NETWORK THEORY 2
Digital Filters

Chapter 2

Design and Realization of non-recursive Filters

UNIVERSITÄT DUISBURG ESSEN

Fachgebiet
Nachrichtentechnische Systeme

2.1 Prefaces to FIR Digital filters

As shown in chapter 1.3, the system function of a not-recursive digital filter looks as follows:

$$H_z(z) = \sum_{m=0}^{M} a_m \, z^{-m}$$

An inverse z-transform gives its impulse response:

$$h(k) = \sum_{m=0}^{M} a_m \, \gamma_0(k-m)$$

Another correspondence is:

$$G_z(z) = H_z(z) \cdot S_z(z)$$

$$\Big\updownarrow \; z$$

$$g(k) = h(k) \; * \; s(k)$$

2.1 Prefaces to FIR Digital filters

Thus it is obtained:

$$G_z(z) = \left(\sum_{m=0}^{M} a_m z^{-m} \right) \cdot S_z(z) = \sum_{m=0}^{M} \left\{ a_m z^{-m} \, S_z(z) \right\}$$

$$\Big\downarrow \; z$$

$$g(k) = \left\{ \sum_{m=0}^{M} a_m \gamma_0 (k-m) \right\} * s(k) = \sum_{m=0}^{M} a_m s(k-m)$$

From these equations, the following ***advantages*** of FIR digital filters result:

1. FIR digital filters are always stable (no feedback-loop)

2. No group delay distortions

2.1 Prefaces to FIR Digital filters

As ***disadvantages*** of the FIR Digitalfilters are to be called:

1. Compared with corresponding recursive digital filters, FIR filters need more delays (memory), multipliers and adders when approximating a desired frequency response (concerning magnitude or phase).

2. Frequency transforms which convert digital low-pass prototypes into appropriate high-passes, band-passes, band stops may not be used, because recursive structures are produced.

2.2 Design with regulations in the frequency range

When designing a digital filter always the Nyquist condition has to be observed, which demands:

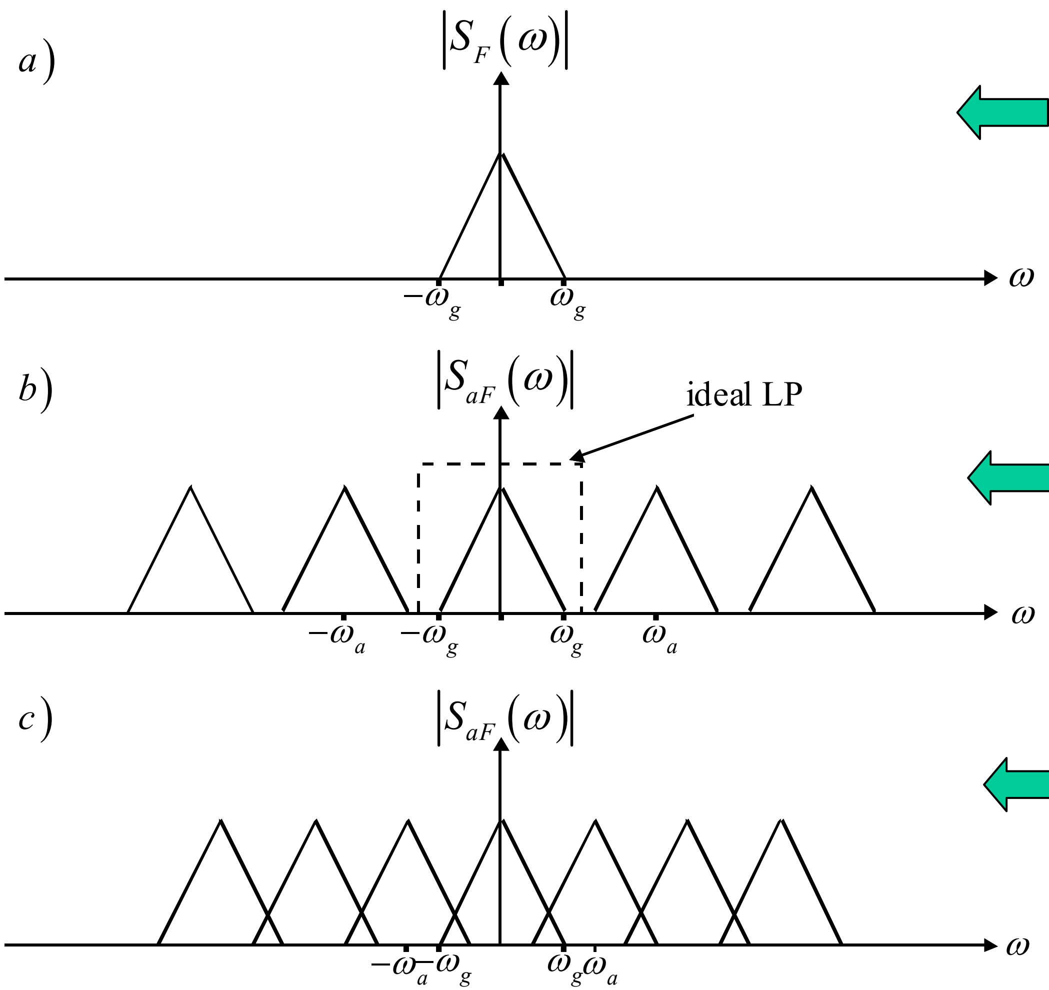
$$\omega_a \geq \omega_{aN} = 2\omega_g$$

This condition limits the frequency range of the desired frequency response, or

otherwise demands a suitably high sampling and operation rate of the system components!

The next diagram shows the different cases of sampling.

2.2 Design with regulations in the frequency range



Spectrum of band-limited signal *s(t)*

Spectrum of oversampled signal

Spectrum of undersampled signal

2.2 Design with regulations in the frequency range

For any digital filter it holds:

$$G_z(e^{j\omega T_a}) = H_z(e^{j\omega T_a}) \cdot S_z(e^{j\omega T_a})$$

with

$$G_{aF}(\omega) = G_z(e^{j\omega T_a})$$

$$S_{aF}(\omega) = S_z(e^{j\omega T_a})$$

$$H_{aF}(\omega) = H_z(e^{j\omega T_a})$$

and

$$G_{aF}(\omega) = S_z(e^{j\omega T_a}) \cdot H_z(e^{j\omega T_a})$$

$$= S_{aF}(\omega) \cdot H_{aF}(\omega)$$

Periodic function

2.2 Design with regulations in the frequency range

The design of any FIR filter leads to determining the filter coefficients such that *h(k)* follows and thus the realised system function follows certain demands:

$$h(k) = \sum_{m=0}^{M} h(m) \cdot \gamma_0(k-m) \quad \text{where} \quad h(m) = a_m$$

$$H_{rz}(z) = \sum_{m=0}^{M} h_m \, z^{-m} = \sum_{m=0}^{M} a_m \, z^{-m}$$

So any realizable FIR transfer function must look as follows:

$$H_{rF}(\omega) = H_{rz}(e^{j\omega T_a}) = \sum_{m=0}^{M} a_m \, e^{-j\omega T_a}$$

2.2 Design with regulations in the frequency range

Thus for any type of FIR filter the amplitude characteristics are given by:

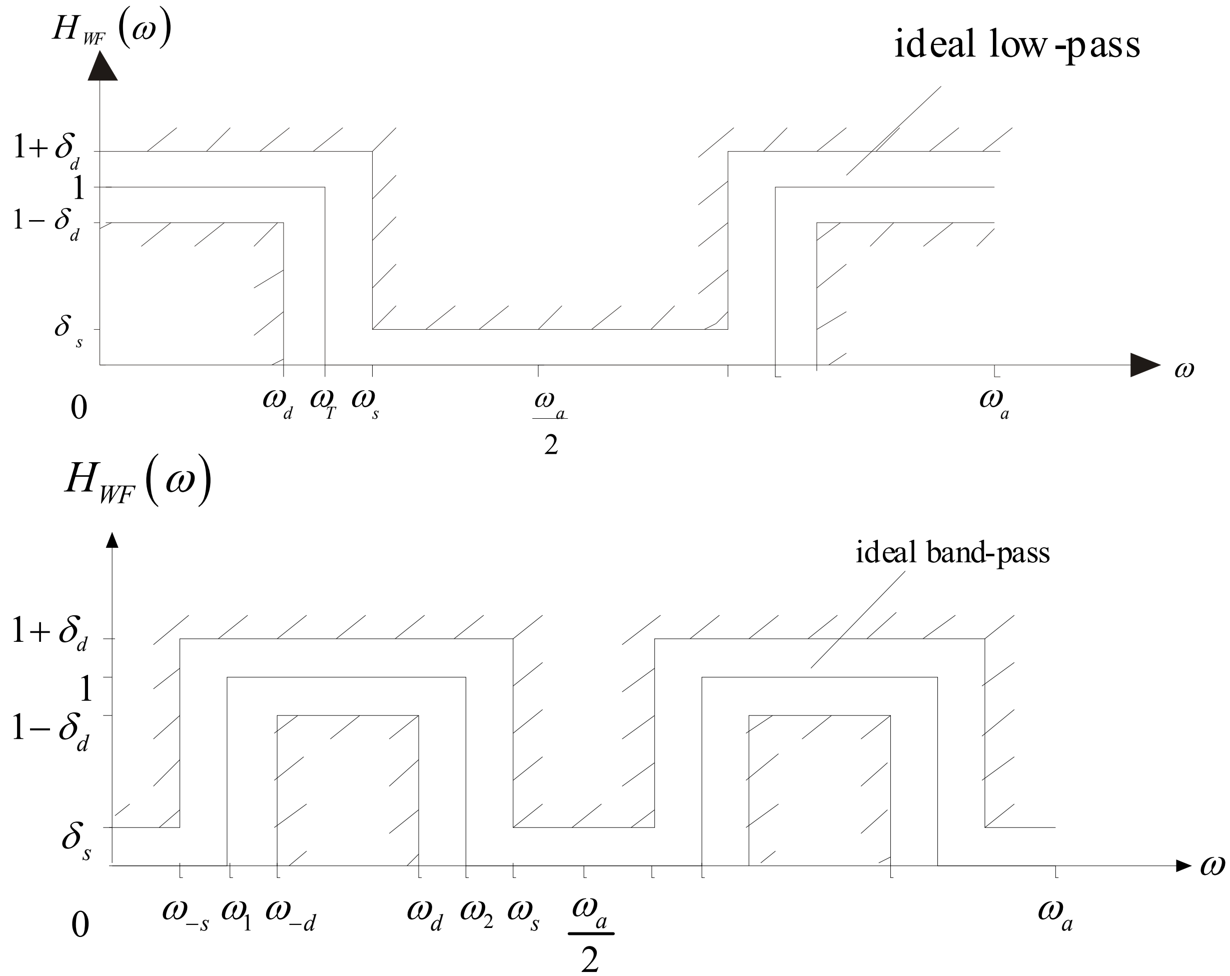
$$\left|H_{rF}(\omega)\right| = \left|H_{rz}(e^{j\omega T_a})\right| = \left|\sum_{m=0}^{M} a_m \, e^{-j\omega T_a}\right|$$

Such a realizable frequency response in principle always is periodic.

Only the two low-passes (in front of the sampling device and for reconstruction of the output signal) lead to a non-periodic behaviour.

