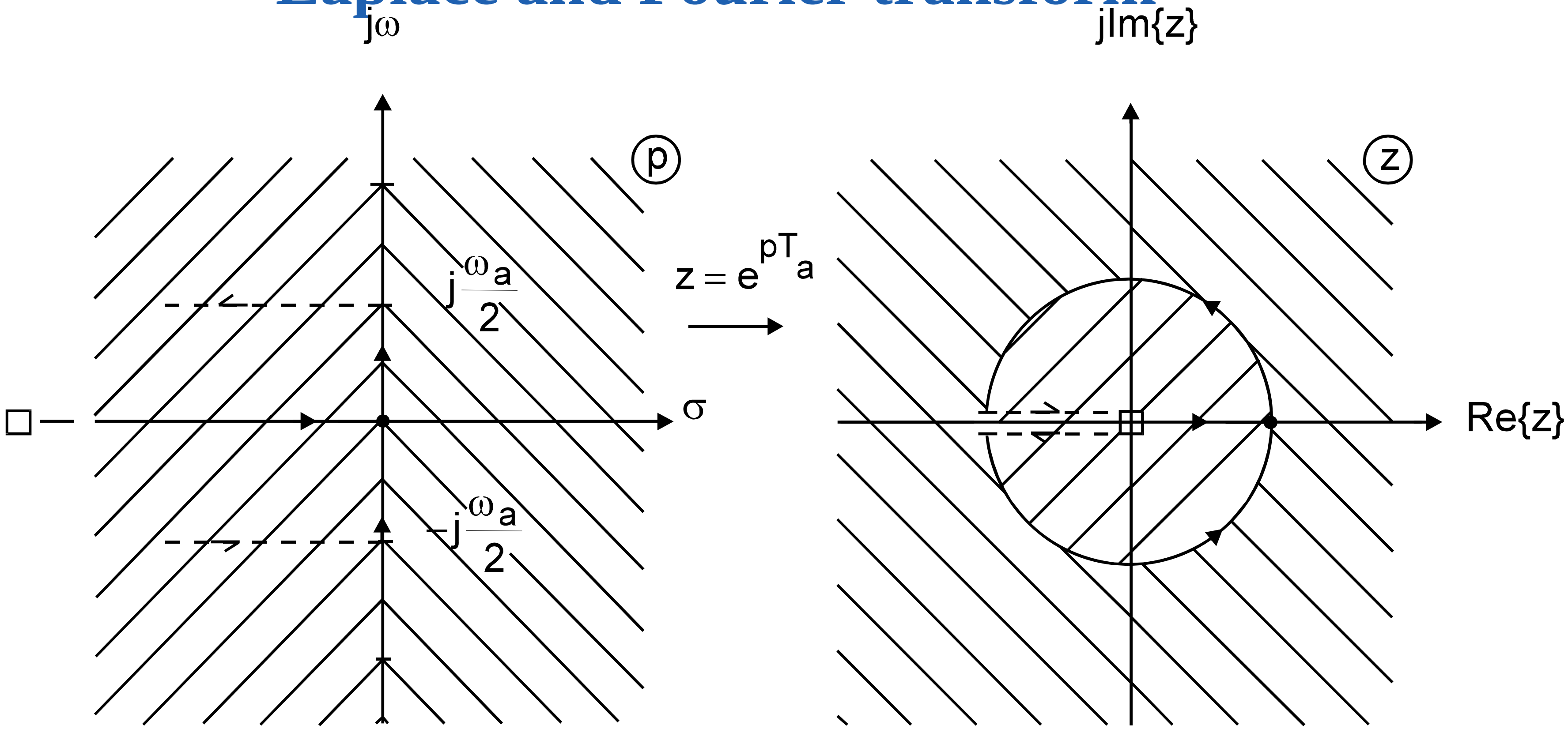
$$z = e^{pT_a} = e^{(|\sigma| + j\omega)T_a} = e^{|\sigma|T_a} \cdot e^{j\omega T_a}$$

From that it is clear that:

$$|z| = e^{|\sigma|T_a} > 1 \quad \text{for all} \, |\sigma| > 0$$

Here z covers the area outside of the unit circle

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

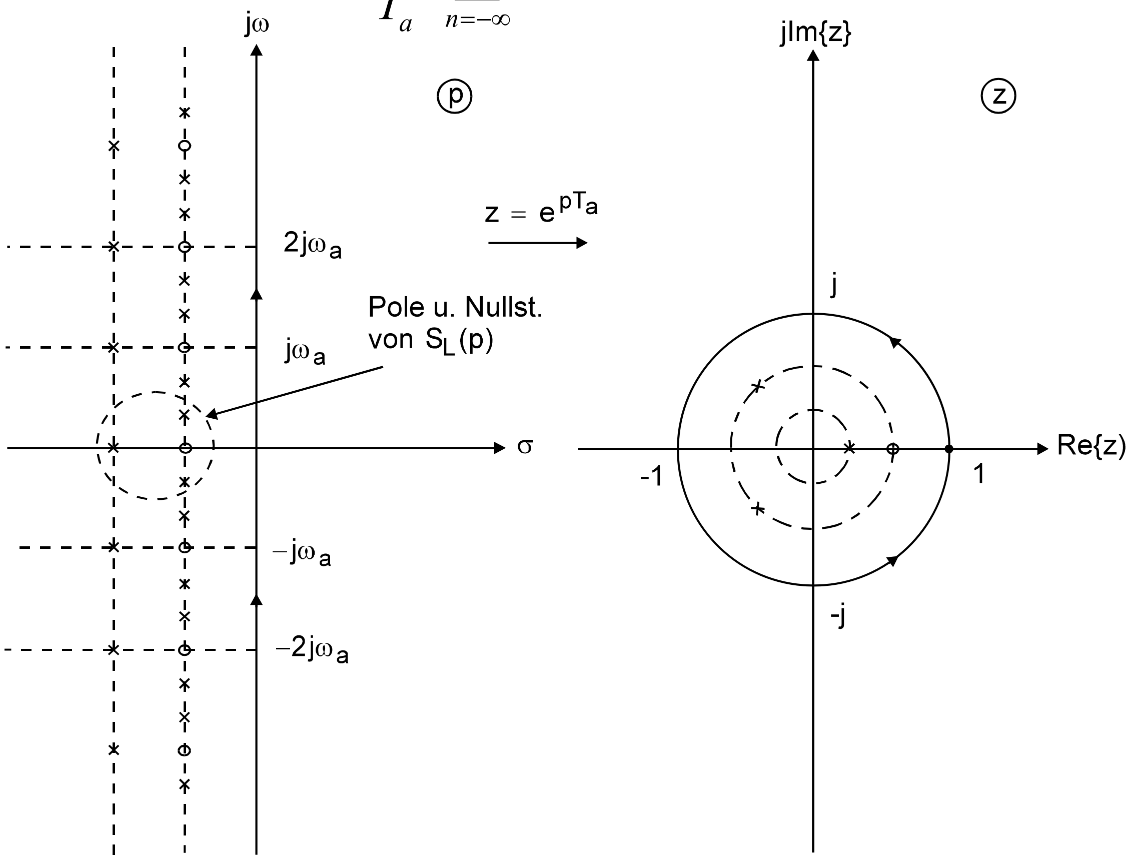


Mapping of p-plane to the z-plane by the function $z = e^{pT_a}$

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

It can be shown that it holds for the relation of transforms of sampled and normal signals:

$$S_{aL}(p) = S_z(e^{pT_a}) = \frac{1}{T_a} \cdot \sum_{n=-\infty}^{n=\infty} S_L(p - jn\omega_a)$$



1.3 The Z-transform and its relationship with the Laplace and Fourier transform

The figure clarifies:
The sampling operation gives a repeating pattern of the original pole-zero plot with infinite number of poles and zeros (looking like a stripe of poles and zeros).

So the z-plane representation is much more compact compared to the p-plane representation!

The repetition period is ω_a.

The mapping reduces the infinitely large number of poles and zeros to a finitely large number of them.

The mapping of each pole and each zero follows these relations:

$$z_{\infty n} = e^{p_{\infty n} T_a} \quad \text{for poles}$$

$$z_{0m} = e^{p_{0m} T_a} \quad \text{for zeros}$$

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

Now the relation of Laplace and Fourier transform of sampled and normal signals are given (provided all poles are in left open p-plane):

$$S_{aF}(\omega) = S_{aL}(j\omega) = \frac{1}{T_a}\cdot\sum_{n=-\infty}^{+\infty} S_L(j\omega - jn\omega_a) = \frac{1}{T_a}\cdot\sum_{n=-\infty}^{+\infty} S_F(\omega - n\omega_a)$$

⇨ Periodicity of $\left|S_{aF}(\omega)\right|, \varphi_{aF}(\omega), \tau_{aF}(\omega)$ can also be deduced

The following example clarifies the periodicity of the Fourier spectrum of the time-discrete signal, which consists of equidistant and weighted Dirac impulses.

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

Example:

For an ideal low-pass and its impulse response holds:

$$H_F(\omega) = rect\left(\frac{\omega}{2\omega_g}\right)$$

$$\bullet\!\!-\!\!\circ \; F$$

$$h(t) = \frac{\omega_g}{\pi}\cdot si\left(\omega_g t\right)$$

If one puts this non-causal signal *h(t)* to the input of an ideal sampler, one observes the pulse series:

$$h_a(t) = h(t)\cdot\sum_{n=-\infty}^{+\infty}\delta(t-nT_a) = \sum_{n=-\infty}^{+\infty} h(nT_a)\cdot\delta(t-nT_a)$$

$$\bullet\!\!-\!\!\circ \; F$$

$$H_{aF}(\omega) = H_F(\omega) * \frac{1}{2\pi}\cdot\omega_a\sum_{n=-\infty}^{n=+\infty}\delta(\omega-n\omega_a) = \frac{1}{T_a}\cdot\sum_{n=-\infty}^{n=+\infty} H_F(\omega-n\omega_a)$$

$$= \frac{1}{T_a}\cdot\sum_{n=-\infty}^{n=+\infty} rect\left(\frac{\omega-n\omega_a}{2\omega_g}\right)$$

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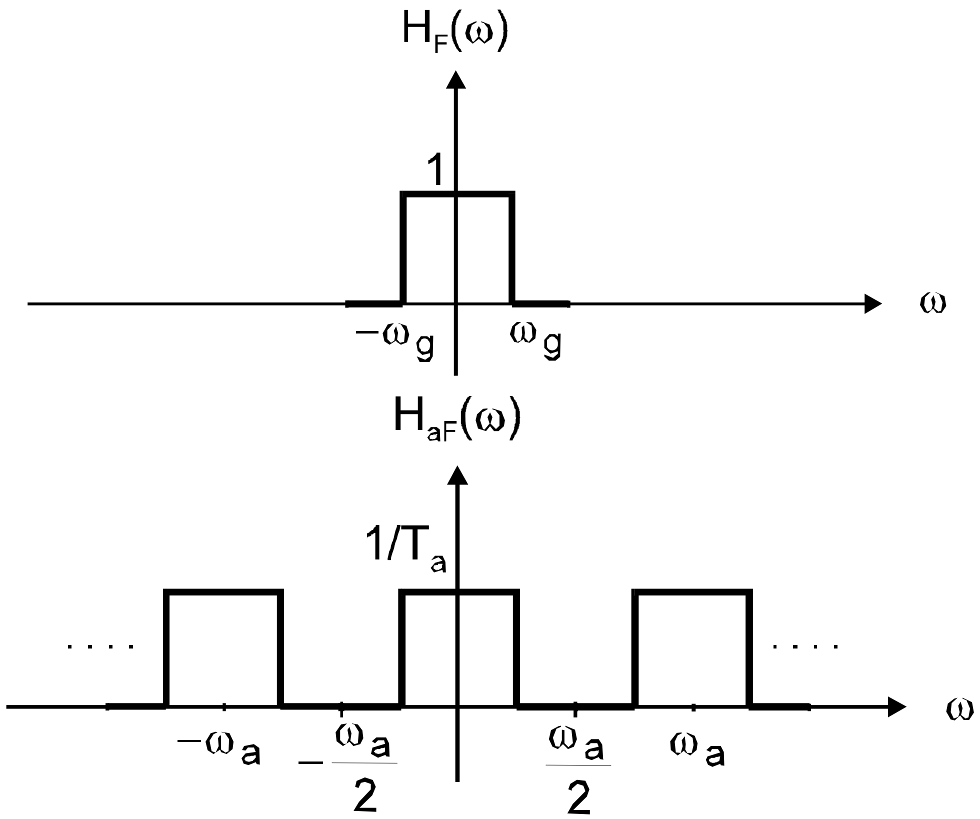
1.3 The Z-transform and its relationship with the Laplace and Fourier transform



Transfer function $H_F(\omega)$ of an analog ideal lowpass and spectrum $H_{aF}(\omega)$ of the associated sampled signal $h_a(t)$ with a sample rate of $\omega_a = 4\omega_g$

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

The preceding figure shows:

$H_F(\omega)$ is band limited to the range $-\omega_g \leq \omega \leq \omega_g$

The part of $H_{aF}(\omega)$ corresponding to $n = 0$ is identical to: $\frac{1}{\mathrm{T_a}} H_F(\omega)$

This is inside of the so-called base band range: $-\omega_a / 2 \leq \omega \leq \omega_a / 2$

Other parts of $H_{aF}(\omega)$, i.e. the parts for other values of $n \neq 0$ are in this example clearly separated from each other.

1.4 The linear, shift-invariant, causal digital filter

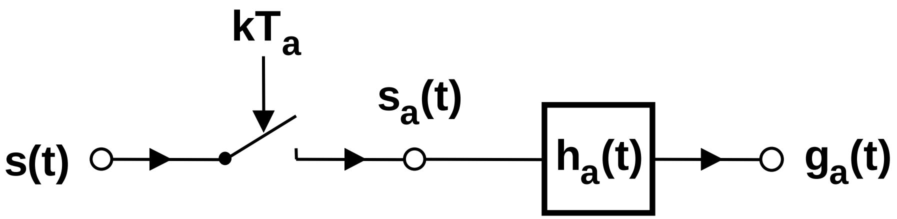
The characteristics of a causal, linear, shift-invariant digital filter can be explained on the basis of a causal, time-discrete and continuous LTI analog filter, which is composed of an analog ideal adder, ideal constant multipliers and ideal analog delay elements with all the same delays.

