NETWORK THEORY 2
Digital Filter

Chapter 3

Design and Realization of Recursive Digital Filters – IIR Filters

3.1 Prefaces to IIR Digital Filter

The design and the realization of digital filters means that magnitude or phase characteristic of the transfer function must be specified (in most cases within the limits of a given tolerance pattern) and from that the filter coefficients have to be determined.

Concerning a recursive digital filters two design methods are available:

Design method 1: Approximate transform of the system function of an analog reference system to a corresponding system function $H_Z(z)$.

Design method 2: The design of the digital filter takes place (with wanted amplitude or phase frequency response) by means of direct approximation of the system function $H_Z(z)$.

In this chapter only the method 1 will be treated!

3.1 Prefaces to IIR Digital Filter

Properties of the design method 1:

- A reference system is chosen
- For this system the approximation task is rewritten suitably
- The approximation task is performed in the p-domain based on a realizable analog system described by means of its system function
- This system function can be specified based on a catalog of filters (including the set of corresponding filter coefficient values)
- Based on the found system function the function $H_Z(z)$ is determined using an „Impulse-invariance method“ or a „Bilinear transform“

3.2 Approximation procedure for analogue low-pass reference systems

Approximation procedures for arbitrary analogue high-passes, band-pass filters and band-stops can often be achieved by a transform of the system function $H_L(p)$ of a low-pass reference filter.

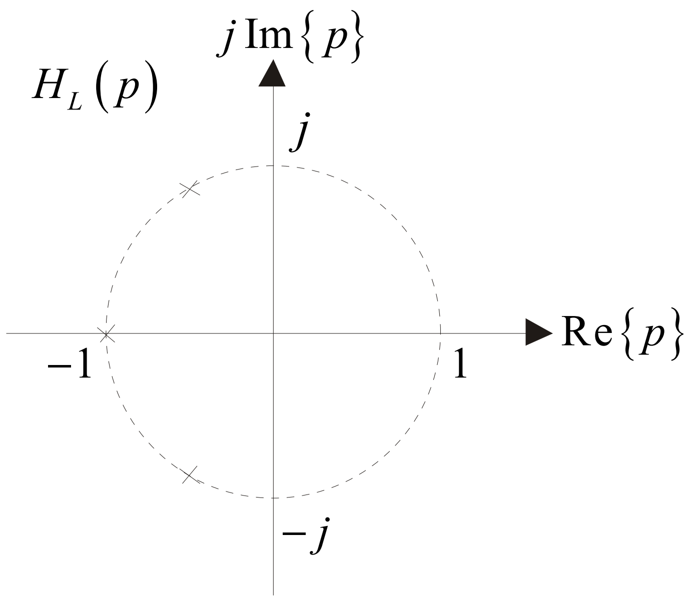
Starting point of a such an onset is the so-called "**Approximation function A(ω)**" gained as follows:

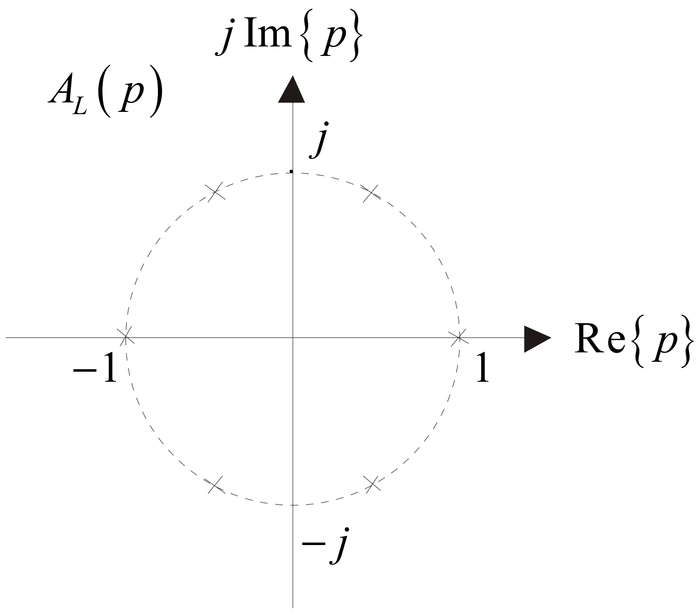
One forms first $$A_L(p) = H_L(p) \cdot H_L(-p)$$

Then the pole-zero plot of this result is determined.

It can be shown that poles and zeros of $H_L(p)$ form a mirror-image compared to $H_L(-p)$.

3.2 Approximation procedure for analogue low-pass reference systems





Pole zero pattern of the system function $H_L(p)$ of a BUTTERWORTH low-pass of 3rd Order and of the associated product function $A_L(p)$

3.2 Approximation procedure for analogue low-pass reference systems

Provided that real filter coefficients and poles being only in the left half p-plane, it follows with

$$H_L(j\omega) = H_F(\omega) \qquad \text{and} \qquad H_L(-j\omega) = H_F^*(\omega)$$

for the approximation function $A(\omega)$ (squared magnitude frequency response):

$$\text{with } A(\omega) = A_L(j\omega) \text{ or } A_L(p) = A(\frac{p}{j})$$

$$A(\omega) = H_L(j\omega) \cdot H_L(-j\omega) = H_F(\omega) \cdot H^*{}_F(\omega) = \left| H_F(\omega) \right|^2$$

Example of a Butterworth approximation:

$$A(\omega) = \frac{1}{1 + \left(\dfrac{\omega}{\omega_T}\right)^{2N}} = \left| H_F(\omega) \right|^2 \text{ with } N = \text{Order of the low-pass system}$$

3.2 Approximation procedure for analogue low-pass reference systems

The reasons for choosing this function are as follows:

$A(\omega)$ is a real and an even function in ω.

Thus $A(\omega)$ can only be represented by a denominator polynomial with purely even exponents leading to

$$A_N(\Omega) = \frac{1}{k_0 + k_2\Omega^2 + k_4\Omega^4 \ldots + k_{2N}\Omega^{2N}} \quad \text{with } \Omega = \frac{\omega}{\omega_T}$$