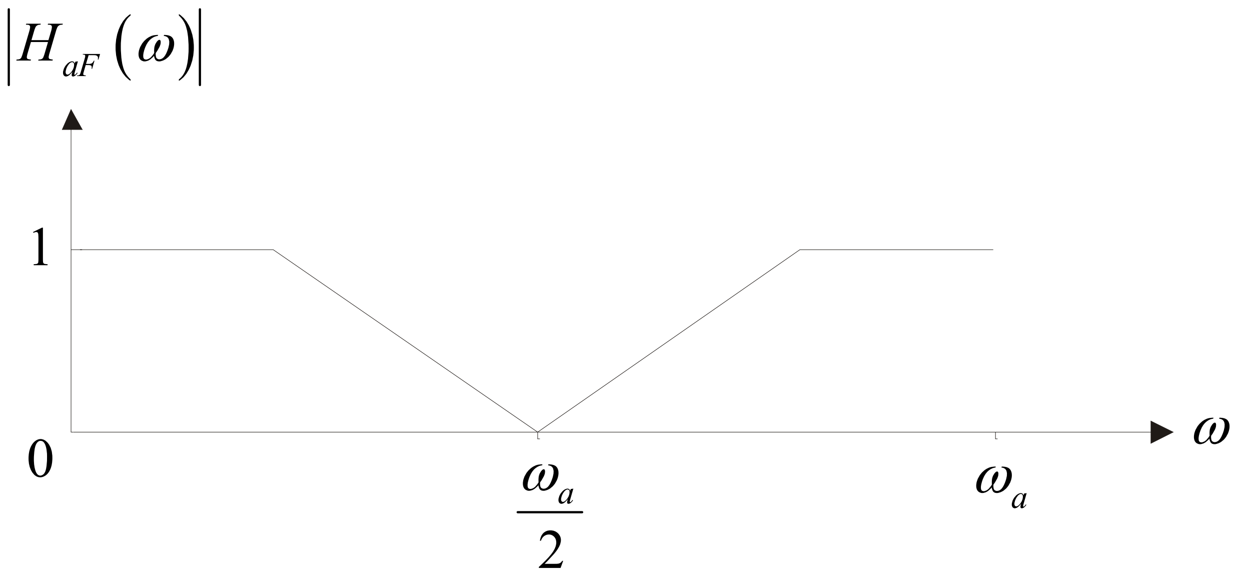
The next slides shows examples of possible (partly ideal) desired frequency responses.

2.2 Design with regulations in the frequency range



Amplitude frequency responses of ideal low-pass and ideal band-pass

2.2 Design with regulations in the frequency range

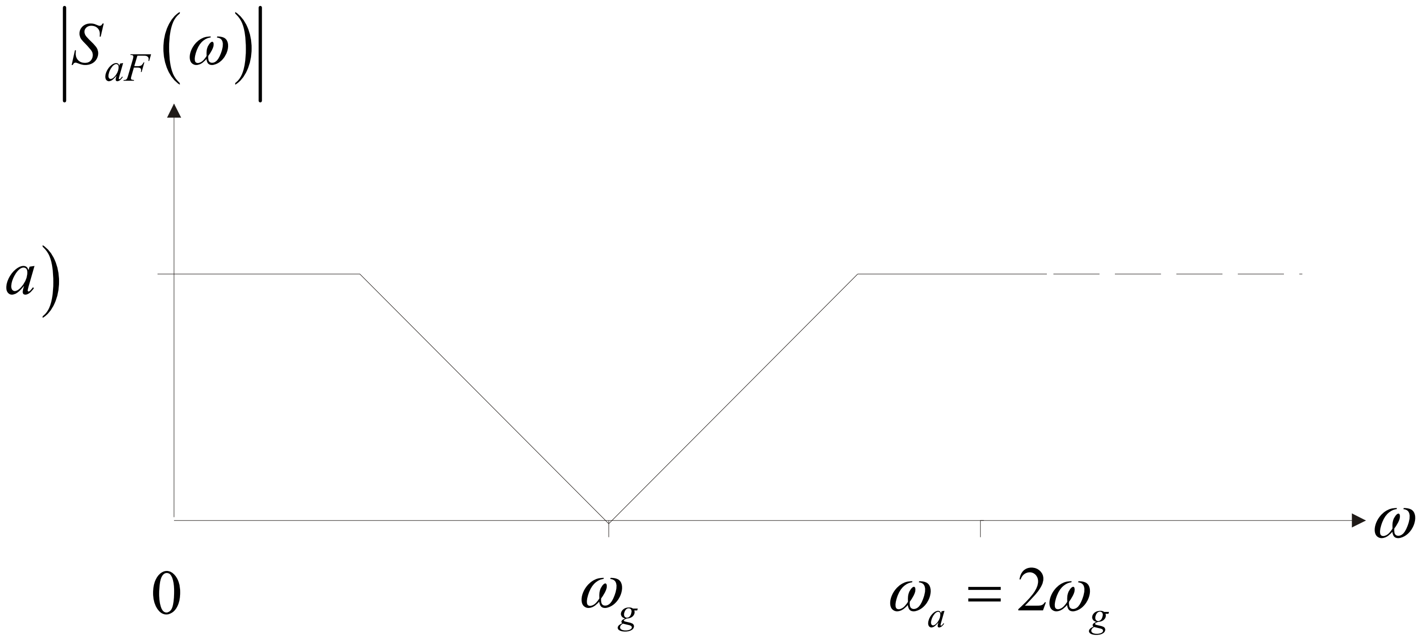


Example: Linear decaying/rising amplitude frequency response of a band-stop

2.2 Design with regulations in the frequency range

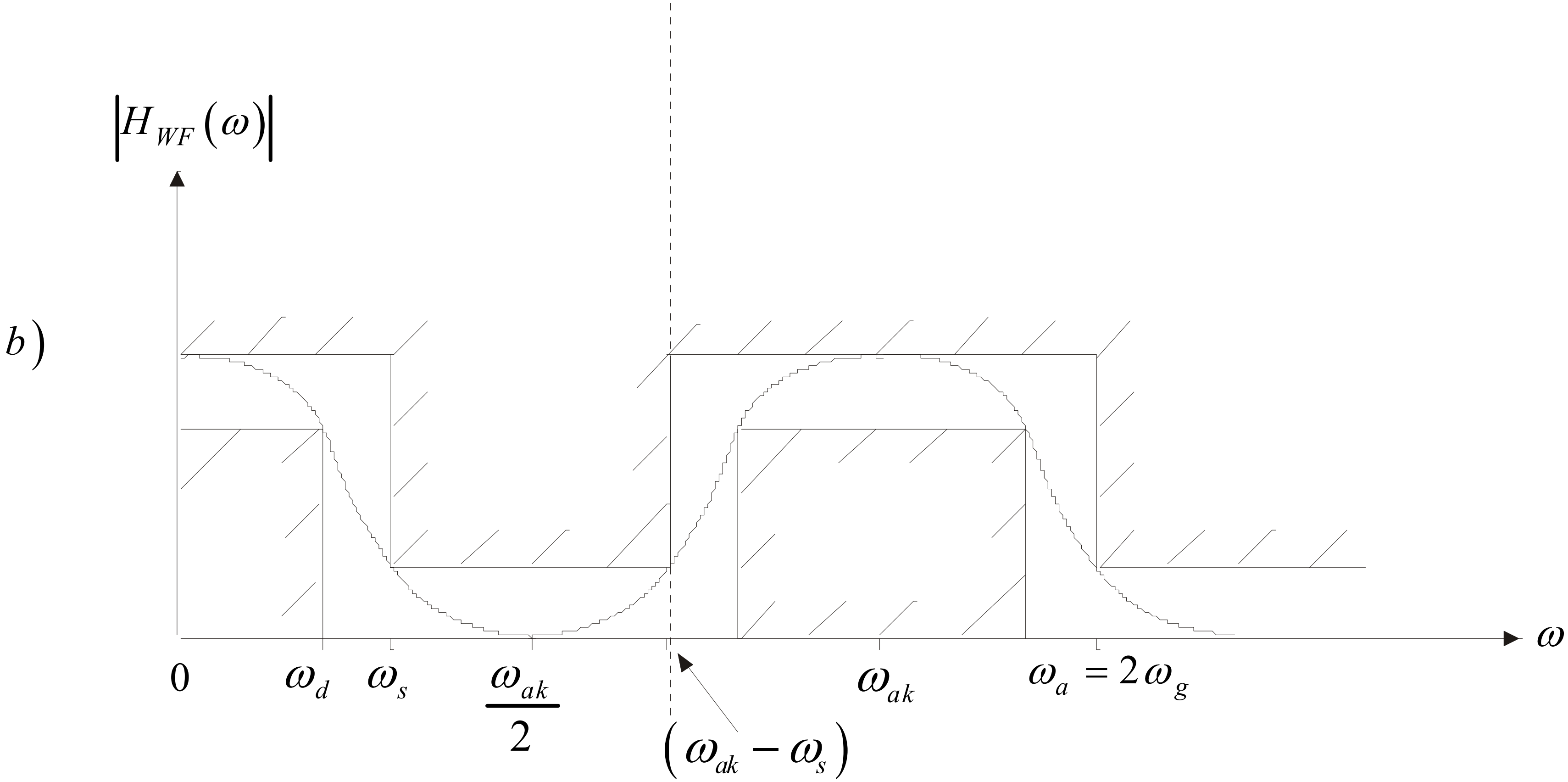
Sometimes (in some seldom cases) a reduced sampling rate can be used if certain conditions hold as shown in the next 3 figures.

Reduced sampling rate ω_{ak} with digital low-pass filters



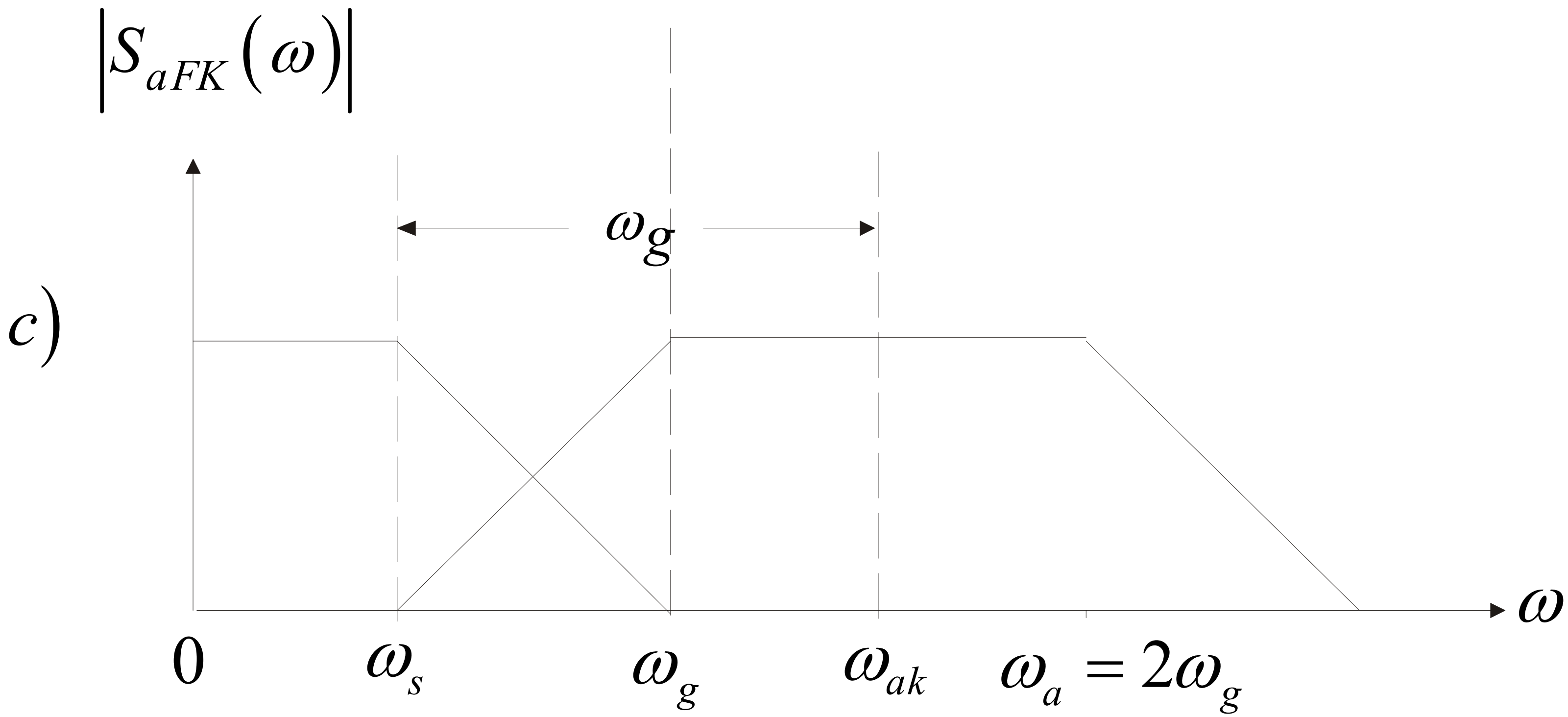
Spectrum of signal sampled at Nyquist rate

2.2 Design with regulations in the frequency range



Spectrum of sampled input signal and frequency response of a low-pass

2.2 Design with regulations in the frequency range



Spectrum of under-sampled signal without filtering effect of the discrete system

For this special signal it does not matter that in the band $\omega_S \leq \omega \leq \omega_g$ alisasing occurs as this is filtered out.