Such a system exhibits a sampled impulse response as follows:

$$h_a(t) = \sum_{k=0}^{\infty} h(kT_a) \cdot \delta(t - kT_a) \text{ in case of IIR Filter}$$

$$h_a(t) = \sum_{k=0}^{M} h(kT_a) \cdot \delta(t - kT_a) \quad \text{in case of FIR filter}$$

To this filter a sampled input signal shall be applied:



$$g_a(t) = s_a(t) * h_a(t)$$

Replacing in this relation $s_a(t)$ and $g_a(t)$ by corresponding relations then gives:

1.4 The linear, shift-invariant, causal digital filter

For the case of the IIR Analogfilter

$$g_a(t) = \left\{\sum_{k=0}^{\infty} s(kT_a)\cdot\delta(t-kT_a)\right\} * \left\{\sum_{k=0}^{\infty} h(kT_a)\cdot\delta(t-kT_a)\right\}$$

$$= \sum_{k=0}^{\infty} g(k)\cdot\delta(t-kT_a)$$

with

$$g(k) = \sum_{\upsilon=0}^{k} s(\upsilon T_a)\cdot h(\{k-\upsilon\}T_a) = \sum_{\upsilon=0}^{k} h(\upsilon T_a)\cdot s(\{k-\upsilon\}T_a)$$

For the case of the FIR Analogfilter

$$g_a(t) = \left\{\sum_{k=0}^{\infty} s(kT_a)\cdot\delta(t-kT_a)\right\} * \left\{\sum_{k=0}^{M} h(kT_a)\cdot\delta(t-kT_a)\right\}$$

$$= \sum_{k=0}^{\infty} g(k)\cdot\delta(t-kT_a)$$

with

$$g(k) = \sum_{\upsilon=k-M}^{k} s(\upsilon T_a)\cdot h(\{k-\upsilon\}T_a) = \sum_{\upsilon=0}^{M} h(\upsilon T_a)\cdot s(\{k-\upsilon\}T_a)$$

1.4 The linear, shift-invariant, causal digital filter

<u>Example</u> (with 3 non-zero samples for both $s_a(t)$ and $h_a(t)$) :

$$s_a(t) = s(0)\delta(t) + s(T_a)\delta(t-T_a) + s(2T_a)\delta(t-2T_a)$$

$$h_a(t) = h(0)\delta(t) + h(T_a)\delta(t-T_a) + h(2T_a)\delta(t-2T_a)$$

Due to $a\cdot\delta(t-c) * b\cdot\delta(t-d) = ab\cdot\delta(t-c-d)$ it follows:

$$g_a(t) = s_a(t) * h_a(t) =$$

$$[h(0)\delta(t) + h(T_a)\delta(t-T_a) + h(2T_a)\delta(t-2T_a)]s(0) * \delta(t) +$$

$$[h(0)\delta(t) + h(T_a)\delta(t-T_a) + h(2T_a)\delta(t-2T_a)]s(T_a) * \delta(t-T_a) +$$

$$[h(0)\delta(t) + h(T_a)\delta(t-T_a) + h(2T_a)\delta(t-2T_a)]s(2T_a) * \delta(t-2T_a)$$

$$\Rightarrow g_a(t) =$$

$$s(0)h(0)\cdot\delta(t) + [s(T_a)h(0) + s(0)h(T_a)]\cdot\delta(t-T_a) +$$

$$[s(0)h(2T_a) + s(T_a)h(T_a) + s(2T_a)h(0)]\cdot\delta(t-2T_a) +$$

$$[s(T_a)h(2T_a) + s(2T_a)h(T_a)]\cdot\delta(t-3T_a) + s(2T_a)h(2T_a)\cdot\delta(t-4T_a)$$

1.4 The linear, shift-invariant, causal digital filter

The last result can be written in shortened form as follows:

$$g(k) =$$
$$s(0)h(0)\cdot\gamma_0(k) + [s(1)h(0) + s(0)h(1)]\cdot\gamma_0(k-1) +$$
$$[s(0)h(2) + s(1)h(1) + s(2)h(0)]\cdot\gamma_0(k-2) +$$
$$[s(1)h(2) + s(2)h(1)]\cdot\gamma_0(k-3) + s(2)h(2)\cdot\gamma_0(k-4)$$

Concernig the result in the rectangular brackets (i.e. a specific value of *g(k)*) the clock shift of the unit-impulses is the same as the sum of the clock shifts of *s(k)* and the clock shifts of *h(k)* according to:

$$g(k) = \sum_{\upsilon=0}^{k} s(\upsilon)\cdot h(w)\gamma_0(k-v-w) \text{ with } k = v + w \text{ or } w = k - v$$

$$\Rightarrow g(k) = \sum_{\upsilon=0}^{k} s(\upsilon)\cdot h(k-\upsilon)\gamma_0(k-v-w)$$

Example for k = 4:

$$g(4) = s(0)h(4) + s(1)h(3) + s(2)h(2) + s(3)h(1) + s(4)h(0)$$

$= s(2)h(2)$ due to non-zero values only for arguments in range 0 to 2

1.4 The linear, shift-invariant, causal digital filter

These results show that also $g_a(t)$ has the properties of a sampled signal!

In order to describe the filter effect of the time-discrete, value-continuous analogue filter in the frequency range, the Laplace transform gives:

$$g_a(t) = s_a(t) \quad * \quad h_a(t)$$

$$G_{aL}(p) = S_{aL}(p) \quad \cdot \quad H_{aL}(p)$$

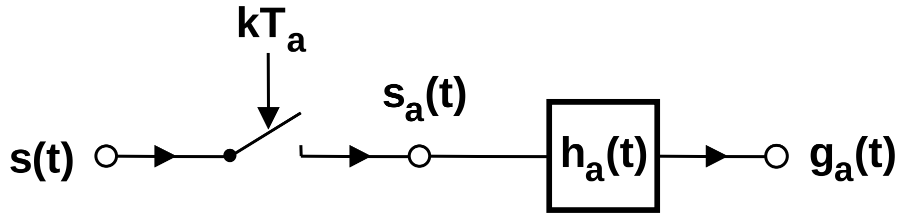
Thus for the IIR filter results: $$H_{aL}(p) = \sum_{k=0}^{\infty} h(kT_a) \cdot e^{-pkT_a}$$

and for the FIR filter: $$H_{aL}(p) = \sum_{k=0}^{M} h(kT_a) \cdot e^{-pkT_a}$$

1.4 The linear, shift-invariant, causal digital filter

Similarly as already shown for a signal in slide 32 it also can be shown for the system function that it features a periodic pole zero diagram:

$$H_{aL}(p) = \frac{1}{T_a} \cdot \sum_{n=-\infty}^{+\infty} H_L(p - jn\omega_a) \quad \text{where} \quad T_a = \frac{2\pi}{\omega_a}$$



System consisting of an ideal sampler and an LTI analog filter with time-discrete and causal impulse response

Another interpretation: $H_{aL}(p)$ is the Laplace transform of an impulse response $h(t)$ which is sampled at a rate of $T_a = 2\pi / \omega_a$.

The Laplace transform $G_{aL}(p)$ of the output $g_a(t)$ can be determined by means of the 2 following methods:

1.4 The linear, shift-invariant, causal digital filter

Method 1: (Multiplication of L-transforms)

1. Compute $H_{aL}(p) = \sum_{k=0}^{\infty} h(kT_a) \cdot e^{-pkT_a}$ and $S_{aL}(p) = \sum_{k=0}^{\infty} s(kT_a) \cdot e^{-pkT_a}$

2. $G_{aL}(p) = S_{aL}(p) \cdot H_{aL}(p)$

Method 2: (Discrete convolution)

1. Calculate $g(k) = \sum_{\upsilon=0}^{k} s(\upsilon T_a) \cdot h(\{k-\upsilon\} T_a)$ using values of $s(kT_a)$ and $h(kT_a)$

2. $G_{aL}(p) = \sum_{k=0}^{\infty} g(k) \cdot e^{-pkT_a}$

Note: If both $S_{aL}(p)$ and $H_{aL}(p)$ shows poles only in the left half p-plane the same property follows for $G_{aL}(p)$.

The corresponding periodic Fourier transforms can thus be determined as follows:

$$H_{aF}(\omega) = H_{aL}(j\omega) \qquad S_{aF}(\omega) = S_{aL}(j\omega) \qquad G_{aF}(\omega) = H_{aF}(\omega) \cdot S_{aF}(\omega)$$

1.4 The linear, shift-invariant, causal digital filter

Substituting in the previous equations z by e^{pT_A}, one receives so called

$z - transform\ of\ the\ impulse\ response\ h(k):$

$$H_{aL}(p) = \sum_{k=0}^{\infty} h(k) \cdot z^{-k} = H_z(z) \qquad \text{for IIR-filters}$$

$$H_{aL}(p) = \sum_{k=0}^{M} h(k) \cdot z^{-k} = H_z(z) \qquad \text{for FIR filters}$$

$z - transform\ G_z(z)\ of\ the\ sequence\ g(k):$

$$G_{aL}(p) = \sum_{k=0}^{\infty} g(k) \cdot z^{-k} = G_z(z)$$

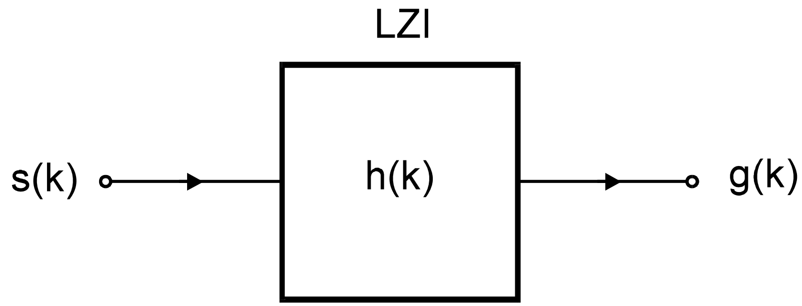
The equation to compute the values of *g(k)* can be written as:

$$g(k) = \sum_{\nu=0}^{k} s(\nu) \cdot h(k-\nu) = \sum_{\nu=0}^{k} h(\nu) \cdot s(k-\nu)$$

1.4 The linear, shift-invariant, causal digital filter

A shorter writing of the preceding formula leads to **the discrete convolution product of the sequences *s(k)* and *h(k)*:**

$$g(k) = s(k) * h(k) = h(k) * s(k)$$



Symbol for a digital filter with the impulse response *h(k)*

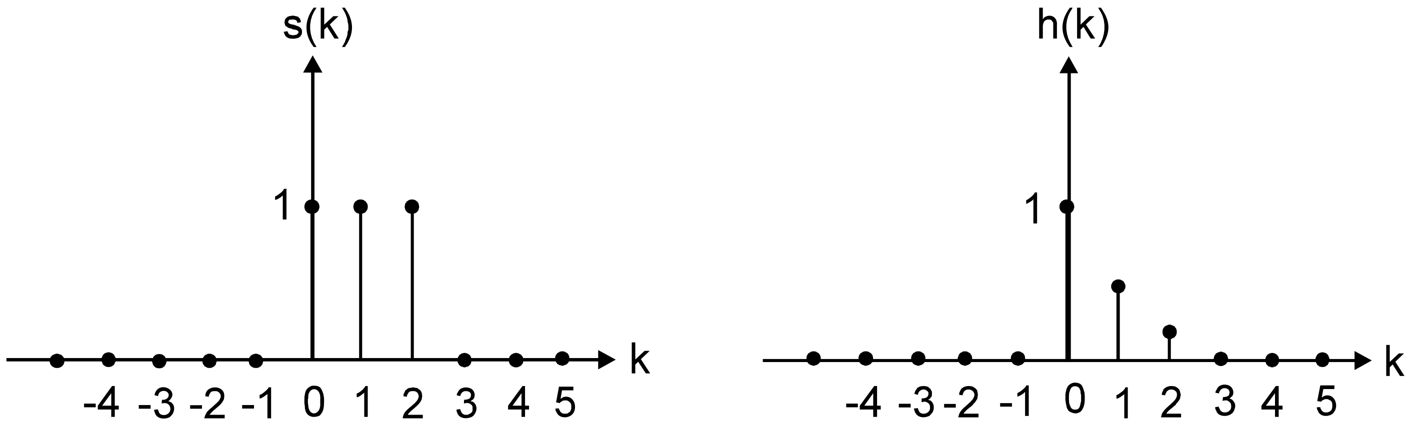
In contrast to the normal convolution product which needs integration, the discrete convolution product just needs algebraic multiplying and adding operations. Thus it can be implemented easily digitally, e.g. by means of suitable digital signal processors (DSP's).