If it is required $A_N(0) = 1$, $A_N(\Omega) \to 0$ for $\Omega \to \infty$

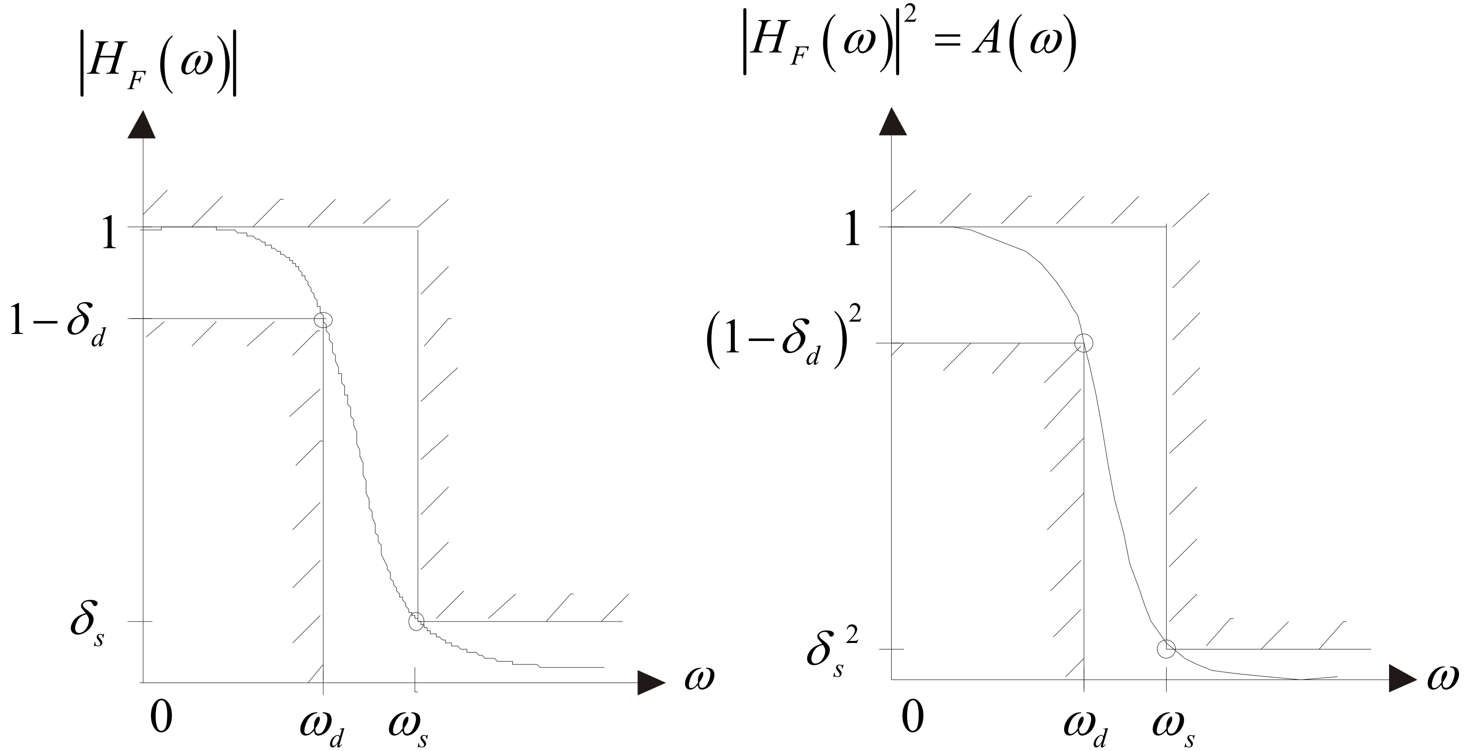
and the sharpest possible drop-off of $A_N(\Omega)$ for $\Omega > 1$ and $A_N^2(1) = 1/2$

it holds:

$$k_0 = 1 \quad k_\nu = 0 \; \forall \nu = 2(1)2N-2 \quad k_{2N} = 1$$

Reason: The expression $1 + k_{2N}\Omega^{2N}$ gives the sharpest drop-off compared with all other expressions $1 + k_{2\nu}\Omega^{2\nu}$ with $\nu = 1(1)N-1$

3.2 Approximation procedure for analogue low-pass reference systems



BUTTERWORTH low-pass tolerance scheme for $|H_F(\omega)|$ and for $|H_F(\omega)|^2 = A(\omega)$

3.2 Approximation procedure for analogue low-pass reference systems

It is now required that the function $A(\omega)$ just touches the point $(\omega_d, (1-\delta_D)^2)$. Thus it holds:

$$A(\omega_d) = (1-\delta_d)^2 = \frac{1}{1+\left(\frac{\omega_d}{\omega_T}\right)^{2N}}$$

Solving for ω_T (the -3dB cut-off frequency) as a function of ω_d, δ_d and N gives:

$$\omega_T = \omega_d \cdot \sqrt[2N]{\frac{(1-\delta_d)^2}{2\delta_d - \delta_d^{\,2}}}$$

Note: Only even values of N are allowed!

3.2 Approximation procedure for analogue low-pass reference systems

The next condition is that the function $A(\omega)$ just touches the point (ω_s, δ_s) leading to the relationship:

$$A(\omega_s) = \frac{1}{1 + \left(\frac{\omega_s}{\omega_T}\right)^{2N}} = \delta_s^2$$

It applies (without proof):

$$N = \frac{\ln(h)}{\ln(r)} \quad \text{with} \quad h = \frac{\sqrt{2\delta_d - \delta_d^{\,2}}}{(1-\delta_d)} \cdot \frac{\delta_s}{\sqrt{1-\delta_s^{\,2}}} \quad \text{and} \quad r = \frac{\omega_d}{\omega_s} = \frac{f_d}{f_s}$$

For the final realization the next larger integer value N must be selected!

3.2 Approximation procedure for analogue low-pass reference systems

After the determination of the filter order *N*, one gets the assigned function:

$$A_L(p) = A\left(\frac{p}{j}\right) = \frac{1}{1 + \left(\frac{p}{j\omega_T}\right)^{2N}}$$

3.2 Approximation procedure for analogue low-pass reference systems

The last formula can also be rewritten as follows:

$$A_L(p) = \frac{1}{1 + \left(\frac{p}{j\omega_T}\right)^{2N}} = \frac{1}{1 + \left(\frac{P}{j}\right)^{2N}} = \frac{1}{1 + \left(-P^2\right)^N} \quad \text{with} \quad P = \frac{p}{\omega_T}$$

In order to obtain the poles of $A_L(p)$ the denominator must be zero leading to:

$$(-1)^N P^{2N} = -1 \quad \text{or} \quad P^{2N} = -1 \text{ for even N} \quad \text{and} \quad P^{2N} = 1 \quad \text{for odd N} \quad \text{or}$$

$$P_{\infty\nu}^{2N} = -1 = e^{j\pi} = e^{j\pi(2\nu+1)} \text{ for even } N \quad \text{and } \nu \geq 0 \qquad \text{as well as}$$

$$P_{\infty\nu}^{2N} = +1 = e^{j\pi 2\nu} \text{ for odd } N \quad \text{and } \nu \geq 0$$

This means:

$$P_{\infty\nu} = e^{j\pi(2\nu+1)/2N} = e^{j\pi(\nu+1/2)/N} \text{for even } N \quad \text{and } \nu \geq 0 \quad \text{as well as}$$

$$P_{\infty\nu} = e^{j\pi 2\nu/2N} = e^{j\pi\nu/N} \quad \text{for odd } N \quad \text{and } \nu \geq 0$$

3.2 Approximation procedure for analogue low-pass reference systems

The index ν needs only to run from 0 to *2N-1*, larger values lead to same pole positions as the exponential function with imaginary exponent is periodic in 2π .

Furthermore it holds:

$$\left|P_{\infty\nu}\right| = 1 \quad \text{or} \quad \left|p_{\infty\nu}\right| = \omega_T$$

→ Thus all normalised poles lie on the unit circle!