2.2 Design with regulations in the frequency range

From the previous slide, it can be inferred that:

$$\omega_s + \omega_g \leq \omega_{ak} \leq \omega_a = 2\omega_g$$

With $\omega_{ak} = \dfrac{2\pi}{T_{ak}}$, the relationship for scanning period T_{ak} is:

$$T_a = \frac{1}{2f_g} < T_{ak} \leq \frac{2\pi}{\omega_s + \omega_g} = \frac{1}{f_s + f_g}$$

Signal processing at reduced sampling rate ω_{ak} and thus with increased sample period T_{ak} makes possible:

- A lower conversion rate (thus cheaper ADC, DAC and discrete system)
- The realization of higher order digital filters.

2.2 Design with regulations in the frequency range

The realization of nearly ideal digital low-passes, band-pass filters, high-passes or band-stops is based on the **desired (wanted) transfer function** $H_{wF}(\omega)$ of a digital filter

$$H_{wF}(\omega) = \sum_{n=-\infty}^{+\infty} H_{bF}(\omega - n\omega_a) \quad \text{being periodic in } \omega_a = 2\pi / T_a$$

with the **baseband spectral function** according to

$$H_{bF}(\omega) = H_{wF}(\omega) \cdot rect\left(\frac{\omega}{\omega_a}\right)$$

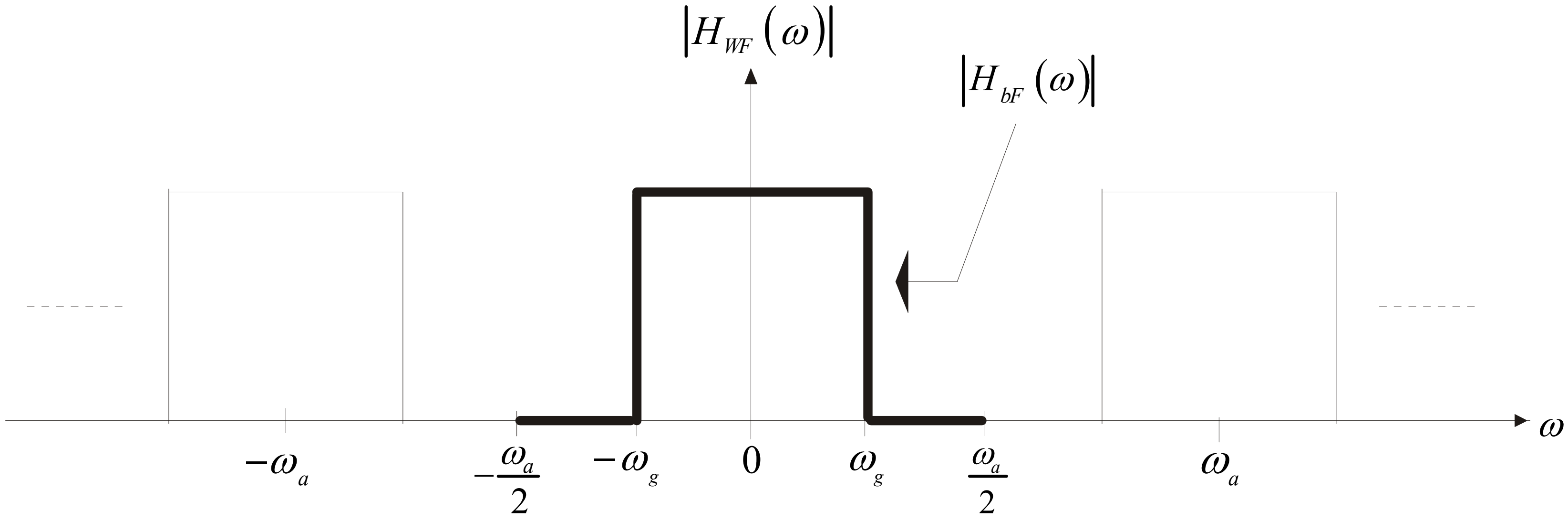
as a spectral part in the frequency range $-\omega_a / 2 \leq \omega \leq +\omega_a / 2$ of $H_{wF}(\omega)$.

Rewriting the equation given gives:

$$H_{wF}(\omega) = H_{bF}(\omega) * \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_a)$$

The next slide shows in an example the relations of these functions.

2.2 Design with regulations in the frequency range



Baseband $H_{BF}(\omega)$ (thick line) and the desired periodic function $H_{wF}(\omega)$

For the further views, only baseband spectral functions $H_{BF}(\omega)$ should be used which fulfil the following conditions:

$\mathrm{Re}\{H_{bF}(\omega)\}$ is even in ω ⟷ $|H_{bF}(\omega)|$ is even in ω

$\mathrm{Im}\{H_{bF}(\omega)\}$ is odd in ω ⟷ $\varphi(\omega) = \arctan\left(\frac{\mathrm{Im}\{H_{bF}(\omega)\}}{\mathrm{Re}\{H_{bF}(\omega)\}}\right)$ is odd in ω

2.2 Design with regulations in the frequency range

If one applies the inverse Fourier transform to

$$H_{wF}(\omega) = H_{bF}(\omega) * \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_a) \quad \text{or} \quad H_{bF}(\omega) = H_{wF}(\omega) \cdot rect(\frac{\omega}{\omega_a})$$

The wanted impulse response is obtained:

$$h_w(t) = 2\pi \cdot h_b(t) \cdot \frac{1}{\omega_a} \sum_{n=-\infty}^{+\infty} \delta(t - nT_a) \quad \text{with} \quad \frac{2\pi}{\omega_a} = T_a$$

and thus:

$$h_w(t) = \sum_{n=-\infty}^{+\infty} T_a \cdot h_b(nT_a) \cdot \delta(t - nT_a)$$

A desired function $H_{bF}(\omega)$ can lead to a non-causal and an over time not limited function $h_w(t)$.

Thus the desired frequency response in general can only be obtained as an approximation!

2.2 Design with regulations in the frequency range

The approximation procedure begins with a windowing of $h_w(t)$ using only a finite number of N Dirac impulses with

$$N = 2N_f + 1 = M + 1$$

This is a window of $h_w(t)$ being realised by means of e.g. the rect function:

$$h_R(t) = rect\left(\frac{t}{2N_f T_a}\right) \quad \text{with} \quad N_f > 0, \ \text{integer}$$

This is used in the following relation which combines a reactangular window and the wanted function to an impulse response of finite length:

$$h_f(t) = h_w(t) \cdot h_R(t) = \sum_{n=-N_f}^{+N_f} T_a \cdot h_b(nT_a) \cdot h_R(nT_a) \cdot \delta(t - nT_a)$$

The windowing effect due to using the rectangular window is now considered by the Fourier transform of $h_f(t)$.

$$H_{fF}(\omega) = \frac{1}{2\pi} H_{wF}(\omega) * H_{RF}(\omega)$$

2.2 Design with regulations in the frequency range

The Fourier transform of the rectangular window is:

$$H_{RF}(\omega) = 2N_f T_a \cdot si(\omega N_f T_a)$$

Thus for $H_{fF}(\omega)$ the following relationship applies:

$$H_{fF}(\omega) = \frac{1}{2\pi} H_{wF}(\omega) * 2N_f T_a si(\omega N_f T_a)$$

$$= \frac{N_f T_a}{\pi} \cdot H_{wF}(\omega) * si(\omega N_f T_a)$$

Thus the limitation in time domain leads in the frequency domain to a convolution with the si-function. So always (even for high values of *N*) an oszillating property in the spectrum of $H_{fF}(\omega)$ is obtained.

Here we see once more the Gibb‘s phenomenon.

<u>Another observation</u>: $h_f(t)$ is in general <u>not causal</u>, which calls for a countermeasure (for being able to realize such filters at all).

2.2 Design with regulations in the frequency range

The countermeasure is to shift $h_f(t)$ by the period $N_f T_a$ so that in any case a causal signal is obtained.

Note: This has no effect on the magnitude frequency response (as it is just a delay).

So the realizable impulse response follows the equation:

$$h_r(t) = h_f(t - N_f T_a) = \sum_{n=-N_f}^{+N_f} T_a \cdot h_b(nT_a) \cdot h_R(nT_a) \cdot \delta\left(t - \left\{n + N_f\right\} T_a\right)$$

After the substitution $k = n + N_f \quad \Rightarrow \quad n = k - N_f$

the following equation results:

$$h_r(t) = \sum_{k=0}^{2N_f} T_a \cdot h_b\left(\left\{k - N_f\right\} T_a\right) \cdot h_R\left(\left\{k - N_f\right\} T_a\right) \cdot \delta(t - kT_a)$$

2.2 Design with regulations in the frequency range

Now the first expression in the summation is abbreviated using a so-called **baseband sequence**:

$$h_{bk}(k) = T_a \cdot h_b\left(\left\{k - N_f\right\} T_a\right)$$

Furthermore a causal and in k finite **rectangular window sequence** with **the window length** $2N_f$ is defined (it is the second expression in the summation):

$$h_{kR}(k) = \begin{cases} h_R\left(\left\{k - N_f\right\} T_a\right) & \text{for } k = 0(1)2N_f \\ 0 & \text{for } k < 0 \text{ and } k > 2N_f \end{cases}$$

Thus $h_r(t)$ can be rewritten in shorter form as follows:

$$h_r(t) = \sum_{k=0}^{2N_f} h_{bk}(k) \cdot h_{kR}(k) \cdot \delta(t - kT_a)$$

2.2 Design with regulations in the frequency range

For rectangular windowing concerning the approximate **impulse response** $h_{rk}(k)$ of the digital filter holds:

$$h_{kr}(k) = h_{bk}(k) \cdot h_{kR}(k) \quad = \begin{cases} h_{bk}(k) & for \quad k = 0(1)2N_f \\ 0 & else \end{cases}$$

$a)$ non causal and infinite

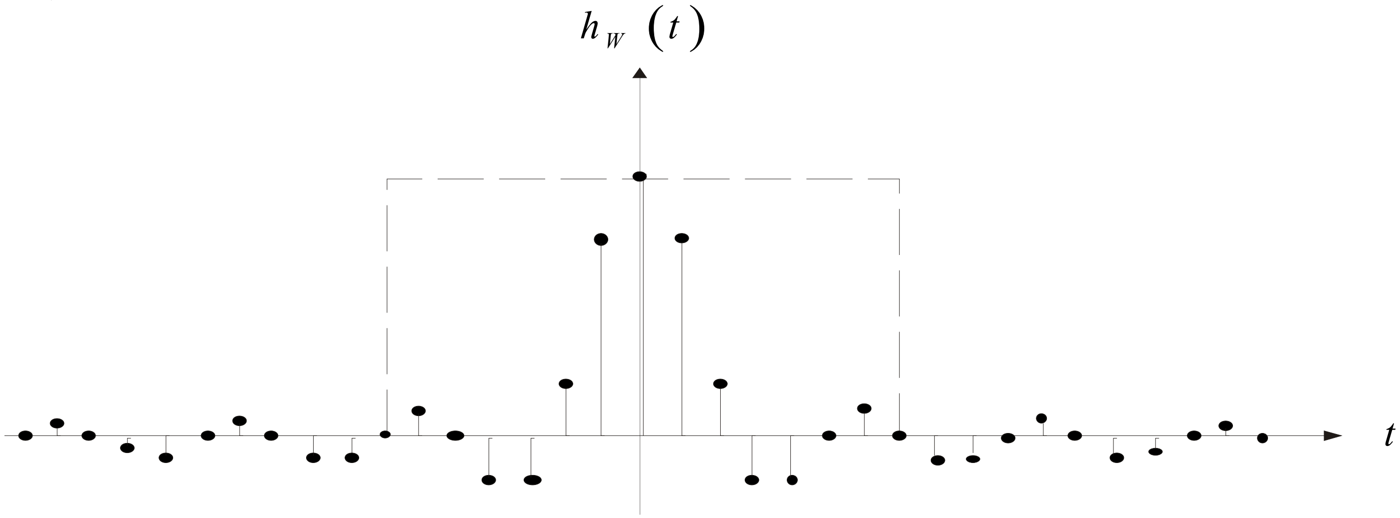


Illustration of sampled functions for the realization of an FIR low-pass filter

2.2 Design with regulations in the frequency range

Here the following steps are performed:

- The infinite and non-causal impulse response $h_w(t)$ must be converted into a finite but non-caual function $h_f(t)$
- Then a shift (delay) of half of the window width is applied leading to the causal impulse response $h_r(t)$

In the following slides also the effect using such a windowing is shown.