1.4 The linear, shift-invariant, causal digital filter

Example: Discrete convolution

Given is a digital filter system with the following properties:

$$s(k) = \sum_{\nu=0}^{2} \gamma_0(k-\nu) \qquad h(k) = \begin{cases} \left(\frac{1}{2}\right)^k & \text{for } 0 \leq k \leq 2 \\ 0 & \text{else} \end{cases}$$



Input sequence *s(k)* and impulse response *h(k)* of a digital filter

Required: Output sequence *g(k)*

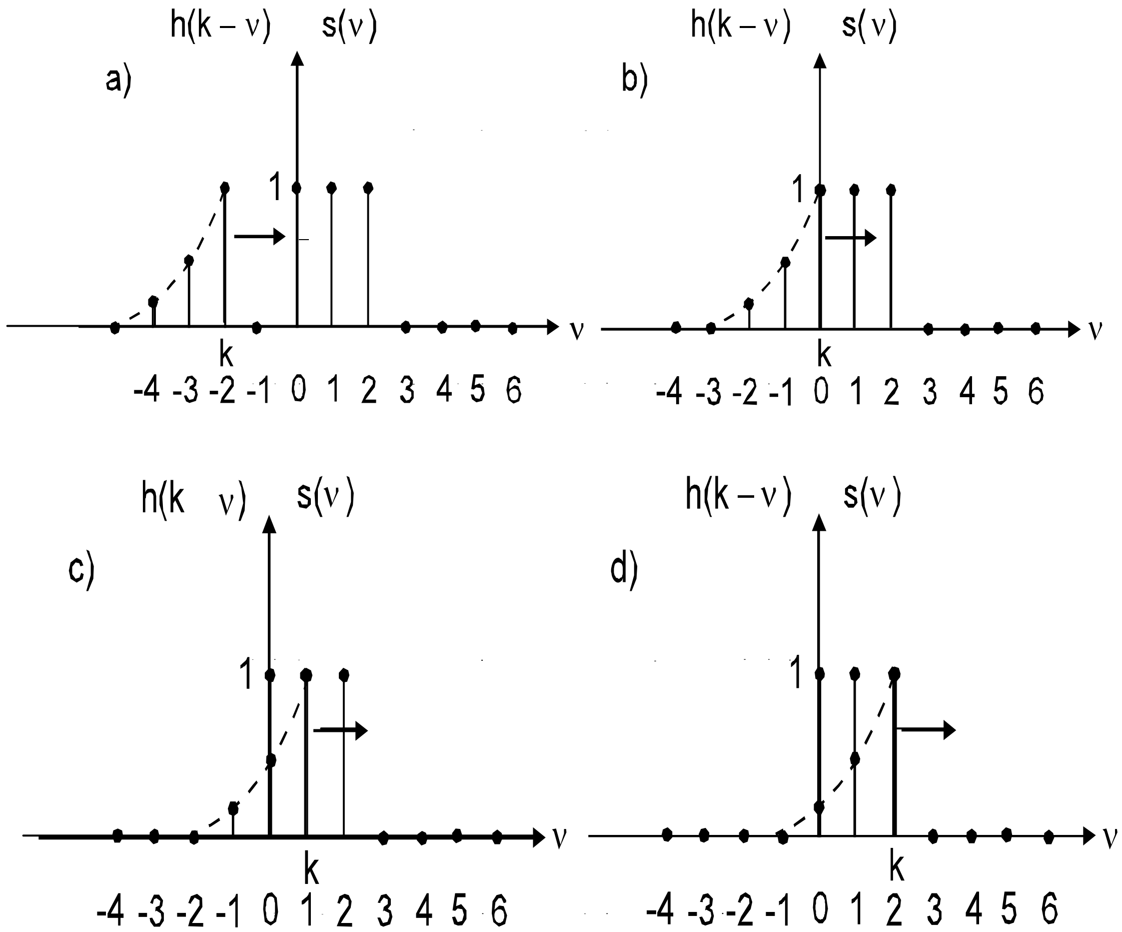
1.4 The linear, shift-invariant, causal digital filter

Using: $g(k)=\sum_{\nu=0}^{k} s(\nu)\cdot h(k-\nu)=\sum_{\nu=0}^{k} h(\nu)\cdot s(k-\nu)$ gives:

1. For $k<0$ $\quad g(k) = 0$
2. For $k=0$ $\quad g(0)=1\cdot 1+1\cdot 0+1\cdot 0=1$
3. For $k=1$ $\quad g(1)=1\cdot 1/2+1\cdot 1+1\cdot 0=3/2$
4. For $k=2$ $\quad g(2) = 1\cdot\frac{1}{4}+1\cdot\frac{1}{2}+1\cdot 1 = \frac{7}{4}$
5. For $k=3$ $\quad g(3)=1\cdot 0+1\cdot 1/4+1\cdot 1/2=3/4$
6. For $k=4$ $\quad g(4)=1\cdot 0+1\cdot 0+1\cdot 1/4=1/4$
7. For $k>4$ $\quad g(k) = 0$

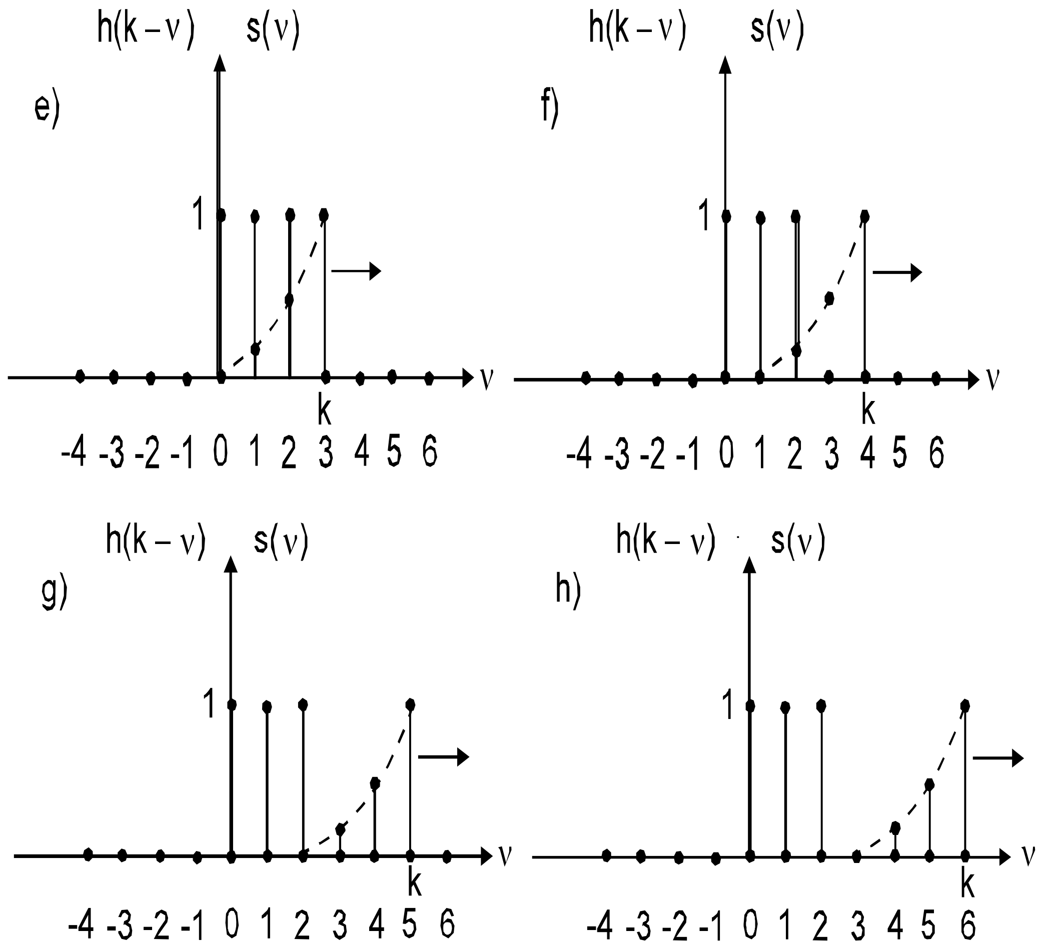
These calculations can also be performed graphically by means of the „foil method“ (comparable to a similar method known from analog convolution operation.

1.4 The linear, shift-invariant, causal digital filter



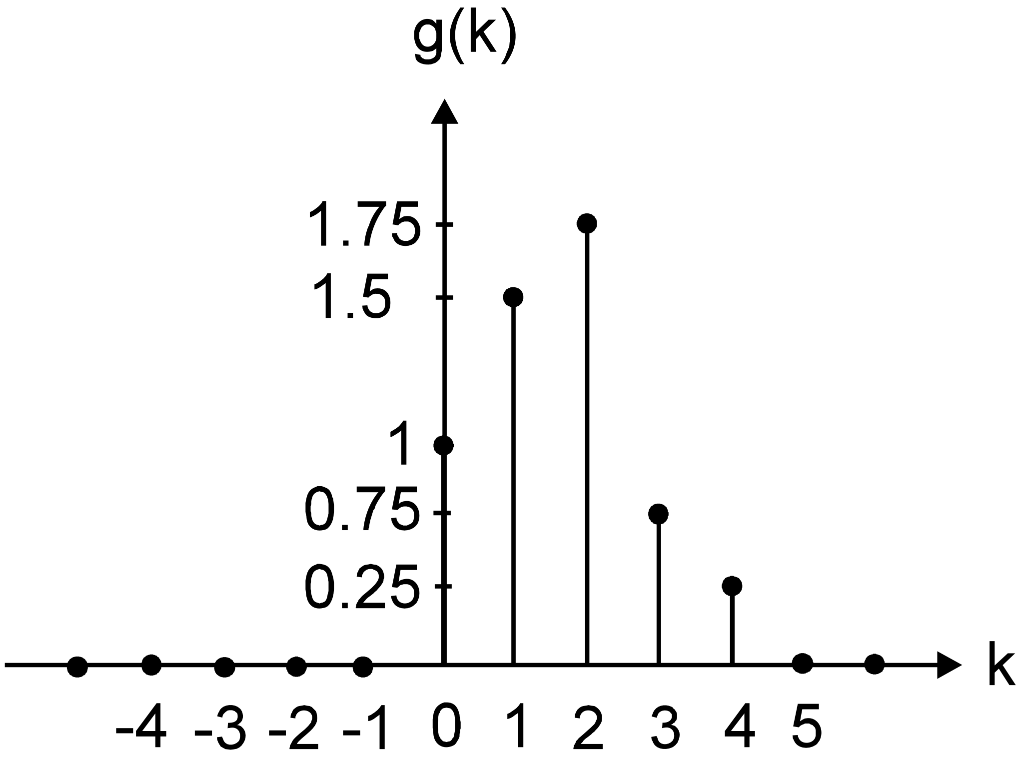
Determination of the output sequence g(k) with the help of the "foil method"

1.4 The linear, shift-invariant, causal digital filter



Determination of the output sequence g(k) with the help of the "foil method"

1.4 The linear, shift-invariant, causal digital filter



Output sequence *g(k)*, determined according to the "foil method"

1.4 The linear, shift-invariant, causal digital filter

Summary:

To determine a z-transform $G_z(z)$ of the sequence $g(k)$, which characterizes the output signal $g_a(t)$, one applies the equation:

$$g_a(t) = s_a(t) * h_a(t)$$

$$\circ\!\!-\!\!\bullet \; L$$

$$G_{aL}(p) = S_{aL}(p) \cdot H_{aL}(p)$$

From this it follows: $G_z(z) = H_z(z) \cdot S_z(z)$ which also corresponds to:

$$g(k) = s(k) * h(k) = h(k) * s(k)$$

„ The multiplication in *z*-domain corresponds to the convolution in *k*-domain“

1.4 The linear, shift-invariant, causal digital filter

Here again the Fourier transforms can be determined from the z-transforms as follows:

$$H_{aF}(\omega) = H_{aL}(j\omega) = H_z(e^{j\omega T_a})$$

$$G_{aF}(\omega) = G_{aL}(j\omega) = G_z(e^{j\omega T_a})$$

Another way of writing for this result is:

$$G_{aF}(\omega) = G_z(e^{j\omega T_a}) = H_z(e^{j\omega T_a}) \cdot S_z(e^{j\omega T_a})$$

Note: Again the condition for all of these equations is that the poles of system function *H(p)* as well as of the Laplace transform *S(p)* must stay in the left open half plane of pole zero diagrams!

1.4 The linear, shift-invariant, causal digital filter

The transition of the analog filter (see S. 17 with its time-discrete and continuous valued impulse response *h(t))* to a digital filter with its discrete impulse response *h(k)* can be achieved with the following definitions:

1) The sampled input signal $s_a(t)$ is replaced by the sequence *s(k)*

2) In reality the values *s(k)* are gained by means of

a) low-pass filtering the input signal with a cut-off frequency of $f_a/2 = 1/2T_a$

b) sampling and binary coding using an „analog to digital converter" ADC which works with *L* bits for each digital word

3) All ideal analog delays are replaced by shift registers operating at the clock rate $fa = 1 / Ta$

4) Analog multipliers are replaced by corresponding digital counterparts

1.4 The linear, shift-invariant, causal digital filter

5) The same is done for the analog adders

6) At the output then the sequence *g(k)* is observed. These values can be interpreted as the weight factors of the Dirac impulses contained in $g_a(t)$.

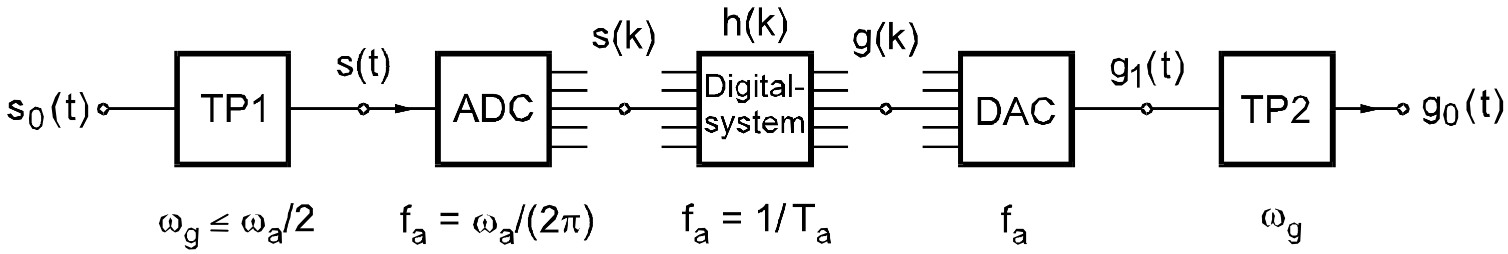
7) Binary coded output values *g(k)* are appearing at the output of the discrete system at the clock rate f_a . A S&H DAC then produces a signal $s_1(t)$.

Each value *s(k)* shows up for a S&H circuit as one step according to:

$$g_1(t) = \sum_{k=0}^{\infty} g(k) \cdot rect\left(\frac{t - kT_a - T_a/2}{T_a}\right)$$

$$= \left\{ \sum_{k=0}^{\infty} g(k) \delta(t - kT_a) \right\} * rect\left(\frac{t}{T_a}\right) * \delta(t - T_a/2)$$

1.4 The linear, shift-invariant, causal digital filter



Block diagram of an arrangement for the digital filtering of analogue signals

TP1,TP2 : Lowpass filter

ADC : Analog-Digital Converter

DAC : Digital-Analog Converter

A certain requirement concerning cut-off frequencies and the sample rate has to be fulfilled:

$$\omega_a > 2\omega_g \text{ (Nyquist condition)} \quad \text{with } 2\omega_g = Nyquist - Rate$$

f_a : sampling or conversion rate

1.4 The linear, shift-invariant, causal digital filter

More details about the Nyquist condition:

- The spectrum $S_{aF}(\omega)$ may not show overlapping (aliasing) with the parts $S_F(\omega - n\omega_a)$
- Otherwise $S_{aF}(\omega)$ will not be identical to $S_F(\omega)$ in the base band.
- No overlapping is given if the Nyquist condition is met.