Now we can rewrite $A_L(p)$ as follows:

$$A_L(p) = \frac{\omega_T^{2N}}{\prod_{\upsilon=0}^{2N-1}(p - p_{\infty\upsilon})} \quad \text{with } p_{\infty\nu} = \omega_T P_{\infty\nu} \quad \text{or} \quad A_{LN}(P) = \frac{1}{\prod_{\upsilon=0}^{2N-1}(P - P_{\infty\upsilon})}$$

3.2 Approximation procedure for analogue low-pass reference systems

So for the *2N* poles holds:

$$p_{\infty\upsilon} = \begin{cases} \omega_T \cdot e^{j\pi\frac{\upsilon}{N}} & \text{for odd } N \\ \omega_T \cdot e^{j\pi(\nu+\frac{1}{2})\frac{1}{N}} & \text{for even } N \end{cases} \quad \text{with } \upsilon = 0(1)(2N-1)$$

The formula above describes the poles of $A_L(p)$ which includes poles from $H_L(p)$ and $H_L(-p)$. For the system function $H_L(p)$ only those poles should be chosen which lead to a stable system (those in the left open p-plane). According to this it must hold:

$$H_L(p) = \frac{\omega_T^N}{\prod_{\upsilon}(p - p_{\infty\upsilon})} \quad \text{and} \quad H_{LN}(P) = \frac{1}{\prod_{\upsilon}(P - P_{\infty\upsilon})}$$

$$\frac{1}{2}\pi < \frac{\upsilon}{N}\pi < \frac{3}{2}\pi \quad \text{for odd } N \quad \text{and} \quad \frac{1}{2}\pi < \frac{\left(\frac{1}{2}+\upsilon\right)}{N}\pi < \frac{3}{2}\pi \quad \text{for even } N$$

3.2 Approximation procedure for analogue low-pass reference systems

The table shows polynomials of first to 6th order:

N	BUTTERWORTH denominator polynomial
1	$(p+\omega_T)$
2	$(p^2+\sqrt{2}\omega_T p+\omega_T^2)$
3	$(p^2+\omega_T p+\omega_T^2)\cdot(p+\omega_T)$
4	$(p^2+0.7653\omega_T p+\omega_T^2)\cdot(p^2+1.84776\omega_T p+\omega_T^2)$
5	$(p+\omega_T)\cdot(p^2+0.6180\omega_T p+\omega_T^2)\cdot(p^2+1.6180\omega_T p+\omega_T^2)$
6	$(p^2+0.5176\omega_T p+\omega_T^2)\cdot(p^2+1.9318\omega_T p+\omega_T^2)\cdot(p^2+\sqrt{2}\omega_T p+\omega_T^2)$

Denominator polynomial of $A_L(p)$

3.2 Approximation procedure for analogue low-pass reference systems

Example: Determination of the low-pass system function $H_L(p)$ from the BUTTERWORTH approximation function $A(\omega)$

The system function for a BUTTERWORTH low pass of 3rd order should be determined.

For $N = 3$ it is obtained:

$$A_N(\Omega) = \frac{1}{1+\Omega^6} = \frac{1}{1+(\frac{\omega}{\omega_T})^6}$$

The corresponding system function $H_L(p)$ results to:

$$H_L(p) = \frac{\omega_T^3}{\prod_{\upsilon=1}^{3}(p - p_{\infty\upsilon})}$$

3.2 Approximation procedure for analogue low-pass reference systems

The poles of $H_L(p)$ can be determined by means of

$$p_{\infty\upsilon} = \omega_T e^{j\pi\frac{\upsilon}{N}} \qquad \text{with the condition of} \qquad \frac{\pi}{2} < \frac{\upsilon\pi}{3} < \frac{3\pi}{2}$$

Thus one receives:

$$p_{\infty 2} = \omega_T \cdot e^{j\pi\frac{2}{3}} = -\frac{\omega_T}{2} + j\frac{\sqrt{3}\,\omega_T}{2}$$

$$p_{\infty 3} = \omega_T \cdot e^{j\pi\frac{3}{3}} = -\omega_T$$

$$p_{\infty 4} = \omega_T \cdot e^{j\pi\frac{4}{3}} = -\frac{\omega_T}{2} - j\frac{\sqrt{3}\,\omega_T}{2}$$

Note the conjugated poles 2 and 4, the other poles 0, 1 and 5 are in the right half p-plane

3.2 Approximation procedure for analogue low-pass reference systems

Thus the system function results as follows:

$$H_L(p) = \frac{\omega_T^3}{(p - p_{\infty 2}) \cdot (p - p_{\infty 3}) \cdot (p - p_{\infty 4})}$$

Combining conjugated poles, the expression can be rewritten as follows:

$$H_L(p) = \frac{\omega_T^3}{\left[p^2 - (p_{\infty 2} + p_{\infty 2}^*)p + p_{\infty 2} p_{\infty 2}^*\right] \cdot (p - p_{\infty 3})}$$

$$= \frac{\omega_T^3}{\left(p^2 + \omega_T p + \omega_T^2\right) \cdot (p + \omega_T)}$$

3.2 Approximation procedure for analogue low-pass reference systems

Other approximation functions are in use:

- the CHEBYSHEV function
- the BESSEL function
- the CAUER function

In the following slides we deal with the CHEBYSHEV function.

There are two types of it:

Type 1 shows in the pass-band a ripple of constant amplitude but no ripple in the stop-band.

Type2 shows no ripple in the pass-band but in the stop-band (also with a constant amplitude).

3.2 Approximation procedure for analogue low-pass reference systems

Here only type 1 will be treated. For it applies:

$$A(\omega) = \frac{1}{1+\varepsilon^2 C_N^2(x)} = \left| H_F(\omega) \right|^2 \qquad \text{with} \quad x = \frac{\omega}{\omega_d}$$

with $C_N(x)$ the CHEBYSHEV polynomial of Nth order:

$$C_N(x) = \begin{cases} \cos(N \cdot ar\cos(x)) & \text{for } |x| \leq 1 \\ \cosh(N \cdot ar\cosh(x)) & \text{for } |x| > 1 \end{cases}$$

CHEBYSHEV polynomials of higher order can be determined based on $C_0(x) = 1$ and $C_1(x) = x$ using the following recursion formula:

$$C_N(x) = 2x \cdot C_{N-1}(x) - C_{N-2}(x)$$

3.2 Approximation procedure for analogue low-pass reference systems

The characteristics of $|H_F(\omega)|$ of CHEBYSEV low-pass are:

1. $|H_F(\omega)|$ ripples in the pass-band $0 \le \omega \le \omega_D$ with a constant amplitude and drops monotonously to 0 in the stop-band

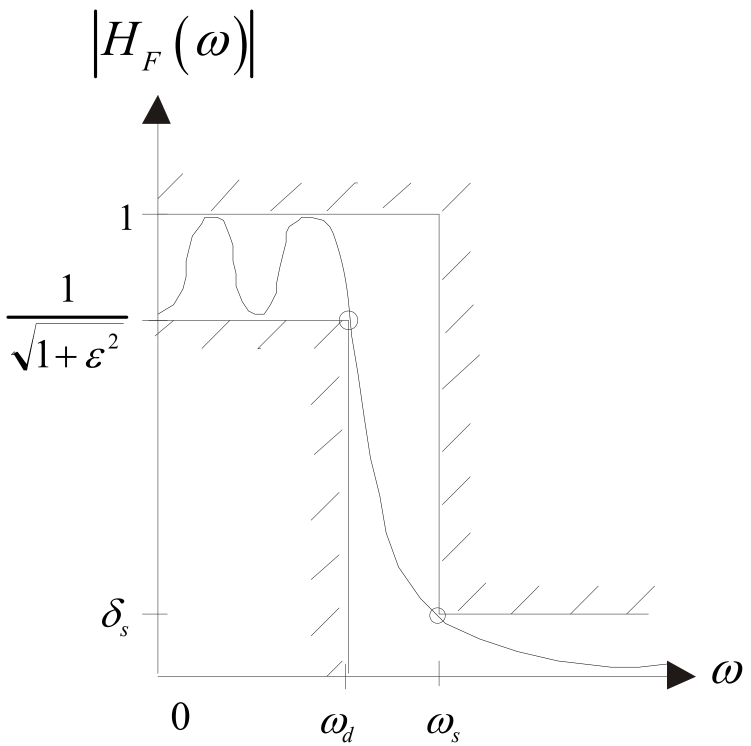
2. The parameter ε specifies the ripple amplitude