It shows the Gibb's effect in the frequency domain and leads to the non-ideal form of the desired amplitude frequency response (with oscillations).

2.2 Design with regulations in the frequency range

$b)$ non causal but finite

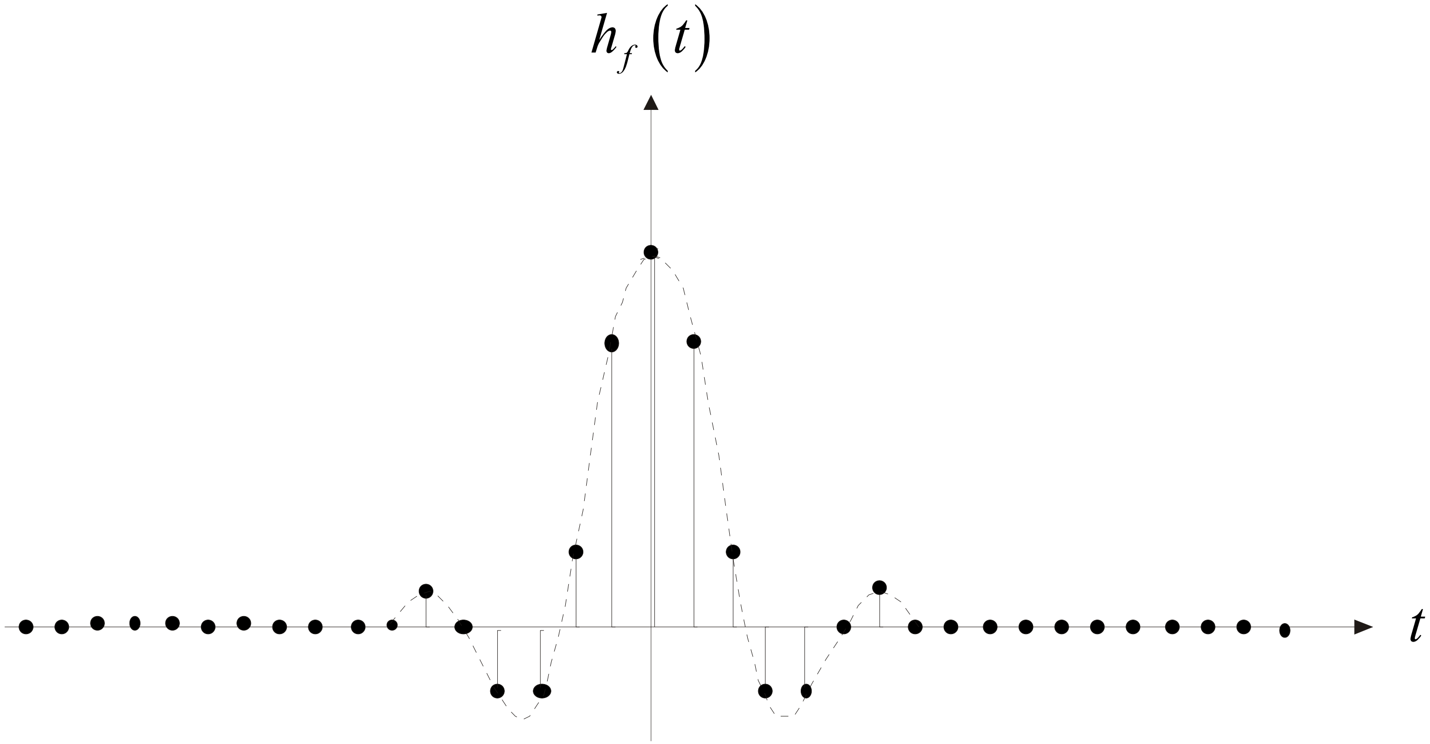


Illustration of sampled functions for the realization of an FIR low-pass filter

2.2 Design with regulations in the frequency range

$c)$ causal and finite

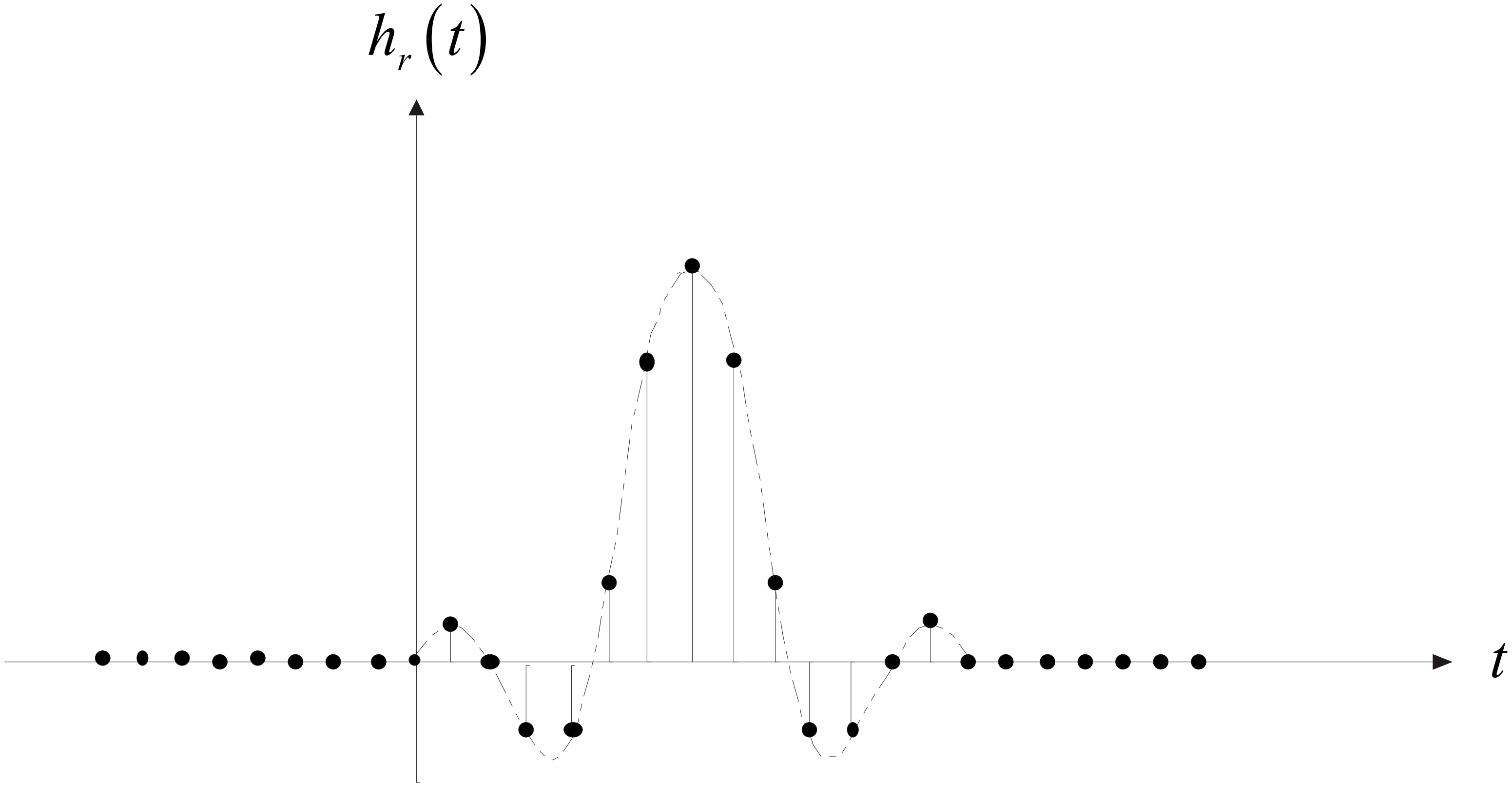
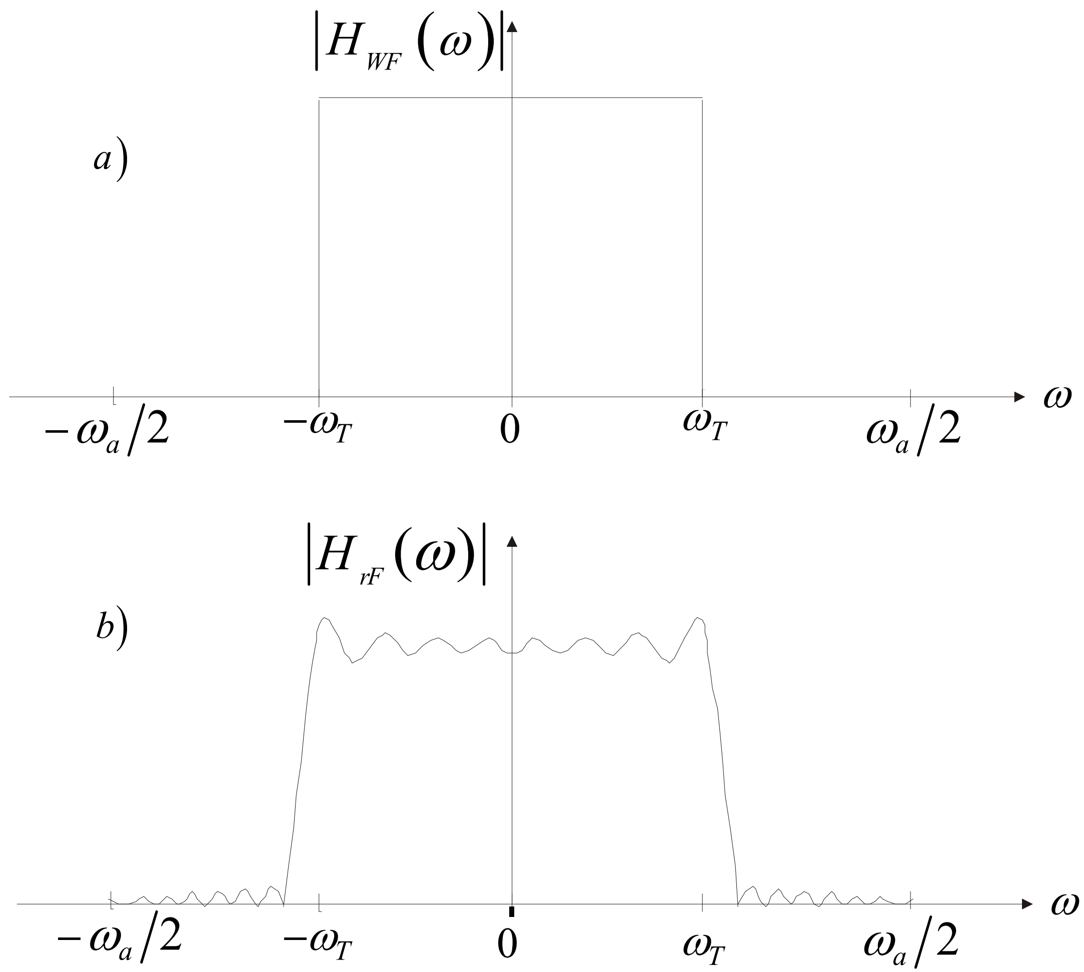