The essential signal processing of the arrangement represented in the figure above takes place in the digital system describable by the linear transformation:

$$s(k) \rightarrow g(k) = T\left[s(k)\right] = s(k) * h(k)$$

$h(k)$ here represents the systems response to the input stimulation

$$\gamma_0(k) = \begin{cases} 1 \; \forall k = 0 \\ 0 \text{ elsewhere} \end{cases}$$

1.4 The linear, shift-invariant, causal digital filter

For the DAC output with S&H function holds:

$$g_1(t) = \left\{\sum_{k=0}^{\infty} s(kT_a)\cdot\delta(t-kT_a)\right\} * \left\{\sum_{k=0}^{\infty} h(kT_a)\cdot\delta(t-kT_a)\right\} * rect\left(\frac{t}{T_a}\right) * \delta\left(t-\frac{T_a}{2}\right)$$

Rewriting the expressions in curly brackets leads to:

$$g_1(t) = \left\{s(t)\cdot\sum_{k=0}^{+\infty}\delta(t-kT_a)\right\} * \left\{h(t)\cdot\sum_{k=0}^{+\infty}\delta(t-kT_a)\right\} * rect\left(\frac{t}{T_a}\right) * \delta\left(t-\frac{T_a}{2}\right)$$

The application of the Fourier transform to this equation results to:

$$G_{1F}(\omega) = \left\{\frac{1}{T_a}\cdot\sum_{n=-\infty}^{+\infty} S_F(\omega-n\omega_a)\right\}\cdot\left\{\frac{1}{T_a}\cdot\sum_{n=-\infty}^{+\infty} H_F(\omega-n\omega_a)\cdot T_a\, si\,(\omega\frac{T_a}{2})\cdot e^{-j\omega\frac{T_a}{2}}\right\}$$

1.4 The linear, shift-invariant, causal digital filter

The reconstruction low-pass TP2 has to filter out only the base band parts of the spectrum (with $n = 0$) and must compensate the si-expression given above. This can be achieved if the transfer function of TP2 approximately looks as follows:

$$H_{RF}(\omega) = \frac{1}{T_a\, si\left(\omega \frac{T_a}{2}\right)} \cdot rect\left(\frac{\omega}{\omega_a}\right) \cdot e^{j\omega \frac{T_a}{2}}$$

$$G_{oF}(\omega) = H_{RF}(\omega) \cdot G_{1F}(\omega) = \left\{\frac{1}{T_a} \cdot S_F(\omega)\right\} \cdot \left\{\frac{1}{T_a} \sum_{n=-\infty}^{\infty} H_F(\omega - n\omega_a)\right\}$$

$$= \left\{\frac{1}{T_a} \cdot S_F(\omega)\right\} H_{aF}(\omega)$$

$$\Rightarrow g_0(t) = \frac{1}{T_a} \cdot s(t) * h_a(t) \quad \text{with}$$

$$s(t) = \sum_{k=0}^{+\infty} s(k) \cdot si\left(\frac{\omega_a}{2}\{t - kT_a\}\right)$$

1.4 The linear, shift-invariant, causal digital filter

Here ideal binary coding in the ADC with $L \to \infty$ is assumed so that

$s(k) = s(kT_a)$ holds.

The same condition should be fulfilled for adders and shift registers.

1.4 The linear, shift-invariant, causal digital filter

Summary: It is required to fulfil the following conditions:

- Perfectly band limited signal *s(t)* exhibiting a cut-off frequency of *fa / 2*
- Infinitely large word length L of digital components
- Suitable transfer function of TP2 according to:

$$H_{RF}(\omega) = \frac{1}{T_a\, si\left(\omega \frac{T_a}{2}\right)} \cdot rect\left(\frac{\omega}{\omega_a}\right) \cdot e^{j\omega \frac{T_a}{2}}$$

Then the digital filter works as an analog filter with the impulse response $h_a(t)$ (apart from the constant $1/T_a$).

1.4 The linear, shift-invariant, causal digital filter

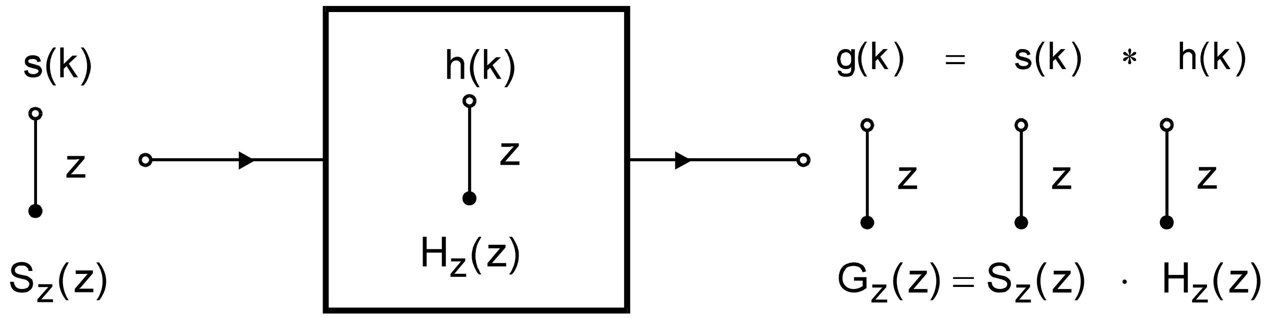
Unfortunately the conditions specified above can only be fulfilled as an approximation.This leads to the following undesired effects:

- Quantisation of signal values $s(k)$ leads to differences to $s(kT_a)$ resulting in a quantisation noise.
- Similar things happen in the multipliers due to finite word length of the filter coefficients and thus to deviations from the desired transfer function of the filter.
- Necessary word length reductions of the results of multiplications and additions lead to additional rounding noise, overflow oszillations and for IIR structure to so-called border cycles.

In the following we neglect these practical problems for easier description of the essential properties of digital filters.

1.4 The linear, shift-invariant, causal digital filter

According to comments mentioned above a digital filter is represented by the following symbol:



Ideal linear, shift-invariant and causal digital filter

For easier calculation of z-transforms suitable tables with already determined transforms are used.

1.4 The linear, shift-invariant, causal digital filter

Table of important correspondences for *z*-transform:

Nr	s(k) operation	S(z) operation	Remarks
1	$\gamma_o(k) = \begin{cases} 1 & for \quad k=0 \\ 0 & for \quad k \neq 0 \end{cases}$	1	Unit Impulse
2	$\gamma_{-1}(k) = \begin{cases} 1 & for \quad k \geq 0 \\ 0 & for \quad k < 0 \end{cases}$	$\frac{z}{z-1}$	Step sequences
3	$e^{p_1 k T_a} \cdot \gamma_{-1}(k) = z_1^k \cdot \gamma_{-1}(k)$	$\frac{z}{z-z_1}$	Exponential Sequences
4	$s(k-\upsilon) \qquad with \qquad \upsilon > 0$	$z^{-\upsilon} \cdot S_z(z)$	Shift
5	$\sum_{\upsilon=0}^{k} h(\upsilon) \cdot s(k-\upsilon) = h(k) * s(k)$	$H_z(z) \cdot S_z(z)$	Convolution ⇕ Multiplication
6	$\sum_{\upsilon} a_\upsilon s_\upsilon(k)$	$\sum_{\upsilon} a_\upsilon S_z(z)$	Linearity

1.4 The linear, shift-invariant, causal digital filter

Example 1 (for showing the use of z-transform tables):

Given is the following equation:

$$g(k) = a_0 \cdot s(k) + a_1 \cdot s(k-1) + a_2 \cdot s(k-2) - b_1 \cdot g(k-1) - b_2 \cdot g(k-2)$$

Required is: $H_z(z) = \dfrac{G_z(z)}{S_z(z)}$

Solution: Nr.4 and Nr.6 from the table above gives:

$$G_z(z) = a_0 \cdot S_z(z) + a_1 \cdot z^{-1} \cdot S_z(z) + a_2 \cdot z^{-2} \cdot S_z(z) - b_1 \cdot z^{-1} \cdot G_z(z) - b_2 \cdot z^{-2} \cdot G_z(z)$$

$$\Rightarrow \quad (1 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}) \cdot G_z(z) = (a_0 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}) \cdot S_z(z)$$

From this the system function results:

$$H_z(z) = \frac{G_z(z)}{S_z(z)} = \frac{a_0 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2}}{1 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2}} = \frac{a_0 \cdot z^2 + a_1 \cdot z + a_2}{z^2 + b_1 \cdot z + b_2}$$

1.4 The linear, shift-invariant, causal digital filter

Example 2: Inverse z-transform with the help of the correspondence table

Given the z-transform $H_z(z) = \dfrac{2}{1-0.5\cdot z^{-1}}$, required is $h(k)$

By extending the nominator and denominator one gets:

$$H_z(z) = \frac{z}{z}\cdot\frac{2}{1-0.5\cdot z^{-1}} = 2\cdot\frac{z}{z-0.5}$$

The application of the correspondence Nr.3 gives:

$$h(k) = 2\cdot\left(\frac{1}{2}\right)^k\cdot\gamma_{-1}(k)$$

As in example 1 a difference equation (as second method) can be applied for the description of the relationship between the input and output sequence.

1.4 The linear, shift-invariant, causal digital filter

Provided the zero-state exists, the output sequence *g(k)* can be computed from:

- The present weighted input value *s(k)*
- A linear combination of earlier, weighted input and output values

These circumstances are described directly by the so-called **difference equation:**

$$g(k) = \sum_{m=0}^{M} a_m s(k-m) - \sum_{n=1}^{N} b_n g(k-n)$$

Weighted input values *s(k)* *(m = 0* for the present value*)* and *s(k-m)* for earlier input values with

$1 \leq m \leq M$

Earlier values are denoted by *g(k- n)* with

$1 \leq n \leq N$

1.4 The linear, shift-invariant, causal digital filter

Example3: Computation of the output sequence from a difference equation

In case of $M = 0, N = 1, a_0 = 1, b_1 = 0.5$ one gets:

$$g(k) = s(k) - 0.5 \cdot g(k-1) \quad with \quad k \geq 1, \; g(-1) = 0$$

$s(k) = \gamma_{-1}(k)$; The zero-state condition requires at least $g(-1) = 0$

	s(k)	$1 \cdot s(k)$	*-0.5 g(k- 1)*	*g(k)*
0	1	1	0	1
1	1	1	-0.5	0.5
2	1	1	-0.25	0.75
3	1	1	-0.375	0.625
4	1	1	-0.3125	0.6875
5	1	1	-0.34375	0.65625

1.4 The linear, shift-invariant, causal digital filter

A <u>third method</u> for the description of the characteristics of a digital filters is represented in the form of a "block diagram”.

The block diagram is a graphical description of the difference equation:

$$g(k) = \sum_{m=0}^{M} a_m s(k-m) - \sum_{n=1}^{N} b_n g(k-n) \quad \textit{with} \quad n, m, M, N \in N_0$$

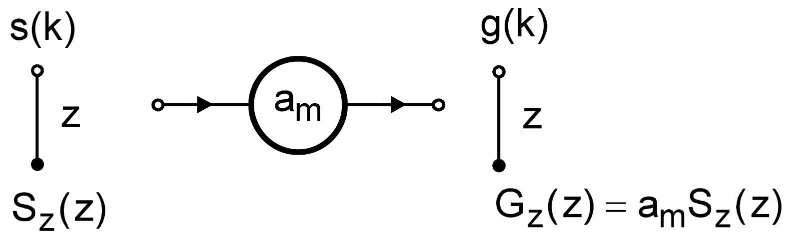
For the execution of the operations "multiplication with a constant", "addition" and "shift" and/or "delay" suitable system components are needed.

The appropriate blocks and the associated input and output relations in k are shown in the following figure.

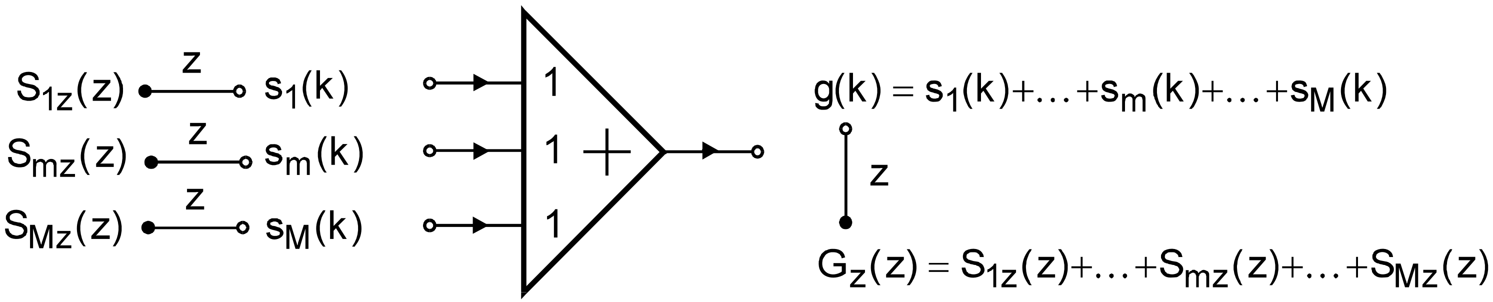
Without loosing generality one could set $N = M$ and sets suitable a_m and b_n values to zero.