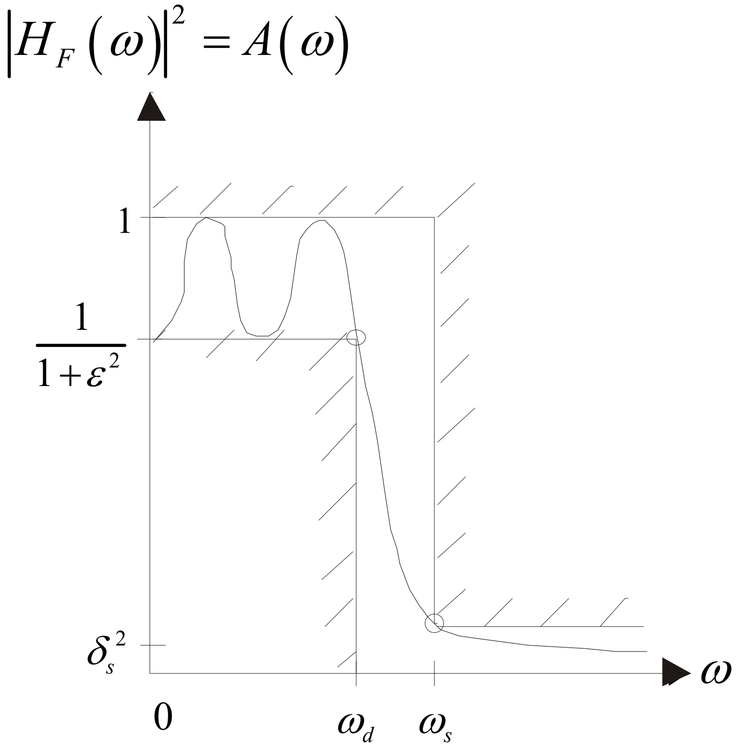
3. $|H_F(\omega)|$ ocillates (N+1) times in the pass-band

   between the values of $1/\sqrt{1+\varepsilon^2}$ and 1

4. $|H_F(0)| = 1/\sqrt{1+\varepsilon^2}$ for even N and $|H_F(0)| = 1$ for odd N

3.2 Approximation procedure for analogue low-pass reference systems





CHEBYSHEV low-pass tolerance schema (type 1, $N = 4$) for $|H_F(\omega)|$ and for $|H_F(\omega)|^2 = A(\omega)$

3.2 Approximation procedure for analogue low-pass reference systems

ε will now be related to the tolerance parameter δ_d by means of the relationship:

$$1-\delta_d = \frac{1}{\sqrt{1+\varepsilon^2}} \Rightarrow \varepsilon^2 = \frac{1}{1-2\delta_d+\delta_d^2} - 1 = \frac{2\delta_d-\delta_d^2}{(1-\delta_d)^2}$$

For given values of ω_D, ω_s *and* δ_s for the necessary minimum filter order N_t holds (without proof):

$$N = \frac{\ln(1+\sqrt{1-h^2}) - \ln(h)}{\ln(1+\sqrt{1-r^2}) - \ln(r)}$$

Of course for N the next larger integer value should be chosen, for h and r the values are according to slide 10.

As specified above, $A_L(p)$ is determined in accordance with equation:

$$A_L(p) = A\left(\frac{p}{j}\right) = \frac{1}{1+\varepsilon^2 C_N^2\left(\frac{p}{j\omega_d}\right)} \quad \text{with } A(\omega) = \left|H_L(j\omega)\right|^2$$

3.2 Approximation procedure for analogue low-pass reference systems

Again for $H_L(p)$ only those poles of $A_L(p)$ are chosen which lie in the left open p-half plane.

$$H_L(p) = \frac{K}{\prod_{\upsilon=1}^{N}(p - p_{\infty\upsilon})}$$

For these poles (without proof) the relationship holds:

$$p_{\infty\upsilon} = -\omega_d \cdot \sinh(a) \cdot \sin\left(\frac{2\upsilon-1}{2N}\pi\right) + j\omega_d \cdot \cosh(a) \cdot \cos\left(\frac{2\upsilon-1}{2N}\pi\right)$$

Here for a applies: $$a = \frac{1}{N} ar\sinh\left(\frac{1}{\varepsilon}\right) = \frac{1}{N} ar\sinh\left(\frac{1-\delta_d}{\sqrt{2\delta_d - \delta_d^2}}\right)$$

3.2 Approximation procedure for analogue low-pass reference systems

For even N one determines the constant K appearing in the system function above by means of:

$$K = \frac{\prod_{\upsilon=1}^{N} \left| p_{\infty\upsilon} \right|}{\sqrt{1+\varepsilon^2}}$$

For odd N it applies:

$$K = \prod_{\upsilon=1}^{N} \left| p_{\infty\upsilon} \right|$$

3.2 Approximation procedure for analogue low-pass reference systems

Example: Determination of a second order low-pass system function $H_L(p)$ using a CHEBYSHEV approximation function $A(\omega)$ with $r_d = 1\ dB$

$$A(\omega) = \frac{1}{1+\varepsilon^2 C_N^2\left(\frac{\omega}{\omega_d}\right)}$$

The ripple in the pass band defined in dB is:

$$r_d = 10 \cdot \log_{10}\left(\frac{A_{\max}}{A_{\min}}\right) [dB]$$

A_{max} is here the maximum amplitude and A_{min} the minimum amplitude inside of the pass-band. For these values and $A_{max} = 1$ it holds:

$$A_{\min} = \frac{1}{1+\varepsilon^2}$$

3.2 Approximation procedure for analogue low-pass reference systems

From this ε can be determined as follows:

$$\varepsilon^2 = 10^{\frac{r_d}{10}} - 1 = 0.26 \text{ for } r_d = 1\,[dB]$$

For the CHEBYSHEV polynomial 2nd order then holds:

$$C_2(x) = 2x \cdot C_1(x) - C_0(x) \quad with\;\; x = \frac{\omega}{\omega_d}$$

$$= 2x^2 - 1$$

$$= 2\left(\frac{\omega}{\omega_d}\right)^2 - 1$$

Thus it results:

$$A(\omega) = \frac{1}{1 + \varepsilon^2 \left[2 \cdot \left(\frac{\omega}{\omega_d}\right)^2 - 1\right]^2} \qquad A_L(p) = A\left(\frac{p}{j}\right) = \frac{1}{1 + \varepsilon^2 [2\left(\frac{p}{j\omega_d}\right)^2 - 1]^2}$$

3.2 Approximation procedure for analogue low-pass reference systems

The system function $H_L(p)$ then gives:

$$H_L(p) = \frac{K}{(p - p_{\infty 1}) \cdot (p - p_{\infty 2})}$$

For the two poles the next equations can be used:

$$p_{\infty 1} = \omega_d \cdot \left[-\sinh(a) \sin\left(\frac{\pi}{4}\right) + j \cosh(a) \cos\left(\frac{\pi}{4}\right) \right]$$

$$p_{\infty 2} = \omega_d \cdot \left[-\sinh(a) \sin\left(\frac{3\pi}{4}\right) + j \cosh(a) \cos\left(\frac{3\pi}{4}\right) \right]$$

with $$a = \frac{1}{2} ar \sinh\left(\frac{1}{\varepsilon}\right)$$

3.2 Approximation procedure for analogue low-pass reference systems

One determines the constant K by considering the behaviour at DC and thus with the equation:

$$K = \frac{|p_{\infty 1}| \cdot |p_{\infty 2}|}{\sqrt{1+\varepsilon^2}} \text{ due to } H_L(0) = \frac{1}{\sqrt{1+\varepsilon^2}}$$