Illustration of sampled functions for the realization of an FIR low-pass filter

2.2 Design with regulations in the frequency range



Spectrum of the ideal and of the approximated digital low-pass

2.2 Design with regulations in the frequency range

The computation of the values of the baseband sequence

$$h_{bk}(k) = T_a \cdot h_b\left(\left\{k - N_f\right\} T_a\right) \quad \text{for } k = -\infty \ (1) + \infty$$

takes place in several steps:

First from a given baseband spectral function $H_{bF}(\omega)$, the <u>impulse response $h_b(t)$ is computed</u>:

$$h_b(t) = \frac{1}{2\pi} \int_{-\omega_a/2}^{+\omega_a/2} H_{bF}(\omega) \cdot e^{j\omega t} d\omega$$

Sampling of this function at $t = \left\{k - N_f\right\} T_a$ gives:

$$h_b\left(\left\{k - N_f\right\} T\right) = \frac{1}{2\pi} \int_{-\omega_a/2}^{+\omega_a/2} H_{bF}(\omega) \cdot e^{j\omega\left\{k - N_f\right\} T_a} d\omega$$

2.2 Design with regulations in the frequency range

Finally the baseband sequences $h_{bk}(k)$ can be computed as:

$$h_{bk}(k) = T_a \cdot h_b\left(\left\{k - N_f\right\} T_a\right)$$

From the time-discrete equation of the impulse response

$$h_r(t) = \sum_{k=0}^{2N_f = M} h_{bk}(k) \cdot h_{kR}(k) \cdot \delta(t - kT_a) = \sum_{k=0}^{2N_f = M} h_{bk}(k) \cdot \delta(t - kT_a)$$

the system function of the assigned analog filter follows:

$$H_{rL}(p) = \sum_{k=0}^{M} h_{kb}(k) \cdot e^{-pkT_a}$$

Thus the formula for the computation of the system function of the realizable FIR digital filter is:

$$H_{rL}(p) = \sum_{k=0}^{M} h_{kb}(k) \cdot z^{-k} = H_{rz}(z)\Big|_{z=e^{pT_a}}$$

2.2 Design with regulations in the frequency range

- Here the important relations for a rectangular window are summarized:

$$h_w(t) = \sum_{n=-\infty}^{+\infty} T_a \cdot h_b(nT_a) \cdot \delta(t - nT_a) \qquad H_{wF}(\omega) = H_{bF}(\omega) * \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_a)$$

$$h_f(t) = h_w(t)\, rect(\frac{t}{2N_f T_a}) \qquad H_{fF}(\omega) = \frac{N_f T_a}{\pi} \cdot H_{wF}(\omega) * si(\omega N_f T_a)$$

$$h_r(t) = h_f(t - N_f T_a) \qquad H_{rF}(\omega) = H_{fF}(\omega) \cdot e^{-j\omega N_f T_a}$$

$$h_{bk}(k) = T_a \cdot h_b\left(\left\{k - N_f\right\} T_a\right) \qquad H_{rL}(p) = \sum_{k=0}^{M} h_{kb}(k) \cdot z^{-k} = H_{rz}(z)\Big|_{z=e^{pT_a}}$$

- In the following improved windows are presented with superior properties compared to the rectangular window.

2.2 Design with regulations in the frequency range

Instead of simple rectangular windows other window functions are known which very much reduce the overshoots due to the Gibb's effect!

The **HANN-window function** is one of these preferred functions**:**

$$h_{HN}(t) = \begin{cases} 0.5 + 0.5 \cdot \cos\left(\dfrac{\pi}{N_f T_a} t \right) & for \;\; -N_f T_a \leq t \leq +N_f T_a \\ 0 & else \end{cases}$$

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2.2 Design with regulations in the frequency range

By multiplying the signal $h_w(t)$ with the Hann window, one receives:

$$h_f(t) = h_w(t)\cdot h_{HN}(t) = \sum_{n=-N_f}^{+N_f} T_a\cdot h_b(nT_a)\cdot\left\{0.5+0.5\cdot\cos\left(\frac{\pi\, n\, T_a}{N_f T_a}\right)\right\}\cdot\delta(t-nT_a)$$

Additional shifting to the right about an amount of $N_f T_a$, the realizable (and causal) impulse response results:

$$h_r(t) = h_f(t-N_f T_a) = \sum_{n=-N_f}^{+N_f} \{T_a\cdot h_b(nT_a)\}\cdot\left\{0.5+0.5\cdot\cos\left(\frac{\pi\, n}{N_f}\right)\right\}\cdot\delta\left(t-\{n+N_f\}T_a\right)$$

From this by substituting $n = k - N_f \Leftrightarrow k = n + N_f$ it is obtained:

$$h_r(t) = \sum_{n=-N_f}^{+N_f} \{T_a\cdot h_b((k-N_f)T_a)\}\cdot\left\{0.5+0.5\cdot\cos\left(\frac{\pi\,(k-N_f)}{N_f}\right)\right\}\cdot\delta\left(t-kT_a\right)$$

$$= \sum_{n=-N_f}^{+N_f} h_{bk}(k)\cdot\left\{0.5+0.5\cdot\cos\left(\frac{\pi\,(k-N_f)}{N_f}\right)\right\}\cdot\delta\left(t-kT_a\right)$$

2.2 Design with regulations in the frequency range

Using the **HANN-discrete sequences:**

$$h_{kHN}(k) = \begin{cases} 0.5 + 0.5 \cdot \cos\left(\dfrac{\pi\left\{k - N_f\right\}}{N_f}\right) & with\ \ k = 0(1)2N_f \\ 0 & \text{else} \end{cases}$$

the realizable impulse response for the analog filter can be described as:

$$h_r(t) = \sum_{k=0}^{M} h_{bk}(k) \cdot h_{HNk}(k) \cdot \delta\left(t - kT_a\right)$$

This corresponds to the following digital filter impulse response:

$$h_{rk}(k) = h_{bk}(k) \cdot h_{HNk}(k) = \sum_{m=0}^{M} h_{bk}(m) \cdot h_{HNk}(m) \cdot \gamma_0(k - m)$$

2.2 Design with regulations in the frequency range

Another related windowing method is based on **the HAMMING function**:

$$h_{HM}(t) = \begin{cases} 0.54 + 0.46 \cdot \cos(\frac{\pi}{N_f T_a} t) & for \quad -N_f T_a \leq t \leq +N_f T_a \\ 0 & else \end{cases}$$

Here the impulse response of the assigned FIR Digitalfilters is as follows:

$$h_{rk}(k) = h_{bk}(k) \cdot h_{HMk}(k) = \sum_{m=0}^{M} h_{bk}(m) \cdot h_{HMk}(m) \cdot \gamma_0(k-m)$$

In this equation the sequence $h_{HMk}(k) = h_{HM}\left(\left\{k - N_f\right\} T_a\right)$ represents the assigned causal HAMMING discrete sequence :

$$h_{HMk}(k) = 0.54 + 0.46 \cdot \cos\left(\frac{\pi\left\{k - N_f\right\}}{N_f}\right) \quad with \quad k = 0(1)2N_f$$

2.2 Design with regulations in the frequency range

The BLACKMAN window function

$$h_B(t) = \begin{cases} 0.42 + 0.5 \cdot \cos\left(\dfrac{\pi}{N_f T_a} t\right) + 0.08 \cdot \cos\left(\dfrac{2\pi}{N_f T_a} t\right) & \text{for } -N_f T_a \le t \le +N_f T_a \\ 0 & \text{else} \end{cases}$$

With the approriate $h_{Bk}(k) = h_B\left(\left\{k - N_f\right\} T_a\right)$, the BLACKMAN discrete sequence follows:

$$h_B(k) = \begin{cases} 0.42 + 0.5 \cos\left(\dfrac{\pi\left\{k - N_f\right\}}{N_f}\right) + 0.08 \cdot \cos\left(\dfrac{2\pi\left\{k - N_f\right\}}{N_f}\right) & \textit{with } k = 0(1)2N_f \\ 0 & \textit{else} \end{cases}$$

2.2 Design with regulations in the frequency range

Compared to the previous windows the **Kaiser window function** offers an additional parameter α for controlling the approximation:

$$h_K(t) = \begin{cases} \dfrac{I_0\left(\alpha \cdot \sqrt{1-\left(\dfrac{t}{N_f T_a}\right)^2}\right)}{I_0(\alpha)} & \mathit{for} \;\; -N_f T_a \le t \le +N_f T_a \\ 0 & \mathit{else} \end{cases}$$

Here $I_0(x)$ represents the modified Bessel function which is defined as:

$$I_0(x) = 1 + \sum_{m=1}^{\infty} \left(\frac{(x/2)^m}{m!} \right)^2$$

In practice one stops the summation when the elements of the sum show a value $< 10^{-8}$.

2.2 Design with regulations in the frequency range

The substitution $h_{Kk}(k) = h_K\left(\left\{k - N_f\right\} T_a\right)$ leads to the KAISER discrete sequence:

$$h_{kK}(k) = \begin{cases} \dfrac{I_0\left(\alpha \cdot \sqrt{1 - \left(\dfrac{k - N_f}{N_f}\right)^2}\right)}{I_0(\alpha)} & \text{for } k = 0(1)2N_f \\ 0 & \text{else} \end{cases}$$

For the realizable impulse response $h_{rk}(k)$ it then applies:

$$H_{rz}(z) = \sum_{k=0}^{2N_f = M} h_{kb}(k) \cdot h_{kK}(k) \cdot z^{-k}$$

2.2 Design with regulations in the frequency range

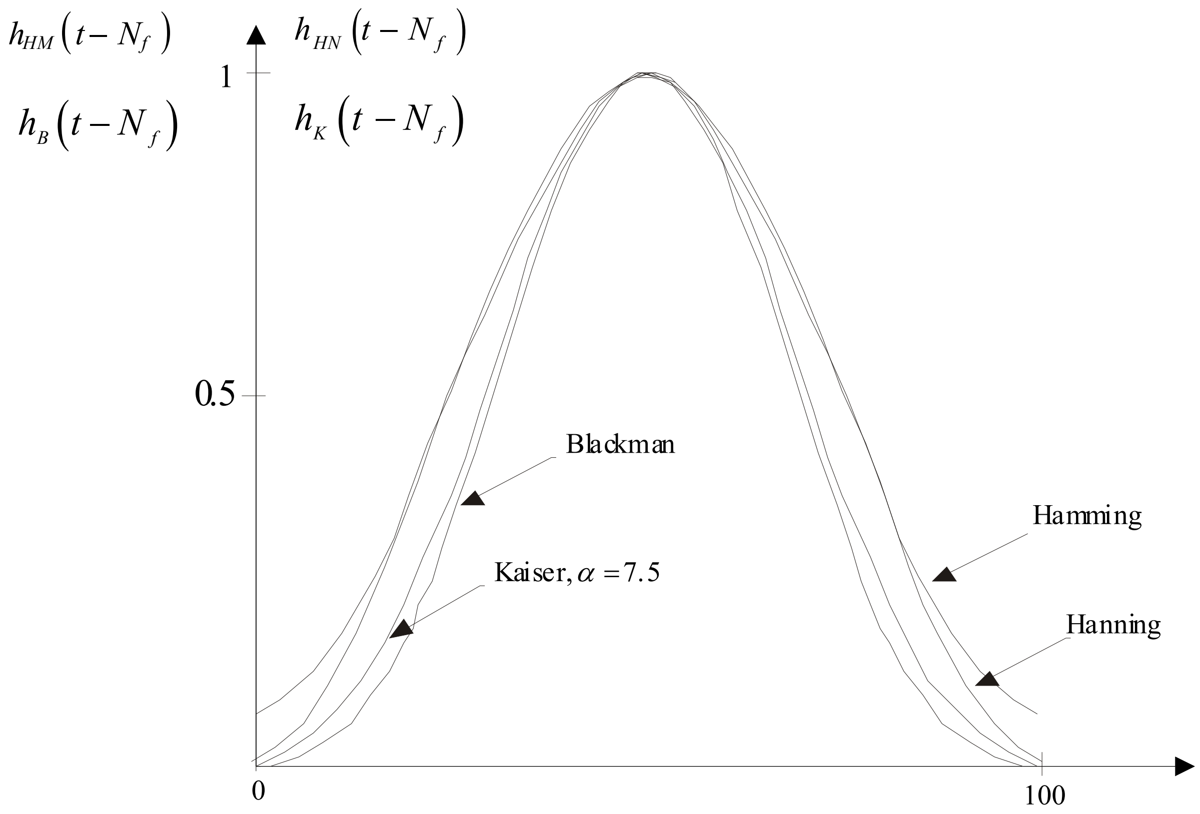
The parameter α is typically set in the range of 4 to 9.