1.4 The linear, shift-invariant, causal digital filter

Multiplizierer :



Addierer :



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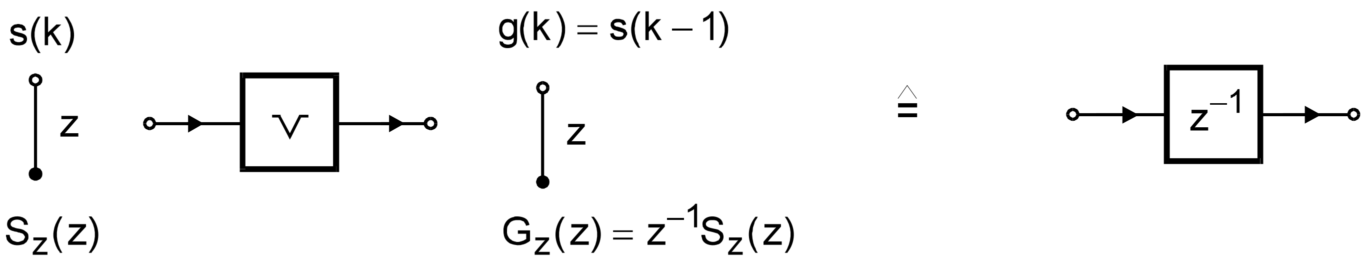
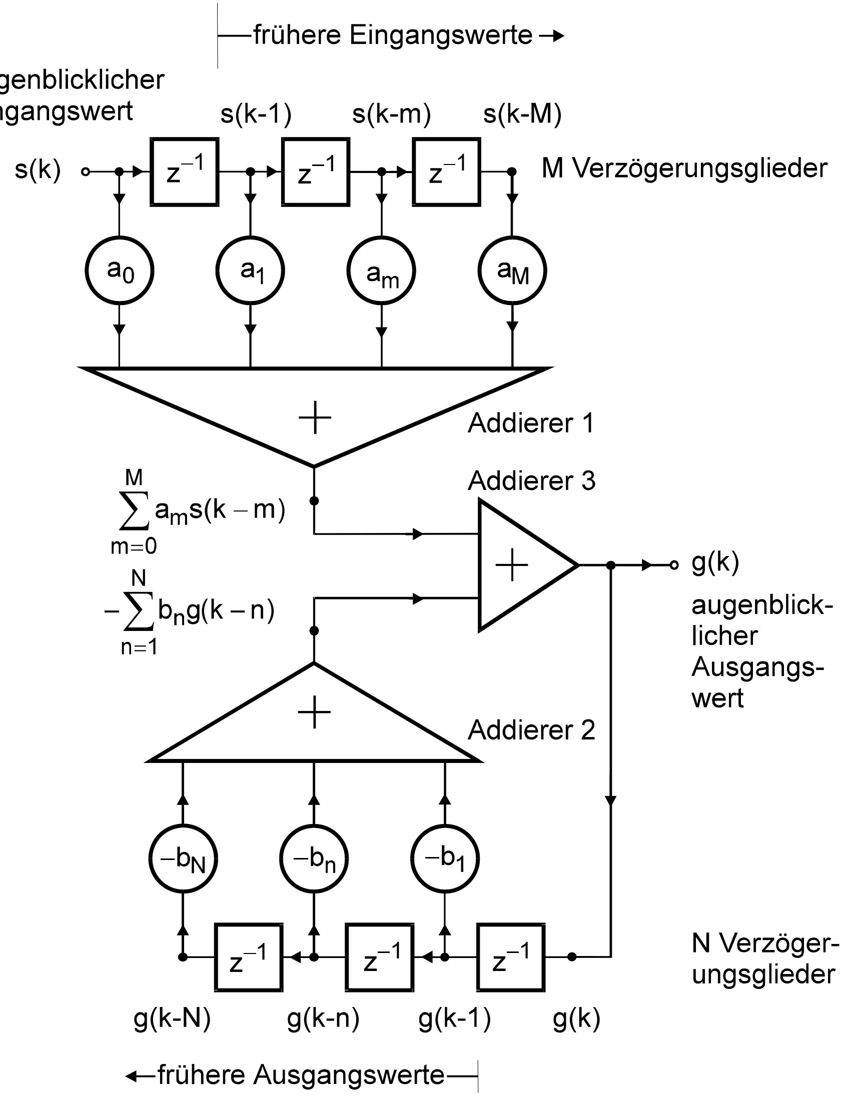


1.4 The linear, shift-invariant, causal digital filter

The conversion of the difference equation into the filter structure can be described in the following steps.

1) Realisation of $M+1$ signals $s(k-m)$ using a chain of M delays
2) Multiplication of $M+1$ signals with constants using $M+1$ multipliers
3) Addition of all $M+1$ signals $a_m\, s(k-m)$ using an adder with $M+1$ inputs
4) Realisation of N signals $g(k-n)$ using a chain of N delays
5) Multiplication of N signals with constants using N multipliers
6) Addition of all N signals $b_\mathrm{n}\, s(k-m)$ using an adder with N inputs
7) Addition of the sums gained in steps 3 and 6 using an adder with two inputs

1.4 The linear, shift-invariant, causal digital filter



Accordingly these steps result in the so-called ***"direct structure"***

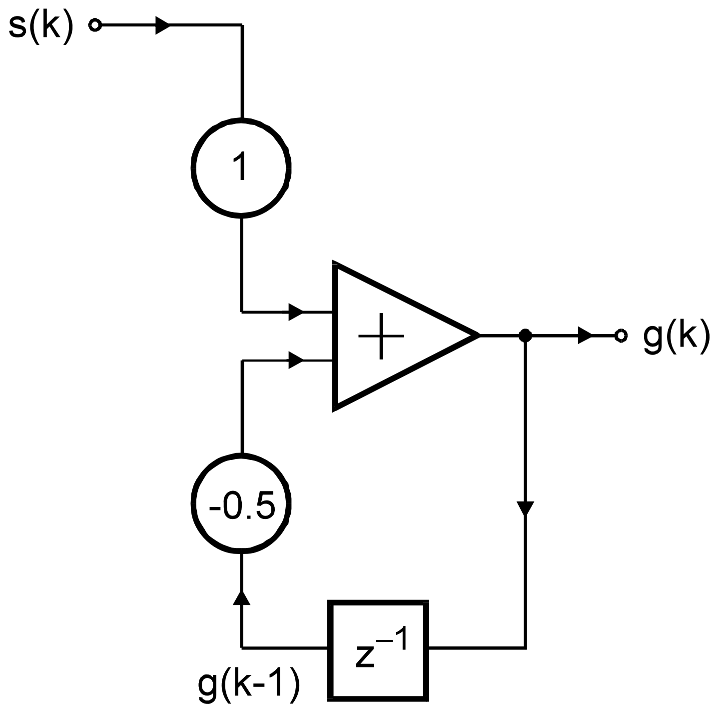
Here $N+M$ delays are needed.

It can be shown that a lower number is sufficient for realising the same filter properties. This leads to so-called canonical structures.

1.4 The linear, shift-invariant, causal digital filter

<u>Example</u>

It shows the conversion of the difference equation presented above into the block diagram.



"Block diagram" for the difference equation $g(k) = s(k) - 0.5g(k\text{-}1)$

1.4 The linear, shift-invariant, causal digital filter

The system function of **a recursive digital filter** is gained by z-transforming the differential equation:

$$g(k) = \sum_{m=0}^{M} a_m\, s(k-m) - \sum_{n=1}^{N} b_n\, g(k-n)$$

Thus it follows:

$$G_z(z) = \sum_{m=0}^{M} a_m\, z^{-m} \cdot S_z(z) - \sum_{n=1}^{N} b_n\, z^{-n} \cdot G_z(z)$$

1.4 The linear, shift-invariant, causal digital filter

Rewriting this equation gives:

$$G_z(z)\cdot\left\{1+\sum_{n=1}^{N} b_n\, z^{-n}\right\}=\sum_{m=0}^{M} a_m\, z^{-m}\cdot S_z(z)$$

$$\Leftrightarrow\; G_z(z) = \frac{\sum_{m=0}^{M} a_m\, z^{-m}}{1+\sum_{n=1}^{N} b_n\, z^{-n}}\cdot S_z(z) = H_z(z)\cdot S_z(z)$$

Thus the **system function** of the general, recursive, linear, shift-invariant and causal digital filter is gained.

$$H_z(z) = \frac{\sum_{m=0}^{M} a_m\, z^{-m}}{1+\sum_{n=1}^{N} b_n\, z^{-n}}$$

1.4 The linear, shift-invariant, causal digital filter

If one sets all constants b_n to zero, one obtain the non-recursive filter:

$$H_z(z) = \sum_{m=0}^{M} a_m z^{-m}$$

Note: As shown earlier it holds in both cases $g(k) = h(k) * s(k)$

For $s(k) = \gamma_0(k)$ it holds:

$$g(k) = h(k) * \gamma_0(k)$$

$$\downarrow z$$

$$G_z(z) = H_z(z) \cdot 1 \longrightarrow g(k) = h(k)$$

The impulse response of the digital filter

1.4 The linear, shift-invariant, causal digital filter

The <u>impulse response</u> *h(k)* can be determined by means of three different procedures:

<u>Procedure 1:</u> Application of the inverse z-transform to $H_Z(z)$

<u>Procedure 2:</u>

1. In case rewrite $H_z(z)$ so that only positive exponents result

2. Develope $H_z(z)$ into a series by continued division of $\dfrac{\text{Nominator}\{H_z(z)\}}{\text{Denominator}\{H_z(z)\}}$

3. Apply inverse transform to each of the simple addends.

<u>Procedure 3:</u>

Develope $H_z(z)$ in a sum of partial fractions and apply the z-inverse transform to all addends.

1.4 The linear, shift-invariant, causal digital filter

Example: Determination of the impulse response *h(k)* by Procedure 2

A system function is given
$$H_z(z) = \frac{\sum_{m=0}^{M} a_m z^{-m}}{1 + \sum_{n=1}^{N} b_n z^{-n}}$$

with $M = 0,\ a_1 = 1,\ N = 1\ and\ b_1 = 0.8$; Required is $h(k)$.

Solution:

With the given information, one directly gets: $H_z(z) = \frac{1}{1 + 0.8\, z^{-1}}$

As we see a negative exponent of -1 , nominator and denominator are multipled with z giving:
$$H_z(z) = \frac{z}{z + 0.8}$$

1.4 The linear, shift-invariant, causal digital filter

Continued polynomial division then gives the following results:

$$H_z(z) = \frac{z}{z+0.8}$$

$$\begin{array}{l} z \;:\; (z+0.8) = 1 - 0.8\cdot z^{-1} + 0.64\cdot z^{-2} - 0.512\cdot z^{-3} + \ldots \\ -(0.8+z) \\ \quad -0.8 \\ -(-0.64\cdot z^{-1} - 0.8) \\ \quad +0.64\cdot z^{-1} \\ -(0.512\cdot z^{-2} + 0.64\cdot z^{-1}) \\ \quad -0.512\cdot z^{-2} \\ -(-0.4096\cdot z^{-3} - 0.512\cdot z^{-2}) \\ \quad +0.4096\cdot z^{-3} \end{array}$$

$-0.64\cdot z^{-1} - 0.8$ → due to $(z+0.8)\cdot\left(-0.8z^{-1}\right) = -0.8 - 0.64z^{-1}$

$0.512\cdot z^{-2} + 0.64\cdot z^{-1}$ → due to $(z+0.8)\cdot\left(0.64z^{-2}\right) = 0.64z^{-1} + 0.512z^{-2}$

$\vdots$

$$H_z(z) = \frac{z}{z+0.8} = 1 - 0.8z^{-1} + 0.64z^{-2} - 0.51z^{-3} + 0.41z^{-4} - 0.33z^{-5} + \ldots$$

1.4 The linear, shift-invariant, causal digital filter

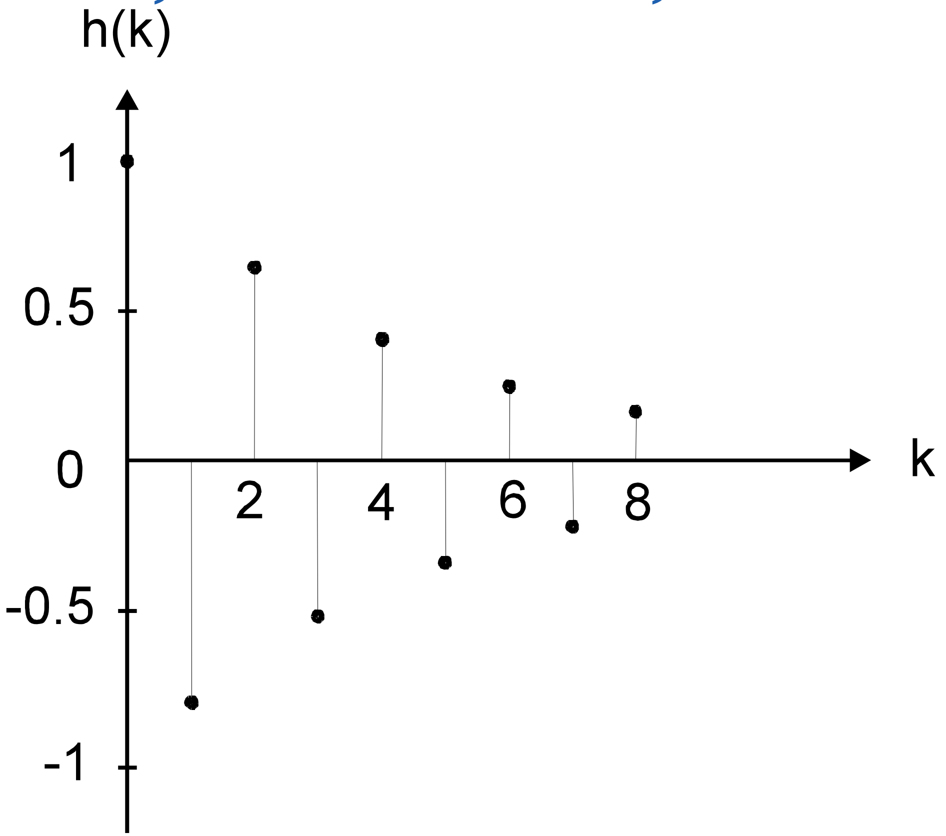
As the continued (polynomial) division resulted in

$$H_z(z) = 1 - 0.8\,z^{-1} + 0.64\,z^{-2} - 0.51\,z^{-3} + 0.41\,z^{-4} - 0.33\,z^{-5} + \ldots$$

applying the correspondences Nr.4, one gets the inverse transform:

$$h(k) = \gamma_0(k) - 0.8\cdot\gamma_0(k-1) + 0.64\cdot\gamma_0(k-2) - 0.51\cdot\gamma_0(k-3) + 0.41\cdot\gamma_0(k-4) - \ldots$$

1.4 The linear, shift-invariant, causal digital filter



Impulse response of a recursive filter with the system function
Note: Decaying absolute coefficient values (stable system)

$$H_z(z) = 1 - 0.8\,z^{-1} + 0.64\,z^{-2} - 0.51 z^{-3} + 0.41 z^{-4} - 0.33\,z^{-5} + \ldots$$

1.4 The linear, shift-invariant, causal digital filter

Note: A recursive system always shows a sequence *h(k)* with infinitely large number of non-zero values. Thus in principle infinitely many divisions need to be made; otherwise the impulse response of an FIR flter is calculated.