After using the values given above one receives in total:

$$a = 0.7131$$

$$p_{\infty 1} = \omega_d \left[-0.548 + j0.8946\right]$$

$$p_{\infty 2} = \omega_d \left[-0.548 - j0.8946\right]$$

$$K = 0.9805 \cdot \omega_d^2$$

$$\rightarrow H_L(p) = \frac{0.9805\, \omega_d^2}{\left(p - \left[-0.548\omega_d + j\, 0.8946\omega_d\right]\right) \cdot \left(p - \left[-0.548\omega_d - j\, 0.8946\omega_d\right]\right)}$$

3.2 Approximation procedure for analogue low-pass reference systems

Note:

The conjugated complex pair of poles can be rewritten as follows leading to real coefficients:

$$H_L(p) = \frac{0.9805\,\omega_d^2}{p^2 + 1.1\cdot\omega_d\,p + \omega_d^2}$$

3.3 Derivation of system functions by frequency transformation

If by means of $H_L(p)$ the low-pass prototype system function is given the system function of another system (low-pass ... band-stop etc.) can be derived which shows the same properties. As a preparation for this $H_L(p)$ is written in the form $H_L(p_1)$

$H_{tL}(p)$ is the symbol for a prototyp function derived from the low-pass prototype function $H_L(p_1)$ by the appropriate transform

For the transform the following relation is used: $H_{tL}(p) = H_L(p_1(p))$

The transformation function $p_1(p)$ is typically chosen so that it maps p-values into p_1-values with the following properties:

1. The $j\omega$-*axis* of the p-level turns into the $j\omega_1$-*axis* of the p_1 domain
2. The left open half p-plane also turns into the left open half p_1 domain

3.3 Derivation of system functions by frequency transformation

In the following the mappings $p_1(p)$ for the transformation of low-pass prototypes are specified:

LP → LP: $$p_1(p) = \frac{p}{\omega_{Tt}}\omega_T$$

LP → HP: $$p_1(p) = \frac{\omega_H}{p}\omega_T \quad \text{or} \quad P_1(P) = P^{-1} \quad \text{for } \omega_T = \omega_H$$

LP → BP: $$p_1(p) = \frac{p^2 + \omega_{Bo}\cdot\omega_{Bu}}{p\cdot(\omega_{Bo} - \omega_{Bu})}\omega_T \quad \text{or} \quad P_1(P) = \frac{\Delta\Omega}{P + P^{-1}}$$

$$\text{for } \Delta\Omega = \frac{\omega_{Bo} - \omega_{Bu}}{\omega_{Bo}\cdot\omega_{Bu}}\omega_T$$

LP → BS: $$p_1(p) = \frac{p\cdot(\omega_{So} - \omega_{Su})}{p^2 + \omega_{So}\cdot\omega_{Su}}\omega_T \quad \text{or} \quad P_1(p) = \frac{P + P^{-1}}{\Delta\Omega}$$

3.3 Derivation of system functions by frequency transformation

Here a list of used symbols:

ω_T: The 3dB(cut-off) frequency of the low-pass prototype

ω_{Tt}: The 3dB(cut-off) frequency of the derived low-pass

ω_H: The 3dB(cut-off) frequency of the derived high-pass

ω_{Bu}, ω_{Bo}: The lower and upper 3dB-Frequency of the pass band

ω_{Su}, ω_{So}: The lower and upper 3dB-Frequency of the stop band

3.3 Derivation of system functions by frequency transformation

Example: Low-pass to Band-pass transformation

The system function of a BUTTERWORTH low-pass 2nd. order with its 3dB-frequency is given:

$$H_{LTP}(p_1) = \frac{\omega_T^2}{p_1^2 + \sqrt{2}\,\omega_T \cdot p_1 + \omega_T^2}$$

Wanted is:

1. The system function of a band-pass filter $H_{tLBP}(p)$

2. The lower and upper 3dB-frequency ω_{Bu}, ω_{Bo}

3.3 Derivation of system functions by frequency transformation

Substituting in the given prototype system function $H_{LTP}(p_1)$ the relation

$$p_1(p) = \frac{p^2 + \omega_{Bo}\omega_{Bu}}{p(\omega_{Bo} - \omega_{Bu})}\,\omega_T$$

leads to the wanted band-pass filter with the system function:

$$H_{tLBP}(p) = \frac{\omega_T^2}{\left[\dfrac{p^2 + \omega_{Bo}\omega_{Bu}}{p(\omega_{Bo} - \omega_{Bu})} \cdot \omega_T\right]^2 + \sqrt{2}\,\omega_T \cdot \left[\dfrac{p^2 + \omega_{Bo}\omega_{Bu}}{p(\omega_{Bo} - \omega_{Bu})} \cdot \omega_T\right] + \omega_T^2}$$

3.3 Derivation of system functions by frequency transformation

Rewriting the formula shown above leads to:

$$H_{tLBP}(p) = \frac{a^2 \cdot p^2}{p^4 + \sqrt{2}\, a \cdot p^3 + (2b + a^2) \cdot p^2 + \sqrt{2}\, ab \cdot p + b^2}$$

with

$$a = \omega_{Bo} - \omega_{Bu}$$

$$b = \omega_{Bo}\omega_{Bu}$$

3.4 Design of IIR Digital filter by means of impulse invariance method

With the help of the impulse invariance method one obtains from a given system function $H_L(p)$ a causal system function $H_z(z)$ of the assigned digital filter as follows:

1. The digital filter impulse response $h(k)$ is determined on the basis of the impulse response $h(t)$ of a continuous (analog) system using the samples $h(kT_A)$

2. Onto $h(k)$ then the z-transform is applied

The described procedure directly explains the name of it.

3.4 Design of IIR Digital filter by means of impulse invariance method

Here the following conditions must be assumed:

1. The given system function $H_L(p)$ has real coefficients
2. In total M single poles lie in the left open p-half plane
3. The nominator order is smaller than the denominator order

In this case $H_L(p)$ can be rewritten as a partial fraction like this:

$$H_L(p) = \sum_{m=1}^{M} \frac{A_m}{p - p_{\infty m}}$$

Each fraction in the p-domain corresponds to an expression $h_m(t)$ in the time domain (i.e. one partial signal of the total impulse response):

$$h_m(t) = \begin{cases} A_m e^{p_{\infty m} t} & \text{for } t \geq 0 \\ 0 & \text{else} \end{cases}$$

3.4 Design of IIR Digital filter by means of impulse invariance method

Sampling this partial signal $h_m(t)$ with an ideal sampler with the sampling period T_A gives for the sampled signal:

$$h_{am}(t) = h_m(t) \cdot \sum_{k=-\infty}^{+\infty} \delta(t - kT_a) = \sum_{k=0}^{\infty} A_m e^{p_{\infty m} kT_a} \cdot \delta(t - kT_a)$$

Laplace transforming gives:

$$H_{Lam}(p) = \sum_{k=0}^{\infty} A_m e^{p_{\infty m} kT_a} \cdot e^{-pkT_a}$$

With $z = e^{pT_a}$ and the constants $z_{\infty m} = e^{p_{\infty m} T_a}$ then one receives:

$$H_{Lam}(p) = A_m \cdot \sum_{k=0}^{\infty} z_{\infty m}^{k} \cdot z^{-k}$$

3.4 Design of IIR Digital filter by means of impulse invariance method

The sum in this equation corresponds to a z-transform of the sequence $z_{\infty m}^{k}$. According to the z-transform table thus it is obtained:

$$H_{aLm}(p) = H_{zm}(z) = \frac{A_m \cdot z}{z - z_{\infty m}}$$

3.4 Design of IIR Digital filter by means of impulse invariance method

For the latter a Laplace transform gives:

$$H_{aL}(p) = \sum_{m=1}^{M} H_{aLm}(p)$$

One thereby obtains the characteristic relations of the impulse invariance method:

$$H_L(p) = \sum_{m=1}^{M} \frac{A_m}{p - p_{\infty m}} \quad \Leftrightarrow \quad H_z(z) = \sum_{m=1}^{M} \frac{A_m \cdot z}{z - z_{\infty m}}$$

3.4 Design of IIR Digital filter by means of impulse invariance method

Again it must be pointed out that the frequency response of the gained digital filter is only approxiately obtained. Therefore

$$\left|H_{aL}(j\omega)\right| = \left|H_z(e^{j\omega T_a})\right|$$

of the digital filter only approximates the frequency response

$$\left|H_L(j\omega)\right|$$

of the corresponding analog prototype system.