For a given window width of $2N_f$ an additional possibility is introduced for further modification of the amplitude frequency response $H_{rF}(\omega)$.

Thus a compromise between window width and acceptable overshoots often can be found.

The next figure shows the important window functions in a comparison.

2.2 Design with regulations in the frequency range



Envelope of the window sequences after Hamming, Hanning, Blackman and Kaiser for a window length of ***$2N_f=100$***

2.2 Design with regulations in the frequency range

Conclusions:

At the transition of pass and stop-band a rectangular window leads to significant overshoots in the frequency response due to the Gibb's effect.

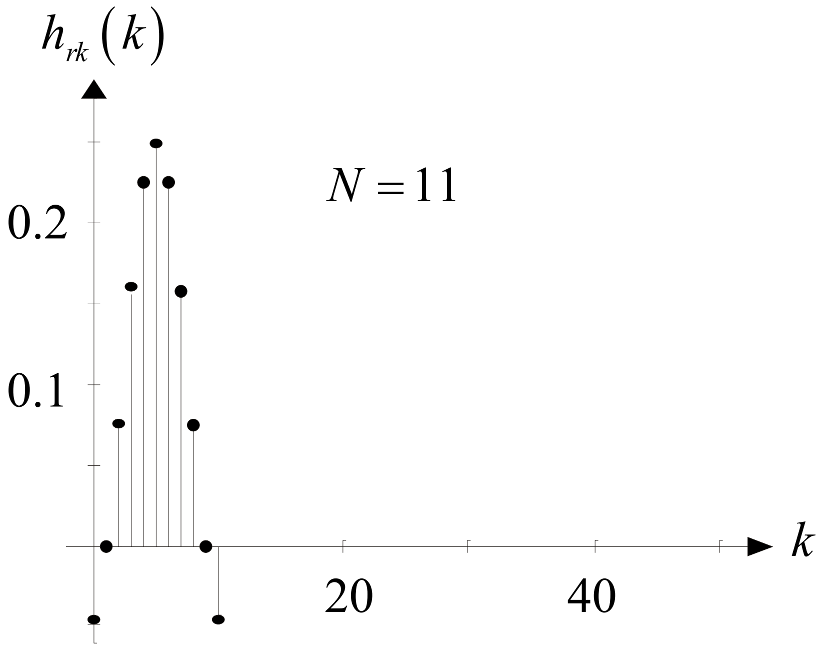
At the transition of pass and stop-band other windows lead to reduced overshoots in the frequency response.

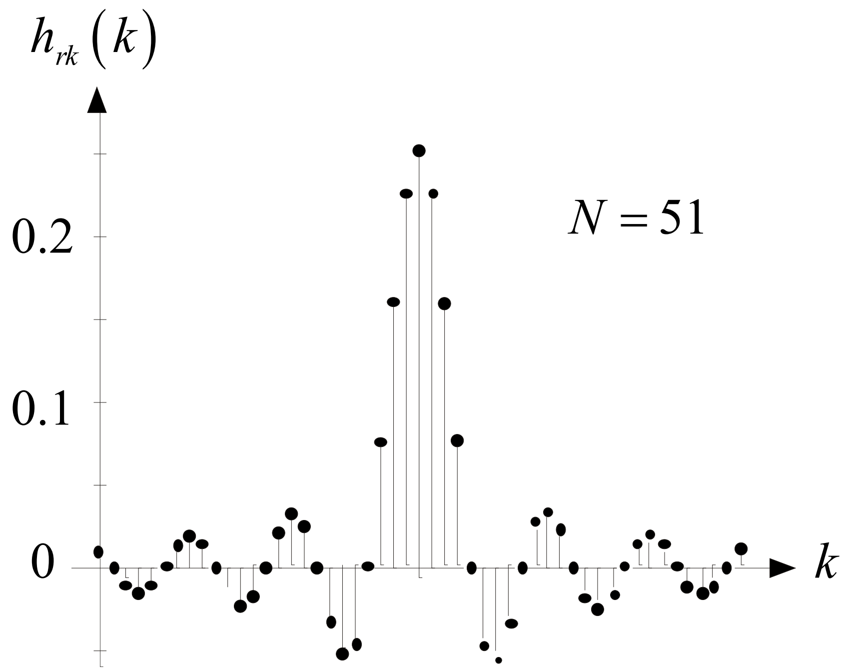
On the other hand the transition range is widened a bit (with smoother transition).

Further reduction of overshoots can only be realised at the cost of increasing the value of N.

This means a corresponding increase in the realisation efforts and costs.

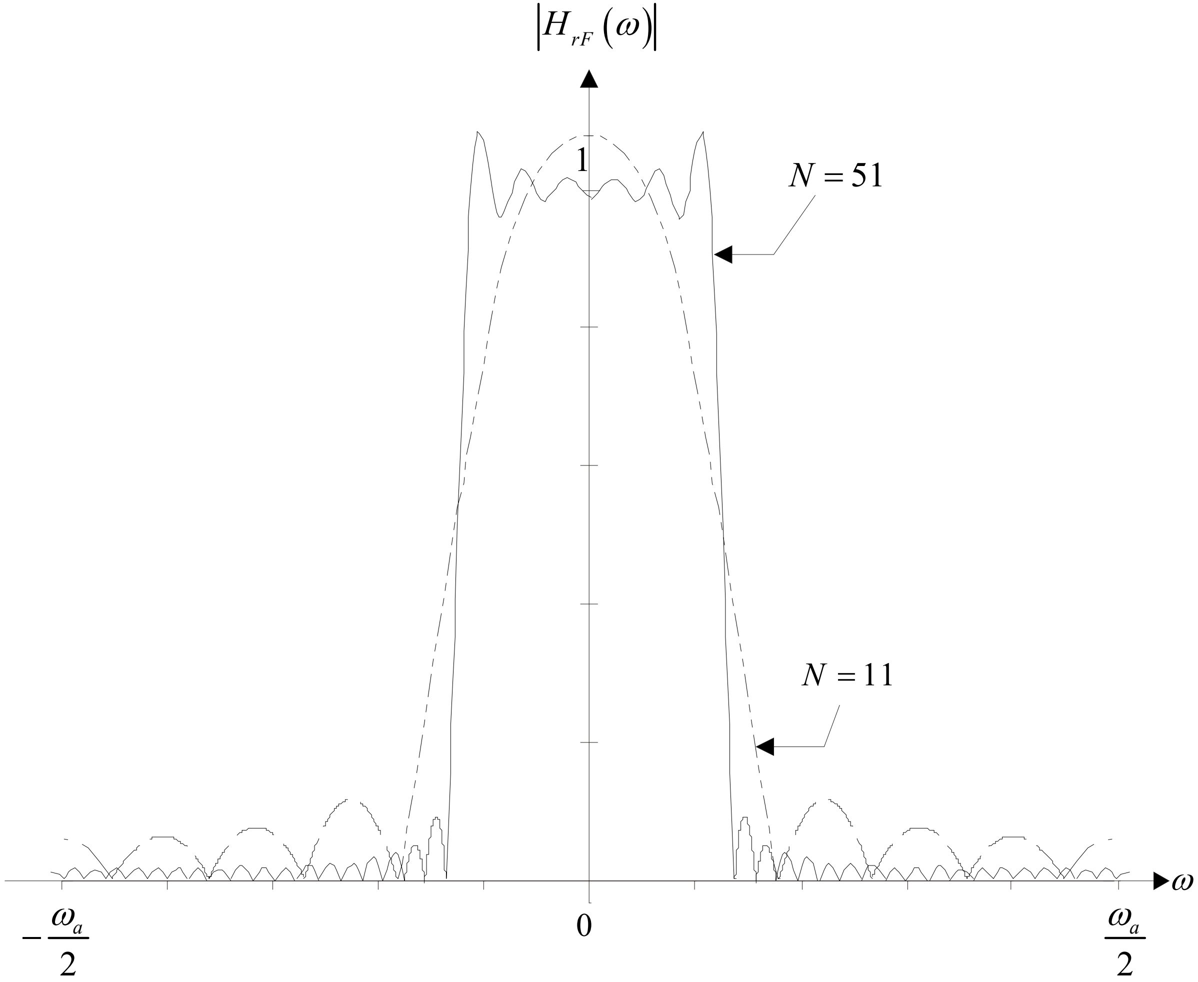
2.2 Design with regulations in the frequency range





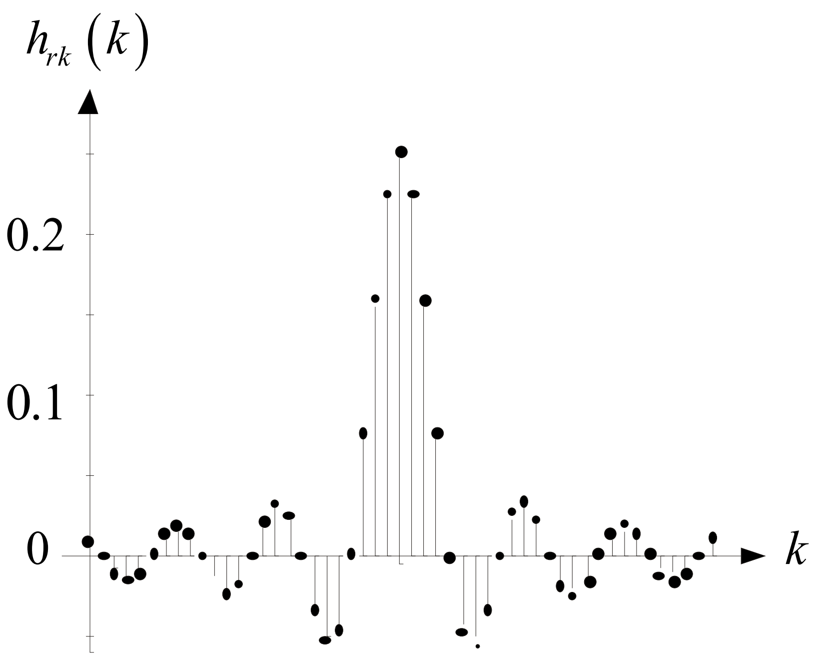
Influence of the number $N = 2N_f + 1 = M + 1$ of filter coefficients on the impulse response $h_{rk}(k)$ and on the amplitude characteristic of a FIR digital low-pass with $\omega_c = \omega_a / 8$