The larger *M* is selected, the more accurate is the recursive IIR filter. Again the following equations clarify these circumstances:

$$H_z(z) = \frac{\sum_{m=0}^{M} a_m z^{-m}}{1+\sum_{n=1}^{N} b_n z^{-n}} \overset{N>M}{=} \frac{\sum_{m=0}^{M} a_m z^{N-m}}{z^N + \sum_{n=1}^{N} b_n z^{N-n}} \approx \sum_{q=0}^{\infty} c_q z^{-q}$$

From this one gets the infinite sequences after inverse z-transforming.

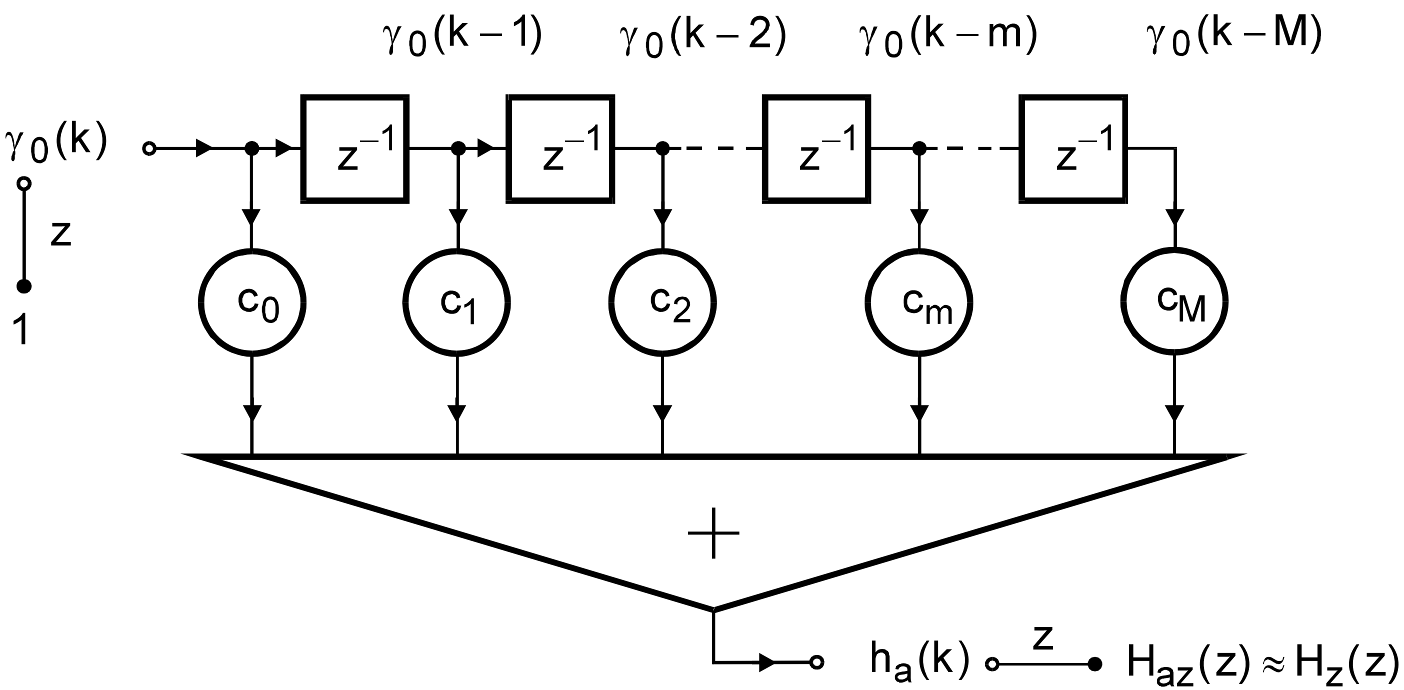
$$h(k) = \sum_{q=0}^{\infty} c_q \gamma_0 (k-q)$$

The restriction on a finite number of M leads to the approximation:

$$H_z(z) \approx H_{az}(z) = \sum_{m=0}^{M} c_m z^{-m}$$

1.4 The linear, shift-invariant, causal digital filter

Its inverse transformation is: $h(k) \approx h_a(k) = \sum_{m=0}^{M} c_m \gamma_0(k-m)$



Structure of an approximated non-recursive digital filter
(Approximation of a recursive by a non-recursive system)

1.4 The linear, shift-invariant, causal digital filter

Procedure 3: For the case of $N = M+1$ the following equation holds:

$$H_z(z) = \frac{\sum_{m=0}^{M} a_m z^{-m}}{1+\sum_{n=1}^{N} b_n z^{-n}} \overset{N=M+1}{=} \frac{\sum_{m=0}^{M} a_m z^{N-m}}{z^N + \sum_{n=1}^{N} b_n z^{N-n}}$$

Assuming that all poles are single, it can be deduced (with real K and $N = M + 1$):

$$H_z(z) = K \cdot \frac{\sum_{m=0}^{M} a_m z^{N-m}}{\prod_{\upsilon=1}^{N} (z - z_{\infty\upsilon})} = K \cdot \frac{a_0 z \cdot \sum_{m=0}^{M} \frac{a_m}{a_0} z^{N-1-m}}{\prod_{\upsilon=1}^{N} (z - z_{\infty\upsilon})}$$

1.4 The linear, shift-invariant, causal digital filter

After decomposition into partial fractions it results:

$$H_z(z) = K \cdot a_0 z \cdot \sum_{\upsilon=1}^{N} \frac{A_\upsilon}{(z - z_{\infty\upsilon})} = K \cdot a_0 \sum_{\upsilon=1}^{N} \frac{A_\upsilon \cdot z}{(z - z_{\infty\upsilon})}$$

With help of the correspondences Nr. 3 from the z-transform table we obtain:

$$h(k) = K \cdot a_0 \sum_{\upsilon=1}^{N} A_\upsilon \cdot z_{\infty\upsilon}^{k} \cdot \gamma_{-1}(k)$$

Example: Determination of the impulse response using Procedure 3

Given is the system function $H_z(z) = \dfrac{\sum_{m=0}^{M} a_m z^{-m}}{1 + \sum_{n=1}^{N} b_n z^{-n}}$

with $M = 0,\ a_0 = 1,\ N = 2,\ b_0 = -0{,}2$ *and* $b_1 = -0.8$

1.4 The linear, shift-invariant, causal digital filter

With the given information it directly results:

$$H_z(z) = \frac{1}{1 - 0.2 \cdot z^{-1} - 0.8 \cdot z^{-2}}$$

Multiplication of the nominator and denominator by $z^N = z^2$ gives only positive exponent values:

$$H_z(z) = \frac{z^2}{z^2 - 0.2 \cdot z - 0.8}$$

Excluding $a_0 \cdot z = z$ from the numerator, it results:

$$H_z(z) = z \cdot \left(\frac{z}{z^2 - 0.2 \cdot z - 0.8} \right)$$

A partial fractions development now leads to following calculations:

1.4 The linear, shift-invariant, causal digital filter

$$\frac{z}{z^2 - 0.2z - 0.8}$$ gives the roots: $z_{\infty 1,2} = 0.1 \pm \sqrt{0.01 + 0.8} = 0.1 \pm 0.9$

$$\Rightarrow \quad \frac{z}{z^2 - 0.2z - 0.8} = \frac{A}{z-1} + \frac{B}{z+0.8}$$

$$\Rightarrow \quad A = \left.\frac{z}{z+0.8}\right|_{z=1} = \frac{1}{9/5} = \frac{5}{9}$$

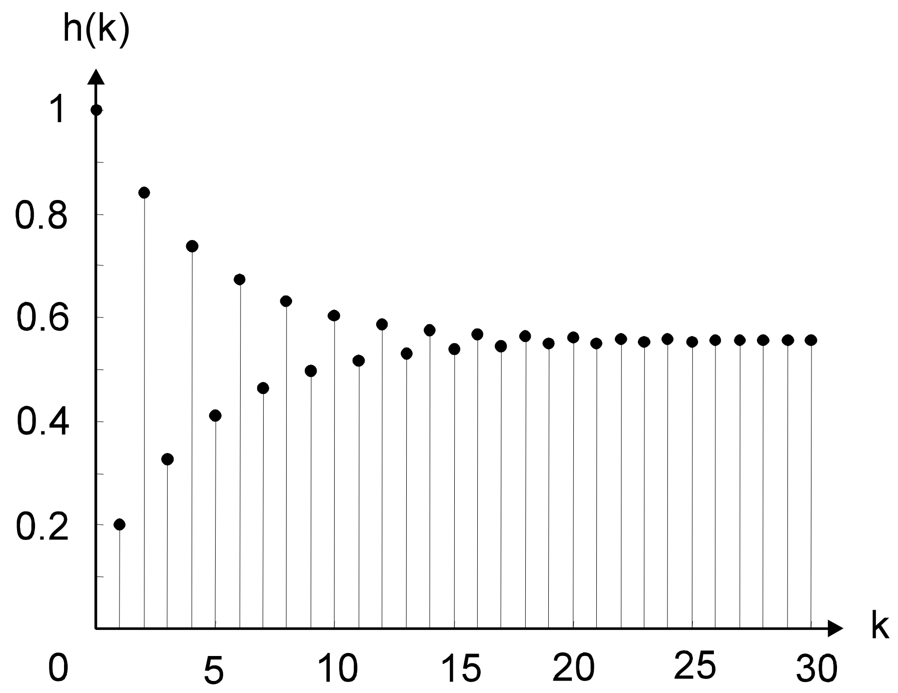
$$B = \left.\frac{z}{z-1}\right|_{z=-0.8} = \frac{-0.8}{-0.8-1} = \frac{-4/5}{-9/5} = \frac{4}{9}$$

Thus we obtain: $$H_z(z) = z \cdot \left(\frac{\frac{5}{9}}{z-1} + \frac{\frac{4}{9}}{z+0.8} \right) = \frac{\frac{5}{9}z}{z-1} + \frac{\frac{4}{9}z}{z-(-0.8)}$$

1.4 The linear, shift-invariant, causal digital filter

Finally an inverse transform leads (with $1^k = 1$) to:

$$h(k) = \frac{5}{9}\gamma_{-1}(k) + \frac{4}{9}(-0.8)^k \cdot \gamma_{-1}(k)$$



Impulse response of the recursive filter.

1.5 Canonical digital filter circuits

With the following relations

$G_z(z) \bullet\!\!\overset{z}{\text{—}}\!\!\circ\; g(k)$ output sequence (reaction)

$S_z(z) \bullet\!\!\overset{z}{\text{—}}\!\!\circ\; s(k)$ input sequence (excitation)

the system function can be presented as:

$$H_z(z) = \frac{G_z(z)}{S_z(z)} = \frac{\sum_{m=0}^{M} a_m z^{-m}}{1+\sum_{n=1}^{N} b_n z^{-n}}$$

Formally extending the system function, one gets:

$$H_z(z) = \frac{\sum_{m=0}^{M} a_m z^{-m}}{1+\sum_{n=1}^{N} b_n z^{-n}} = \frac{G_z(z)}{S_z(z)} = \frac{G_z(z)}{X_z(z)} \cdot \frac{X_z(z)}{S_z(z)}$$

1.5 Canonical digital filter circuits

One sets now $\frac{G_z(z)}{X_z(z)}$ to the numerator of $H_z(z)$: $\frac{G_z(z)}{X_z(z)} = \sum_{m=0}^{M} a_m z^{-m}$

Solving for $G_z(z)$ gives:

$$G_z(z) = \left[\sum_{m=0}^{M} a_m z^{-m}\right] \cdot X_z(z) = \sum_{m=0}^{M} a_m z^{-m} X_z(z)$$

This leads to the description of a non-recursive subsystem:

$$g(k) = \sum_{m=0}^{M} a_m x(k-m)$$

1.5 Canonical digital filter circuits

Similiarly, one sets
$$\frac{X_z(z)}{S_z(z)} = \frac{1}{1+\sum_{n=1}^{N} b_n z^{-n}}$$

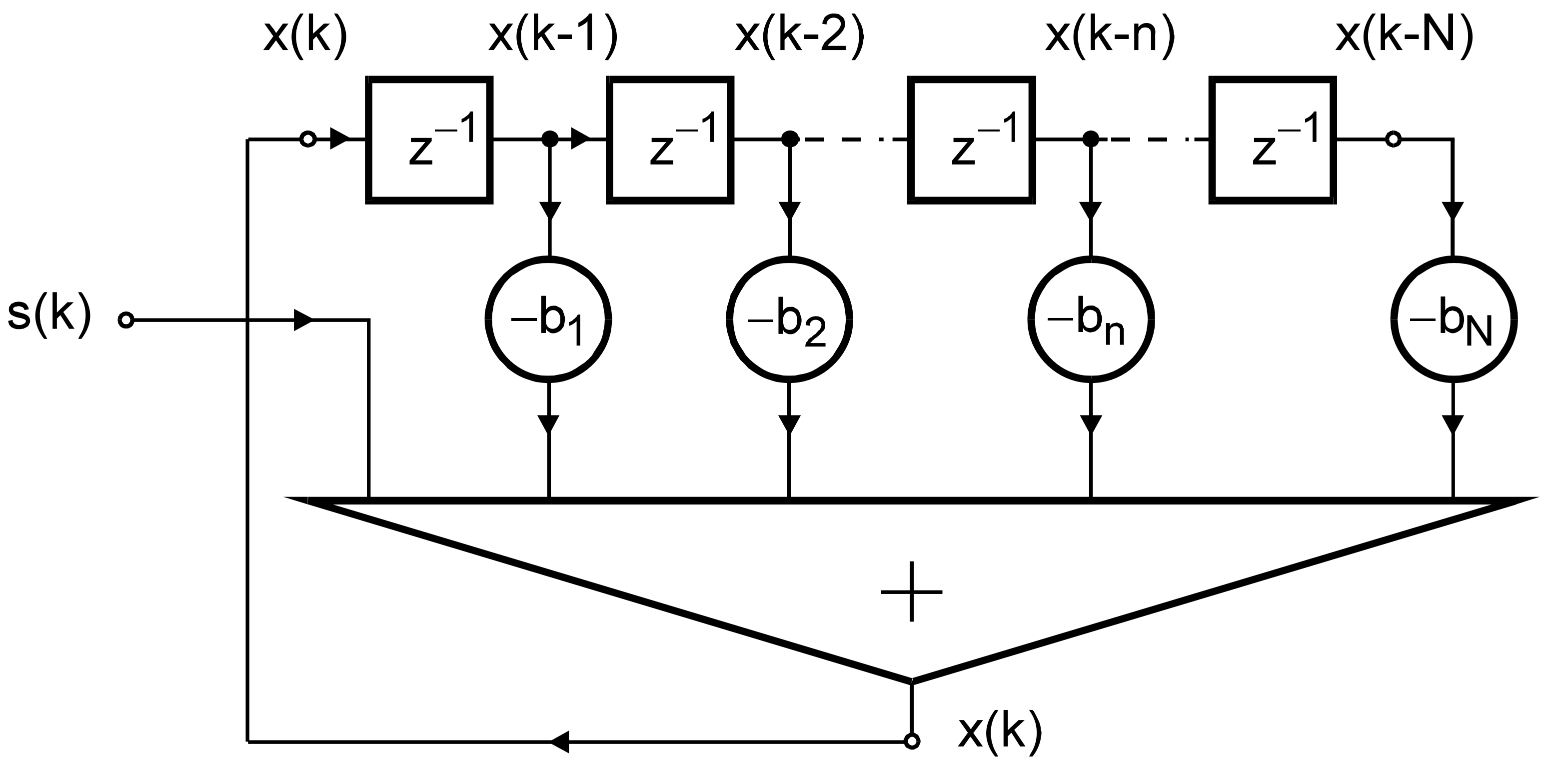
to the denominator
which gives:
$$X_z(z)\cdot(1+\sum_{n=1}^{N} b_n z^{-n}) = S_z(z)$$
$$\Rightarrow X_z(z) = S_z(z) - \sum_{n=1}^{N} b_n z^{-n} X_z(z)$$

Finally it is obtained:
$$x(k) = s(k) - \sum_{n=1}^{N} b_n x(k-n)$$

Now *x(k)* describes the reaction of another recursive subsystem (as part of the whole digital filter).