3.4 Design of IIR Digital filter by means of impulse invariance method

Reasons for obtaining only an approximation:

Each discrete Signal shows periodic properties in the frequency domain.

Thus any continuous frequency response shows oberlapping (aliasing) when turned into a discrete system.

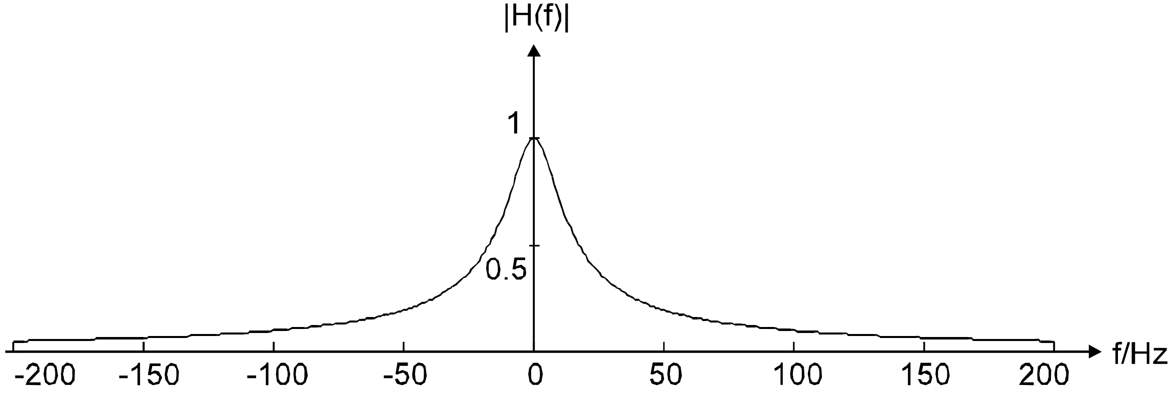
The errors due to aliasing can be reduced by the following measures:

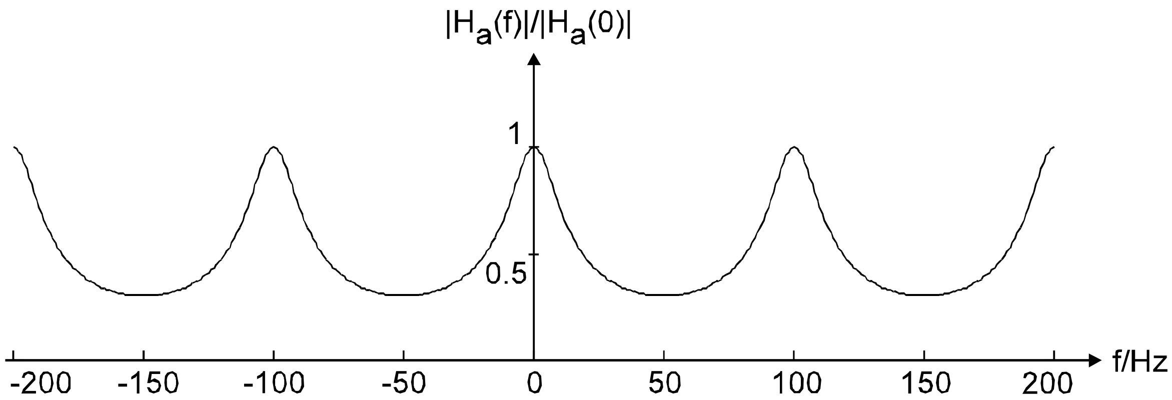
- Increase of the sampling rate (leading to an increase of the filter order)
- Increase of the order of the analog reference system for reducing values in the stop-band (where aliasing takes place)

The next figure shows that only to band limited analog systems the impulse invariance method should be applied.

Otherwise infinitely high values of $H_{aL}(j\omega)$ result.

3.4 Design of IIR Digital filter by means of impulse invariance method





Transfer function of an analog prototype with $f_g = 10Hz$
and of the corresponding digital filter with $f_a = 100\ Hz$

3.4 Design of IIR Digital filter by means of impulse invariance method

Summarizing the method:

The realization of a digital filter according to the impulse invariance method happens in the following steps:

Step 1: The given tolerance pattern for the digital filter is used as a tolerance pattern for the analog reference system

Step 2: A realizable system function $H_L(p)$ is looked for which fulfils the tolerance pattern

Step 3: This system function $H_L(p)$ is rewritten into a partial fraction according to (in case of only single poles):

$$H_L(p) = \sum_{m=1}^{M} \frac{A_m}{p - p_{\infty m}} = \sum_{m=1}^{M} H_{Lm}(p)$$

3.4 Design of IIR Digital filter by means of impulse invariance method

Step 4: All partial fraction addends $H_{Lm}(p)$ are converted into the assigned $H_{zm}(z)$ corresponding to:

$$H_L(p) = \sum_{m=1}^{M} \frac{A_m}{p - p_{\infty m}} \quad \Leftrightarrow \quad H_z(z) = \sum_{m=1}^{M} \frac{A_m \cdot z}{z - z_{\infty m}}$$

Step 5: The addends $H_{zm}(z)$ are interpreted as system functions of digital filters of first order. Addends with a conjugated-complex pair of poles have to be interpreted as a system function of 2nd order.

Step 6: From the system functions $H_{zm}(z)$ the recursive subsystems of first and/or 2nd order are developed.

3.4 Design of IIR Digital filter by means of impulse invariance method

Example: Determination of a digital low-pass system with $H_z(z)$ from the analog low-pass system with $H_L(p)$ according to the impulse invariance method.

It is assumed that after the first two steps described above the following continuous system function (BUTTERWORTH low-pass second order) was found:

$$H_L(p) = \frac{\omega_T^2}{p^2 + \sqrt{2}\,\omega_T p + \omega_T^2}$$

According to step 3 the partial fraction description is as follows:

$$H_L(p) = \frac{-j\dfrac{\omega_T}{\sqrt{2}}}{p - \left[-\dfrac{\omega_T}{\sqrt{2}} + j\dfrac{\omega_T}{\sqrt{2}}\right]} + \frac{j\dfrac{\omega_T}{\sqrt{2}}}{p - \left[-\dfrac{\omega_T}{\sqrt{2}} - j\dfrac{\omega_T}{\sqrt{2}}\right]}$$