2.2 Design with regulations in the frequency range

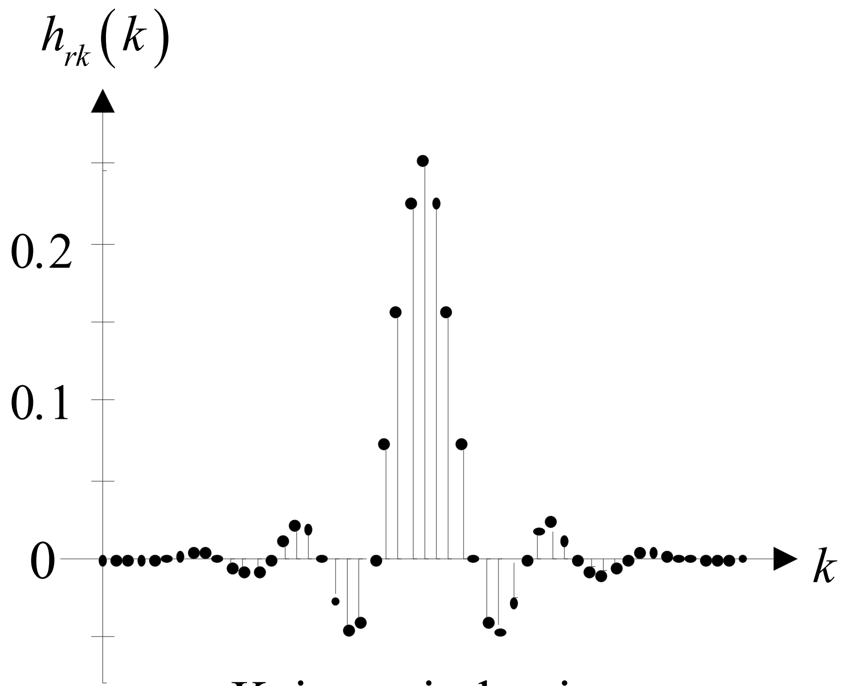


Influence of different values for the parameter N (or different window width) for an approximation of an ideal low-pass with $\omega_c = \omega_a / 8$

2.2 Design with regulations in the frequency range



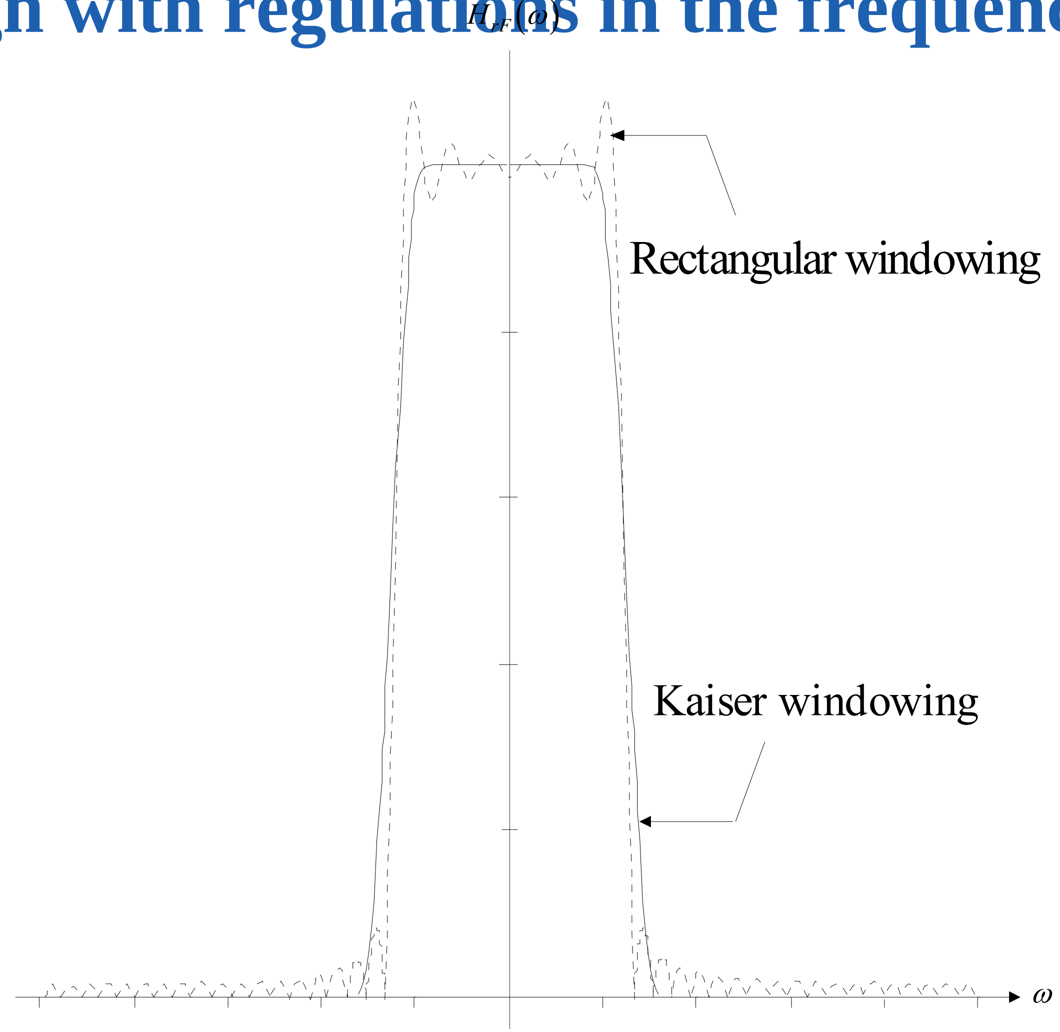
Rectangular windowing



Kaiser windowing

Influence of the rectangular and of the Kaiser windows on impulse response and on amplitude characteristic of an ideal low-pass filter

2.2 Design with regulations in the frequency range



Frequency response for rectangular and Kaiser windowing

2.2 Design with regulations in the frequency range

Example 1: **Low-pass baseband sequence**

The assigned baseband spectral function for the ideal low-pass is:

$$H_{bF}(\omega) = rect\left(\frac{\omega}{2\omega_T}\right) with \quad 0 < \omega_T < \frac{\omega_a}{2}$$

and $h_b(t) = \dfrac{\omega_T}{\pi} si(\omega_T t)$

The appropriate non-causal, infinite low-pass baseband sequence can be computed as follows:

$$h_{bk}(k) = T_a h_b((k - N_f)T_a) = \frac{2\pi}{\omega_a} h_b((k - N_f)\frac{2\pi}{\omega_a})$$

$$= 2\frac{\omega_T}{\omega_a} \cdot si\left(2\pi\frac{\omega_T}{\omega_a}(k - N_f)\right)$$

2.2 Design with regulations in the frequency range

Example 2: **Band-pass filter baseband sequence**

The baseband spectrum of an ideal band-pass filter can be written as:

$$H_{bF}(\omega) = rect\left(\frac{\omega}{\Delta\omega}\right) * \left\{\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right\}$$

with $\omega_0 = \dfrac{\omega_2+\omega_1}{2}$ as the centre frequency

and $\Delta\omega = \omega_2 - \omega_1$ as the width of the pass band

and $$h_b(t) = 2\pi\frac{\Delta\omega}{2\pi} si\left(\frac{\Delta\omega t}{2}\right)\cdot\frac{1}{\pi}\cos(\omega_0 t)$$

$$= \Delta\omega\, si\left(\frac{\Delta\omega t}{2}\right)\cdot\frac{1}{\pi}\cos(\omega_0 t)$$

2.2 Design with regulations in the frequency range

The assigned infinite band-pass filter baseband sequence then results as follows:

$$h_{kb}(k) = T_a h_b((k-N_f)T_a) = \frac{2\pi}{\omega_a} h_b((k-N_f)\frac{2\pi}{\omega_a})$$

$$= \frac{2\pi}{\omega_a} \cdot \Delta\omega si(\frac{\Delta\omega t}{2}) \cdot \frac{1}{\pi} \cos(\omega_0 t)\Big|_{t=(k-N_f)\frac{2\pi}{\omega_a}}$$

$$= \frac{2\Delta\omega}{\omega_a} \cdot si\left(\frac{\pi\Delta\omega}{\omega_a}\left\{k-N_f\right\}\right) \cdot \cos\left(\frac{2\pi\omega_0}{\omega_a}\left\{k-N_f\right\}\right)$$

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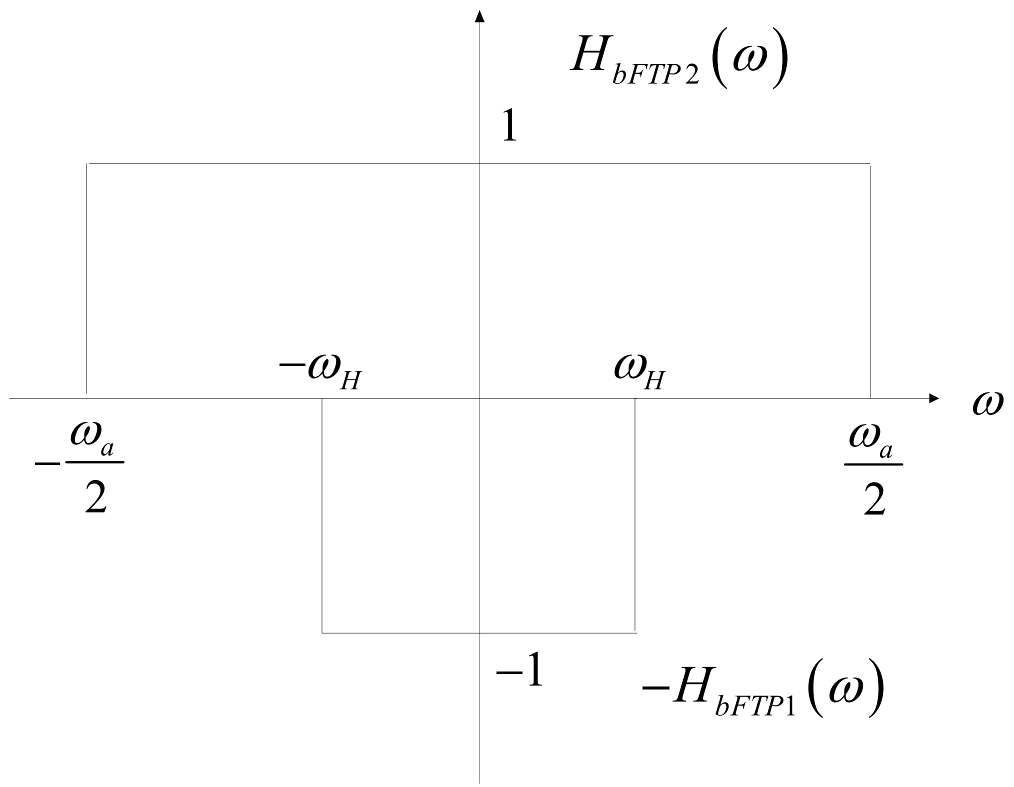
2.2 Design with regulations in the frequency range

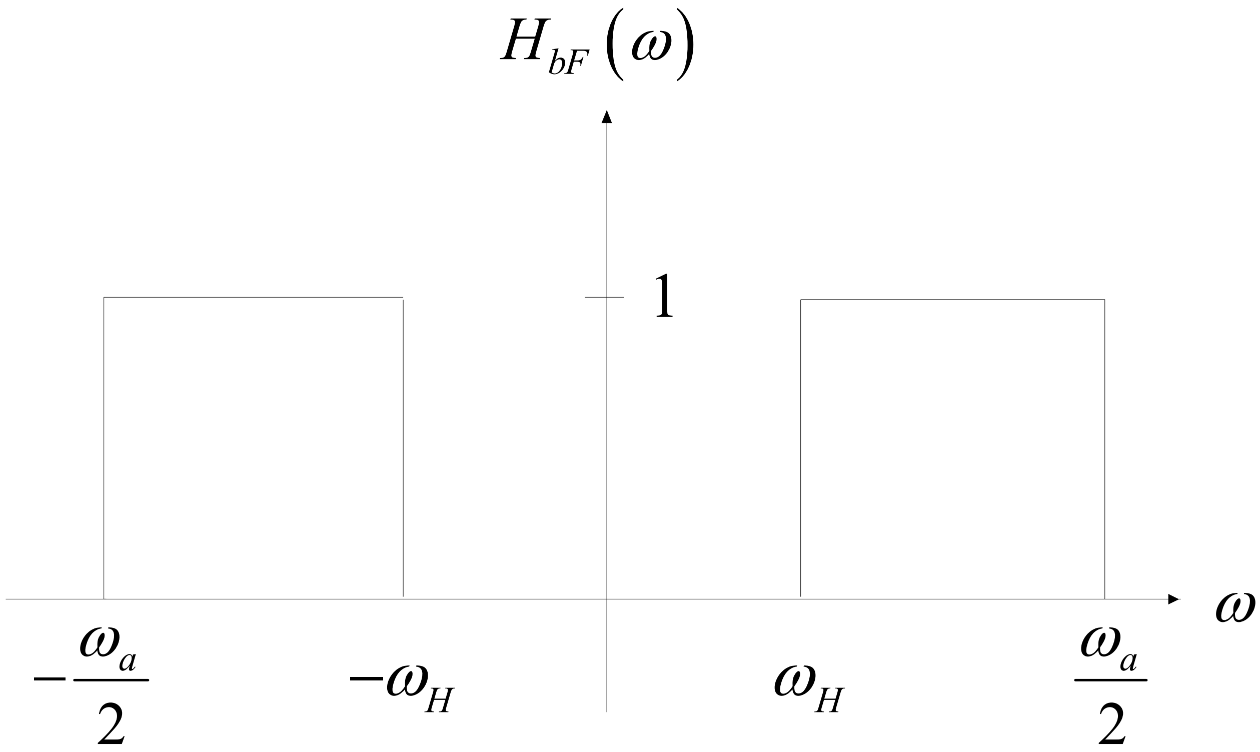
Example 3: High-pass baseband sequence

Determination of the baseband spectrum for the ideal digital high-pass based on the spectral difference of two ideal digital low-passes.