The realisation follows the procedures described above.

1.5 Canonical digital filter circuits



Recursive subsystem

1.5 Canonical digital filter circuits

The realisation of g(k) follows the relation described above: $g(k) = \sum_{m=0}^{M} a_m \, x(k-m)$

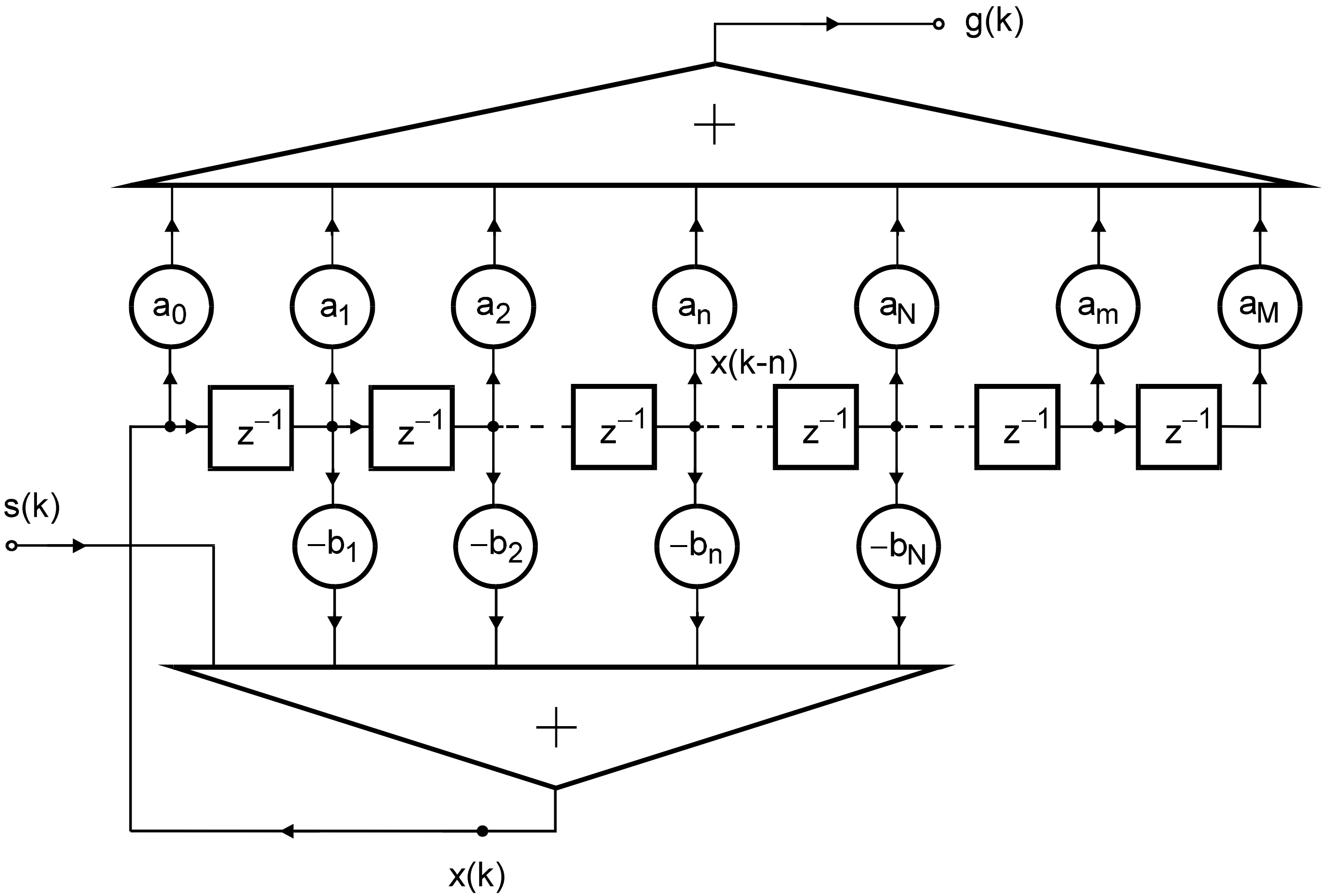
Doing so from the recursive subsystem the sequences $x(k-m)$
are used and multiplied by a_m.
Thus the following diagram on the next slide results.

Moreover: The amount of delays is reduced to a minimum.

Please note:

There are additional methods for finding other canonical structures.

1.5 Canonical digital filter circuits



Canonical realization of a general recursive digital filter for the case M > N

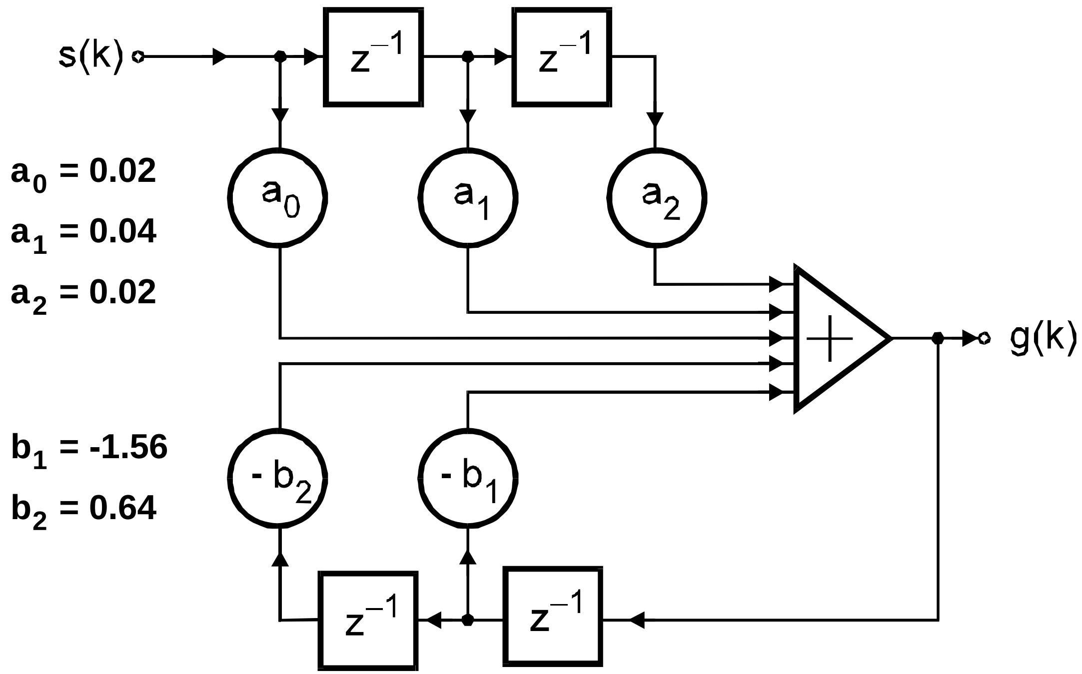
1.5 Canonical digital filter circuits

Example: Derivation of the "direct structure" and of the "canonical structure"

Given is the differential equation:

$$g(k) = 1.56 \cdot g(k-1) - 0.64 \cdot g(k-2) + 0.02 \cdot s(k) + 0.04 \cdot s(k-1) + 0.02 \cdot s(k-2)$$

Direct structure resulting from this:



1.5 Canonical digital filter circuits

"**Canonical filter structure**"

The first step is to z-transform the difference equation:

$$G_z(z) = 1.56 \cdot z^{-1} G_z(z) - 0.64 \cdot z^{-2} G_z(z) + 0.02 \cdot S_z(z) + 0.04 \cdot z^{-1} S_z(z) + 0.02 \cdot z^{-2} S_z(z)$$

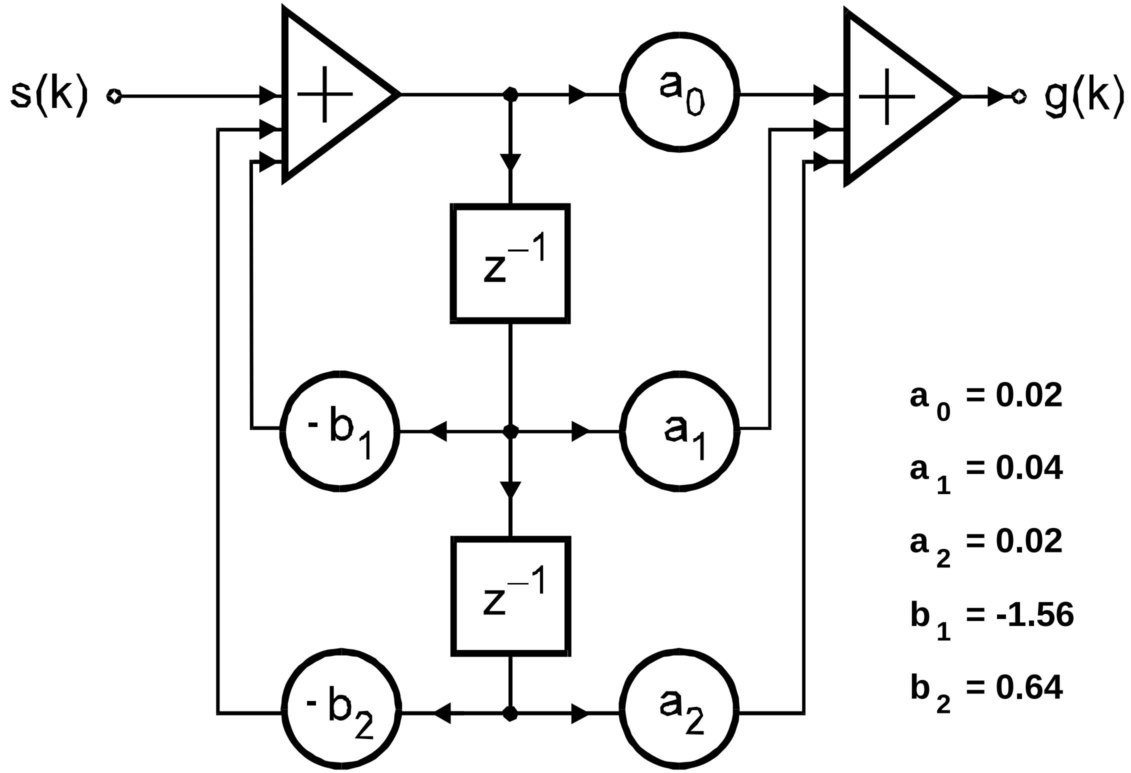
$$\Rightarrow \quad G_z(z) \cdot (1 - 1.56 \cdot z^{-1} + 0.64 \cdot z^{-2}) = S_z(z) \cdot (0.02 + 0.04 \cdot z^{-1} + 0.02 \cdot z^{-2})$$

Thus It applies:

$$H_z(z) = \frac{G_z(z)}{S_z(z)} = \frac{0.02 + 0.04 \cdot z^{-1} + 0.02 \cdot z^{-2}}{1 - 1.56 \cdot z^{-1} + 0.64 \cdot z^{-2}}$$

By a comparison of filter coefficients with the general canonical filter structure (see slide 92) the following diagram is obtained.

1.5 Canonical digital filter circuits



$a_0 = 0.02$

$a_1 = 0.04$

$a_2 = 0.02$

$b_1 = -1.56$

$b_2 = 0.64$

“Canonical structure" for the realization of the system function

1.5 Canonical digital filter circuits

- There is one disadvantage of this canonical structure: A significant sensitivity to parameter values exists.
- Small deviations of coefficients a_m and b_n lead to large deviations in the position of poles and zeros in the z-plane (and to corresponding changes in filter properties).
- Therefore more robust structures are often desired.

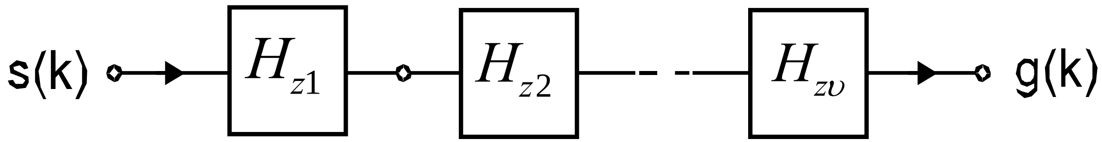
1.5 Canonical digital filter circuits

a) Robust filters using a cascade structure

Using a cascade one divides the system function into a product of appropriate product terms of first and second order subsystems.