3.4 Design of IIR Digital filter by means of impulse invariance method

In Step 4 the partial addends are converted into:

$$H_{L1}(p) = \frac{-j\frac{\omega_T}{\sqrt{2}}}{p - \left[-\frac{\omega_T}{\sqrt{2}} + j\frac{\omega_T}{\sqrt{2}}\right]} \Leftrightarrow H_{z1}(z) = \frac{-j\frac{\omega_T}{\sqrt{2}} \cdot z}{z - e^{\left(-\frac{\omega_T}{\sqrt{2}} + j\frac{\omega_T}{\sqrt{2}}\right)T_a}}$$

$$H_{L2}(p) = \frac{j\frac{\omega_T}{\sqrt{2}}}{p - \left[-\frac{\omega_T}{\sqrt{2}} - j\frac{\omega_T}{\sqrt{2}}\right]} \Leftrightarrow H_{z2}(z) = \frac{j\frac{\omega_T}{\sqrt{2}} \cdot z}{z - e^{\left(-\frac{\omega_T}{\sqrt{2}} - j\frac{\omega_T}{\sqrt{2}}\right)T_a}}$$

3.4 Design of IIR Digital filter by means of impulse invariance method

The sum of $H_{z1}(z)$ and $H_{z2}(z)$ results then to the digital low-pass system function:

$$H_z(z) = H_{z1}(z) + H_{z2}(z) = \frac{-j\frac{\omega_T}{\sqrt{2}} \cdot z}{z - e^{\left(-\frac{\omega_T}{\sqrt{2}} + j\frac{\omega_T}{\sqrt{2}}\right)T_a}} + \frac{j\frac{\omega_T}{\sqrt{2}} \cdot z}{z - e^{\left(-\frac{\omega_T}{\sqrt{2}} - j\frac{\omega_T}{\sqrt{2}}\right)T_a}}$$

After some rewritings one receives:

$$H_z(z) = \frac{j\frac{\omega_T}{\sqrt{2}} \cdot e^{-\frac{\omega_T}{\sqrt{2}}T_a}\left(e^{-j\frac{\omega_T}{\sqrt{2}}T_a} - e^{j\frac{\omega_T}{\sqrt{2}}T_a}\right) \cdot z^{-1}}{1 - e^{-\frac{\omega_T}{\sqrt{2}}T_a}\left(e^{j\frac{\omega_T}{\sqrt{2}}T_a} + e^{-j\frac{\omega_T}{\sqrt{2}}T_a}\right) \cdot z^{-1} + e^{-\frac{\omega_T}{\sqrt{2}}T_a} z^{-2}}$$

3.4 Design of IIR Digital filter by means of impulse invariance method

Using the relations

$$\left(e^{jx}-e^{-jx}\right)=-2j\sin(x) \quad \text{and} \quad \left(e^{jx}+e^{-jx}\right)=2\cos(x)$$

and applying a coefficient comparison it is obtained:

$$H_z(z)=\frac{a_1 z^{-1}}{1+b_1 z^{-1}+b_2 z^{-2}}=\frac{a_1 z}{z^2+b_1 z+b_2}$$

Thus the filter coefficients are determined as follows:

$$a_1=\sqrt{2}\,\omega_T e^{-\frac{\omega_T}{\sqrt{2}}T_a}\sin\left(\frac{\omega_T}{\sqrt{2}}T_a\right)$$

$$b_1=2\,e^{-\frac{\omega_T}{\sqrt{2}}T_a}\cos\left(\frac{\omega_T}{\sqrt{2}}T_a\right)$$

$$b_2=-e^{-\frac{2\omega_T}{\sqrt{2}}T_a}$$

This digital system function can be realized e.g. directly by the canonical structures presented in chapter 1.4

3.5 Design of IIR Digital filter by means of bi-linear Z-transform

Applying another method, called the bi-linear z-transform, one starts with a given system function $H_L(p)$ written as $H_L(p_k)$ of a realizable analog system, which should also be a causal and stable system with a real impulse response $h(t)$.

- $H_L(p_k)$ now is given as a function of the complex frequency:

$$p_k = \sigma_k + j\omega_k$$

- All poles and zeros of $H_L(p_k)$ must lie on the σ_k-axis or appear as conjugated complex pairs, which provides a real impulse response $h(t)$.

3.5 Design of IIR Digital filter by means of bi-linear Z-transform

From the system function $H_L(p_k)$ a stable and causal digital system with the system function $H_Z(z)$ can be developed if one replaces the argument p_k in $H_L(p_k)$ with a function of z.

For this function $p_k(z)$ holds:

- It is rational in z
- It is given by $$p_k(z) = \frac{2}{T_a} \cdot \frac{z-1}{z+1}$$ **Formula for bi-linear Z-transform**

The inverse function to $p_k(z)$ is: $$z = \frac{1 + \frac{T_a}{2} p_k}{1 - \frac{T_a}{2} p_k}$$

3.5 Design of IIR Digital filter by means of bi-linear Z-transform

By using $z = e^{j\omega Ta}$ in the argument of $p_k(z)$ it results:

$$p_k(e^{j\omega T_a}) = \frac{2}{T_a} \cdot \frac{e^{j\omega T_a} - 1}{e^{j\omega T_a} + 1} = j\frac{2}{T_a} \cdot \tan\left(\frac{T_a}{2}\omega\right) = j\omega_k$$

with $\omega_k = \frac{2}{T_a} \cdot \tan\left(\frac{T_a}{2}\omega\right) \quad \Leftrightarrow \quad \omega = \frac{2}{T_a} \cdot arc\tan\left(\frac{T_a}{2}\omega_k\right)$

Note:

ω is the frequency for the digital filter, describing with T_a the angle of a point on the unit circle in the z-domain.

ω_k represents the frequency in the p-domain

3.5 Design of IIR Digital filter by means of bi-linear Z-transform

From this one obtains the periodic transfer function $H_{aF}(\omega)$ of the digital filter, by using the following relation:

$$H_{aF}(\omega) = H_z(e^{j\omega T_a}) = H_L(j\,\omega_k) = H_F(\omega_k)$$

Summarizing:

The determination of the system function of the digital filter is thus obtained on the basis of the system function of the analog filter!

So by means of $p_k(z) = \frac{2}{T_a} \cdot \frac{z-1}{z+1}$ the function $H_L(p)\big|_{p \to p_k}$ is turned into

$H_Z(z)$ according to $H_Z(z) = H_L(\frac{2}{T_a} \cdot \frac{z-1}{z+1})$

3.5 Design of IIR Digital filter by means of bi-linear Z-transform

Please note:

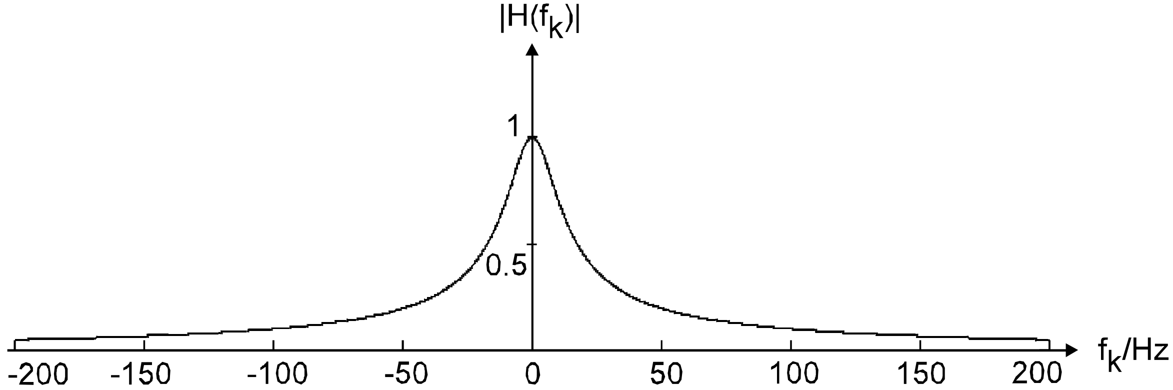
If the variable ω of $H_{aF}(\omega)$ of the digital system is varied from $-\infty$ to $+\infty$, $H_F(\omega_k)$ of the analog system is run through infinetely countable much times.