2.2 Design with regulations in the frequency range

Thus the ideal digital high-pass can be represented as the difference of two appropriate analog low-pass transfer functions:



2.2 Design with regulations in the frequency range

Therefor the following relationship for the baseband spectrum can be determined:

$$H_{bF}(\omega) = H_{bFTP2}(\omega) - H_{bFTP1}(\omega)$$
$$= rect\left(\frac{\omega}{\omega_a}\right) - rect\left(\frac{\omega}{2\omega_H}\right) \quad \text{with} \quad 0 < \omega_H < \frac{\omega_a}{2}$$

After inverse Z-transform the assigned non-causal infinite high-pass sequence results:

$$h_{bk}(k) = T_a h_b((k-N_f)T_a) = \frac{2\pi}{\omega_a} h_b((k-N_f)\frac{2\pi}{\omega_a})$$
$$= \frac{2\pi}{\omega_a} \cdot \left(\frac{\omega_a}{2\pi} si(\frac{\omega_a t}{2}) - \frac{2\omega_H}{2\pi} si(\frac{2\omega_H t}{2})\right)\Bigg|_{t=(k-N_f)\frac{2\pi}{\omega_a}}$$
$$= si\left(\pi\left\{k-N_f\right\}\right) - 2\frac{\omega_H}{\omega_a} si\left(2\pi \cdot \frac{\omega_H}{\omega_a}\left\{k-N_f\right\}\right)$$

2.2 Design with regulations in the frequency range

Example 4: **Bandstop baseband sequence**

The baseband spectral function of the transfer function of an ideal band stop can be formulated like this:

$$H_{bF}(\omega) = rect\left(\frac{\omega}{\omega_a}\right) - \left[rect\left(\frac{\omega}{\Delta\omega}\right) * \left\{\delta(\omega+\omega_0)\right\} + \delta(\omega-\omega_0)\right\}\right]$$

with $\omega_0 = \frac{\omega_2+\omega_1}{2}$, $\Delta\omega = \omega_2 - \omega_1$ and $0 < \omega_1 < \omega_2 < \frac{\omega_a}{2}$

The corresponding non-causal band stop baseband infinite sequence is:

$$h_{kb}(k) = si\left(\pi\left\{k-N_f\right\}\right) - \frac{2\Delta\omega}{\omega_a}\cdot\cos\left(\frac{2\pi\omega_0}{\omega_a}\left\{k-N_f\right\}\right)\cdot si\left(\frac{\pi\Delta\omega}{\omega_a}\left\{k-N_f\right\}\right)$$

2.2 Design with regulations in the frequency range

In the following **two special cases** are examined regarding characteristic of $H_{wF}(\omega)$ and the characteristics of the realizable system function of a digital filter :

Case 1: $\mathrm{Re}\{H_{wF}(\omega)\}$ real valued and even in ω

$$\mathrm{Im}\{H_{wF}(\omega)\} = 0 \qquad \forall\,\omega$$

Under these conditions also the corresponding baseband spectrum is even and real valued. From this it is clear that its inverse Fourier transform $h_b(t)$ is also real valued and even.

If also the window function $h_{wi}(t)$ is real and even in t, then (without proof) it holds:

$$h_{rk}(k) = h_{kb}(k) \cdot h_{kwi}(k)$$

and moreover a **point symmetry relationship** is given:

$$h_{rk}(k) = h_{rk}(2N_f - k) \qquad \text{for} \qquad k = 0(1)2N_f$$

2.2 Design with regulations in the frequency range

Another consideration:

For the transfer function holds:

$$H_{rF}(\omega) = H_{rz}(e^{j\omega T_a}) = \sum_{k=0}^{2N_f} h_r(k)\cdot e^{-j\omega k T_a} = e^{-j\omega N_f T_a}\cdot \sum_{k=0}^{2N_f} h_r(k)\cdot e^{-j\omega(k-N_f)T_a}$$

$$= e^{-j\omega N_f T_a}\left(\sum_{k=0}^{N_f-1} h_r(k)\cdot e^{-j\omega(k-N_f)T_a} + h_r(N_f)\cdot 1 + \sum_{k=N_f+1}^{2N_f} h_r(k)\cdot e^{-j\omega(k-N_f)T_a}\right)$$

$$= e^{-j\omega N_f T_a}\left(h_r(N_f) + \sum_{k=0}^{N_f-1} h_r(k)\cdot e^{-j\omega(k-N_f)T_a} + \sum_{k=0}^{N_f-1} h_r(k)\cdot e^{+j\omega(k-N_f)T_a}\right)$$

In accordance with the EULER' formula it results:

$$H_{rF}(\omega) = \left[h_{rk}\left(N_f\right) + 2\cdot \sum_{m=0}^{N_f-1} h_r(m)\cdot \cos\left(\left\{N_f - m\right\}\omega T_a\right)\right]\cdot e^{-j\omega N_f T_a}$$

with $\left|H_{rF}(\omega)\right| = [\ldots]$ and $\sphericalangle H_{rF}(\omega) = -\omega N_f T_a$

2.2 Design with regulations in the frequency range

From this equation some remarkable characteristics of the realizable digital filter transfer function can be read off:

1. The expression in rectangular brackets

$$h_{rk}\left(N_f\right) + 2 \cdot \sum_{m=0}^{N_f - 1} h_r(m) \cdot \cos\left(\left\{N_f - m\right\} \omega T_a\right)$$

is real and even in ω.

2. The phase angle $\sphericalangle H_{rF}(\omega) = \varphi_r(\omega)$ is linear depending on ω:

$$\varphi_r(\omega) = -\omega N_f T_a$$

and thus the group envelope delay of the digital filter is constant all over:

$$\tau_{gr}(\omega) = -\frac{d\varphi_r(\omega)}{d\omega} = N_f T_a$$

2.2 Design with regulations in the frequency range

Case 2: $\mathrm{Re}\{H_{wF}(\omega)\} = 0 \quad \forall\omega$

$\mathrm{Im}\{H_{wF}(\omega)\}$ real and odd in ω

Similiar to case 1 one gets the so called **antisymmetry** relationship:

$$h_r(k) = -h_r(2N_f - k) \qquad \text{for } k = 0(1)2N_f \text{ with } k \neq N_f$$

$$h_r(k) = 0 \qquad \text{for } k = N_f$$

The assigned transfer function is determined as follows

$$H_{rF}(\omega) = H_{rz}(e^{j\omega T_a}) = \sum_{k=0}^{2N_f} h_r(k) \cdot e^{-j\omega k T_a}$$

$$= e^{-j\omega N_f T_a} \cdot \sum_{k=0}^{2N_f} h_r(k) \cdot e^{-j\omega (k - N_f) T_a}$$

2.2 Design with regulations in the frequency range

Based on the antisymmetry and using EULER‘s formula it results in a similar procedure as before:

$$H_{rF}(\omega) = \left|H_{rF}(\omega)\right| \cdot e^{j\varphi_r(\omega)} = \left[2 \cdot \sum_{m=0}^{N_f-1} h_r(m) \cdot \sin\left(\left\{N_f - m\right\} \omega T_a\right)\right] \cdot j \cdot e^{-j\omega N_f T_a}$$

The expression in square brackets is odd and real.

Thus $H_{rF}(\omega)$ is imaginary and odd.

Similar to case1 $H_{rF}(\omega)$ shows a perfect linear phase (or a constant group delay).

2.2 Design with regulations in the frequency range

Example: Construction of a FIR low-pass with $\omega_T = \omega_a / 4$ and $N_f = 4$ and an ideal low-pass as reference filter

The following is unknown:

- the filters coefficients a_k,
- the phase function $\varphi_r(\omega)$,
- the envelope delay $\tau_{gr}(\omega)$

Step 1: $a_k = h_{rk}(k) = h_{kbTP}(k) \cdot h_{kHM}(k)$

Step 2: The desired transfer function of the reference digital low-pass in accordance with

$$H_{bF}(\omega) = H_{wF}(\omega) \cdot rect\left(\frac{\omega}{\omega_a}\right) = rect\left(\frac{\omega}{2\omega_T}\right)$$

2.2 Design with regulations in the frequency range

Step 3: The last equation is inversely Fourier transformed and thus gives $h_b(t)$
This intermediate result is then used in the next equation

$$h_{bk}(k) = T_a \cdot h_b((k - N_f)T_a)$$

which then leads to:

$$h_{kbTP}(k) = \frac{2\omega_T}{\omega_a} \cdot si\left(\frac{2\pi\omega_T}{\omega_a}\left\{k - N_f\right\}\right) \quad \text{for} \quad k = -\infty(1)\infty$$

Step 4: Now the HAMMING window sequence is computed:

$$h_{kHM}(k) = 0.54 + 0.46 \cdot \cos\left(\frac{\pi\left\{k - N_f\right\}}{N_f}\right) \quad \text{for} \quad k = 0(1)2N_f$$

2.2 Design with regulations in the frequency range

Step 5:

Using $\omega_T = \omega_a / 4$ and $N_f = 4$, the coefficients a_k can be computed as follows:

$$a_k = h_{rk}(k) = h_{bkTP}(k) \cdot h_{HMk}(k)$$

$$= \frac{1}{2} \mathrm{si}\left(0.5\pi\{k-4\}\right) \cdot \left(0.54 + 0.46 \cdot \cos\left(\frac{\pi\{k-4\}}{4}\right)\right), \quad k = 0(1)8$$

2.2 Design with regulations in the frequency range

Step 6: Final determination of filter coefficients uses the following table:

k	$h_{bkTP}(k)$	$h_{HMk}(k)$	$a_k = h_{rk}(k) = h_{bkTP}(k) \cdot h_{HMk}(k)$
0	0	0.54	0
1	-0.1061	0.215	-0.0228
2	0	0.08	0
3	0.3183	0.865	0.2754
4	0.5	1	0.5
5	0.3183	0.865	0.2754
6	0	0.08	0
7	-0.1061	0.215	-0.0228
8	0	0.54	0

Table of the values for the filter coefficients

2.2 Design with regulations in the frequency range

On the basis the table one recognizes that the impulse response of the realizable digital low-pass **is real** and is **symmetrical** regarding the point $k = Nf = 4$.

So the symmetry condition is fulfilled:

$$h_{rk}(k) = h_{rk}(2N_f - k) = h_{rk}(8 - k)$$

Step 7: One determines ***the phase function***

$$\varphi_r(\omega) = -\omega N_f T_a = -\omega \cdot 4T_a \quad \text{with} \quad T_a = \frac{2\pi}{\omega_a}$$

and ***the envelope delay***

$$\tau_{gr}(\omega) = 4T_a$$

2.2 Design with regulations in the frequency range

Important note:

A formal conversion of the filter coefficients represented in the table of the values of a_k to the assigned FIR Digitalfilter structure would lead to a filter circuit with 9 constant multipliers and 8 delay units.

However as a_0, a_2, a_6 *and* a_8 show the value of zero, the number of constant multipliers can be reduced to 5.

Moreover 2 delay units are not used here.