The following equation describes this onset:

$$H_z(z) = \frac{\prod_{m=1}^{M_1} (z^2 + c_m z + d_m)}{\prod_{n=1}^{N_1} (z^2 + u_n z + v_n)} \bullet \frac{\prod_{m=1}^{M_2} (z + e_m)}{\prod_{n=1}^{N_2} (z + l_n)} \bullet \ldots = H_{z1} \cdot H_{z2} \cdots H_{z\upsilon}$$



Realization of a digital filter as a cascade structure

1.5 Canonical digital filter circuits

Example: Realization of a digital filter in a cascade structure

Given is the system function

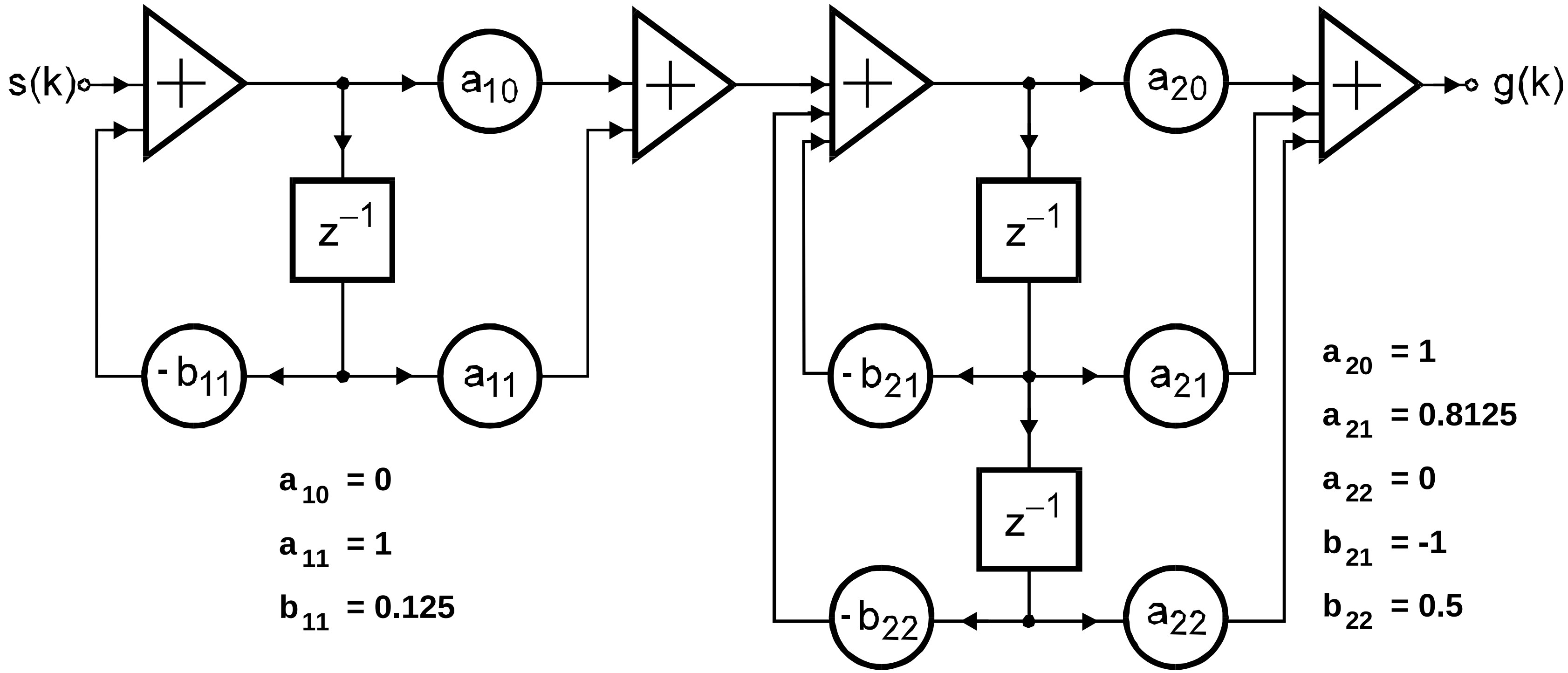
$$H_z(z) = \frac{z^{-1} + 0.8125 \cdot z^{-2}}{1 - 0.875 \cdot z^{-1} + 0.375 \cdot z^{-2} + 0.0625 \cdot z^{-3}}$$

A separation of $H_z(z)$ of the degree of 3 into a product of two subsystems of the order of 1 and 2 gives:

$$H_z(z) = \frac{z^{-1}}{1 + 0.125 \cdot z^{-1}} \cdot \frac{1 + 0.8125 \cdot z^{-1}}{1 - z^{-1} + 0.5 \cdot z^{-2}}$$

The following diagram shows the corresponding filter structure.

1.5 Canonical digital filter circuits



$a_{10} = 0$

$a_{11} = 1$

$b_{11} = 0.125$

$a_{20} = 1$

$a_{21} = 0.8125$

$a_{22} = 0$

$b_{21} = -1$

$b_{22} = 0.5$

Cascade structure of digital filters of 1st and 2nd Order (with canonical structures also for subsystems)

1.5 Canonical digital filter circuits

b) Robust filters with a parallel structure:

Another onset is to develop the system function $H_z(z)$ into a sum of partial fractions (assuming $N \geq M$). Each fraction then represents a subsystem of lower order (1 or 2).

For the case of simple poles this sum can be written as follows:

$$H_z(z) = K_0 + \sum_{\nu=1}^{N_1} \frac{K_\nu}{z - z_{\infty\nu}}$$

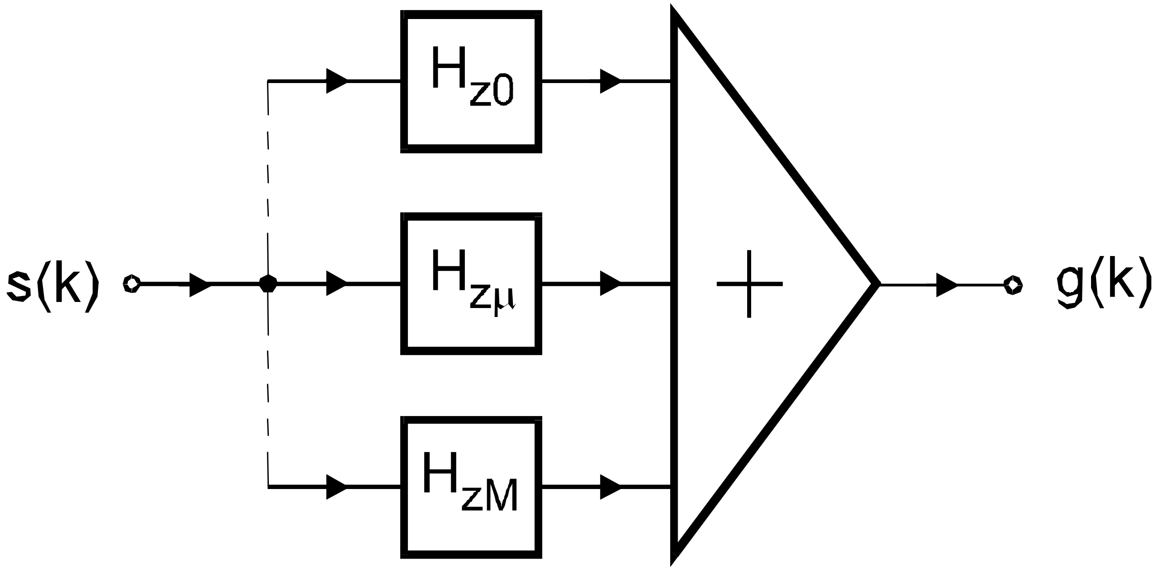
Each of these addends corresponds to a pole. In case of two conjugated complex poles such a pair has to be converted into a real fraction of second order with:

$$\frac{K_\nu}{z - z_{\infty\nu}} \text{ and } \frac{K_{\nu+1}}{z - z_{\infty\nu+1}} \text{ with } K_{\nu+1} = K_\nu^* \text{ and } z_{\infty\nu+1} = z_{\infty\nu}^* \text{ leading to}$$

$$H_z(z) = K_0 + \sum_{\nu=1}^{N_1} \frac{K_\nu}{z - z_{\infty\nu}} = H_{z0}(z) + H_{z1}(z) + H_{z2}(z) + \cdots + H_{z\mu}(z) + \ldots + H_{zM}(z)$$

1.5 Canonical digital filter circuits

The following diagram shows the general filter structure:



Realization of a digital filter using parallel connections

1.5 Canonical digital filter circuits

Example: Realization of a digital filter using parallel connections

Given is one known root of the denominator (*z = -1/8*) and the system function:

$$H_z(z) = \frac{z^{-1} + 0.8125 \cdot z^{-2}}{1 - 0.875 \cdot z^{-1} + 0.375 \cdot z^{-2} + 0.0625 \cdot z^{-3}} = \frac{z^2 + 0.8125 \cdot z}{z^3 - 0.875 \cdot z^2 + 0.375 \cdot z + 0.0625}$$

A decomposition into partial fractions starts with:

$$\begin{array}{l} \quad z^3 - 0.875 \cdot z^2 + 0.375 \cdot z + 0.0625 \; : \; z + 1/8 = z^2 - z + 0.5 \\ - \; \underline{z^3 + 0.125 \cdot z^2} \\ \qquad -z^2 + 0.375 \cdot z + 0.0625 \\ \quad - \; \underline{-z^2 - 0.125 \cdot z} \\ \qquad\qquad 0.5 \cdot z + 0.0625 \\ \quad\quad - \; \underline{0.5 \cdot z + 0.0625} \\ \qquad\qquad\qquad 0 \end{array}$$

1.5 Canonical digital filter circuits

Thus it holds: $H_z(z) = \dfrac{z^2 + 0.8125z}{(z + 1/8)\cdot(z^2 - z + 1/2)}$

$$H_z(z) \overset{!}{=} \frac{A}{(z + 1/8)} + \frac{B \cdot z + D}{(z^2 - z + 1/2)}$$

with $A = \left.\dfrac{z^2 + 0.8125z}{z^2 - z + 0.5}\right|_{z=-1/8} = \dfrac{0.016 - 0.102}{0.016 + 0.125 + 0.5} = -0.134$ and

$$\left.H_z(z)\right|_{z=0} = \frac{-0.134}{0.125} + \frac{D}{1/2} = 0 \Rightarrow -1.073 + 2D = 0 \quad \Rightarrow D = 0.537$$

$$\left.H_z(z)\right|_{z=1} = \frac{-0.134}{1.125} + \frac{B + 0.537}{1/2} = \frac{1.8125}{1 - 0.875 + 0.375 + 0.0625}$$

$$\Rightarrow -0.1191 + 2(B + 0.537) = 3.222 \quad \Rightarrow \mathrm{B} = 1.1337$$

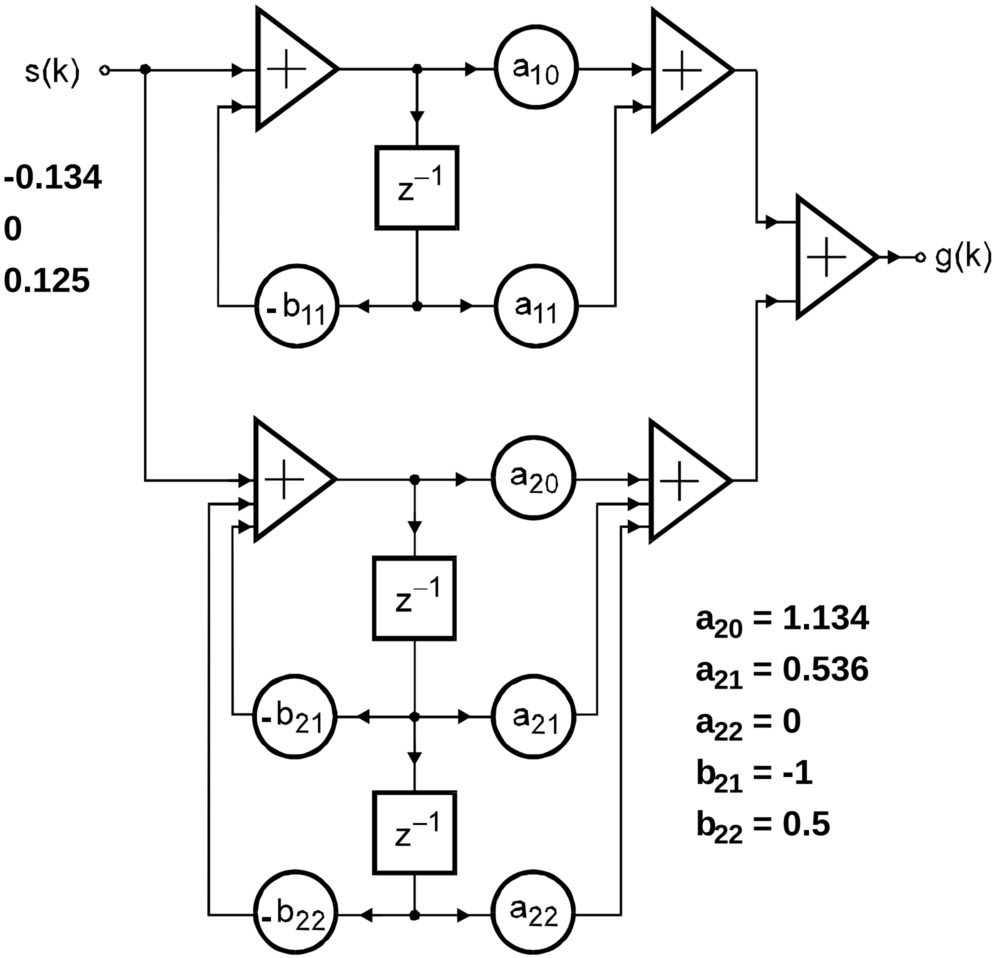
1.5 Canonical digital filter circuits

So finally the partial fractions result in:

$$H_z(z) = \frac{-0.134}{1+0.125\cdot z^{-1}} + \frac{1.134+0.536\cdot z^{-1}}{1-z^{-1}+0.5\cdot z^{-2}}$$

The coefficients of the polynoms of nominator and denominator polynoms then turn into the values of the filter coefficients.

1.5 Canonical digital filter circuits



a_{10} = -0.134
a_{11} = 0
b_{11} = 0.125

a_{20} = 1.134
a_{21} = 0.536
a_{22} = 0
b_{21} = -1
b_{22} = 0.5

Digital filters for parallel connections of systems of 1st and 2nd order