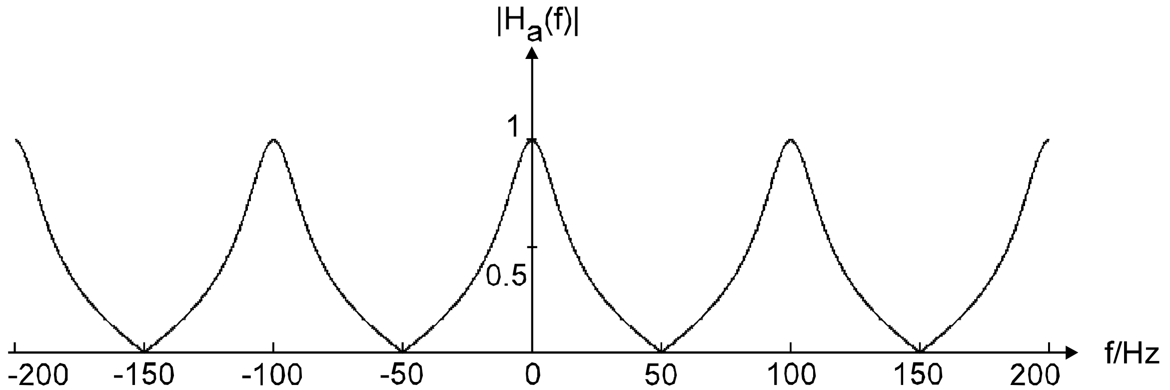
At the same time the system funktion $H_Z(z)$ is evaluated infinetely much countable times along the unit circle.

Also at the same time $H_L(p_k)$ is evaluated once along the whole $j\omega$-axis.

The next slide compares the frequency responses of the analog reference filter and the corresponding digital filter.

3.5 Design of IIR Digital filter by means of bi-linear Z-transform





Transfer function of an analog prototype with $f_g = 10Hz$
and of the corresponding digital filter with $f_a = 100\ Hz$

3.5 Design of IIR Digital filter by means of bi-linear Z-transform

Further characteristics of the bi-linear transform are:

1. The order of the derived digital filter agrees with the order of the given analog reference filter.

2. If the tolerance scheme for the analog filter includes piecewise horizontal parts, the same property will be true for the digital filter.

3. The frequency distortion especially for higher frequencies (close to half of the sampling rate) leads to the fact that the phase and envelope delay characteristics of the digital filter do not agree with those of the reference system.

For lower frequencies the distortion is very low so that the frequency response of the digital filter is close to that of the analog filter.

3.5 Design of IIR Digital filter by means of bi-linear Z-transform

The realization of digital low-pass, high-pass, band-pass filters or band-stops using the bi-linear transform is performed in the following steps:

Step 1: An approximation procedure is used and leads together with a LP-transform (to another LP, to a HP, BP or BS) to the system function $H_L(p)$ of the reference analog filter – or $H_L(p)$ is specified in another way.

Step 2: In $H_L(p)$, p is replaced by p_k which is then turned into $p_k(z)$. The result gives $H_z(z)$.

Step 3: A modification of the cut-off frequencies are needed due to frequency distortions produced by the bi-linear transform.

3.5 Design of IIR Digital filter by means of bi-linear Z-transform

These distortions can be avoided for the low-pass (without proof) by using a modified function for $p_k(z)$ as follows:

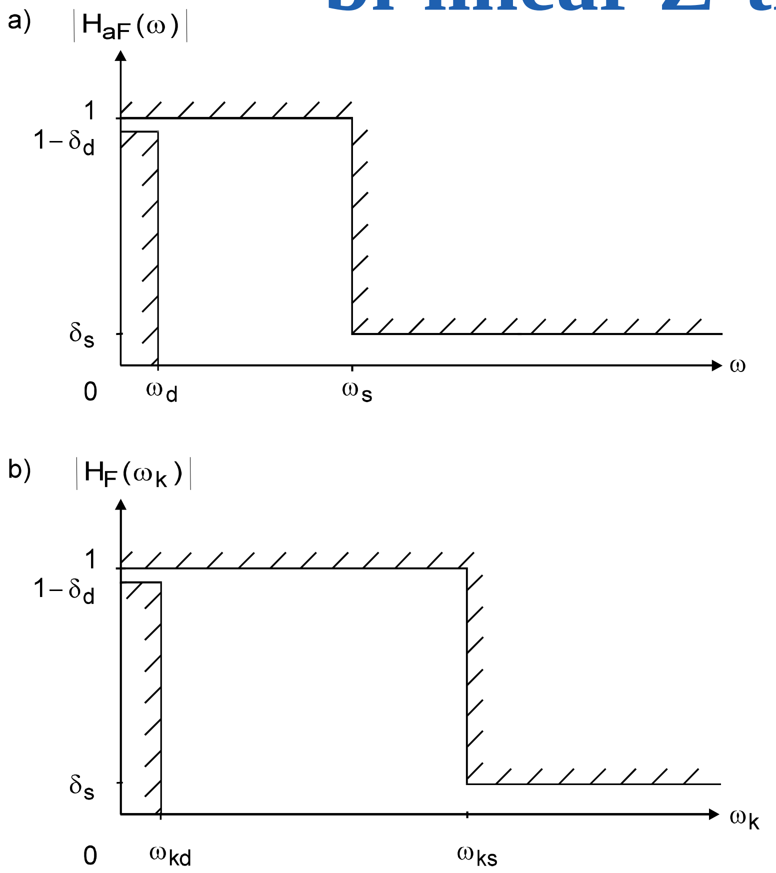
$$p_k(z) = \omega_T \cdot \cot(\pi \frac{\omega_T}{\omega_a}) \cdot \frac{z-1}{z+1} \quad \text{instead of} \quad p_k(z) = \frac{2}{T_a} \cdot \frac{z-1}{z+1}$$

Step 4: For the system function $H_z(z)$ a canonical filter structure is selected (see chapter 1.4).

Example: Digital low-pass filter synthesis by means of the bi-linear transform

The tolerance pattern for the desired amplitude characteristic $|H_{aF}(\omega)|$ is given. It is represented in the following figure a).

3.5 Design of IIR Digital filter by means of bi-linear Z-transform



Given values:

$\omega_d = 2\pi \cdot 1\ Hz = 6.28$ rad/s

$\omega_s = 2\pi \cdot 6\ Hz = 37.7$ rad/s

$\delta_d = 0.026$

$\delta_s = 0.15$

$\omega_a = 2\pi \cdot 20 Hz$

Tolerance pattern of the digital low-pass (figure a) and the frequency pre-distorted analog low-pass (figure b)

3.5 Design of IIR Digital filter by means of bi-linear Z-transform

For the sampling rate and the corresponding period it is given:

$$\omega_a = 2\pi \cdot 20\ Hz = 125.66 \quad \text{rad/s}$$

$$T_a = \frac{2\pi}{\omega_a} = \frac{1}{20\ Hz} = 0.05\ s$$

Convertion of the cut-off frequencies gives:

$$\omega_{kd} = \frac{2}{T_a} \cdot \tan\left(\frac{T_a}{2}\omega_d\right) = 6.33 \quad \text{rad/s}$$

$$\omega_{ks} = \frac{2}{T_a} \cdot \tan\left(\frac{T_a}{2}\omega_s\right) = 55.06 \quad \text{rad/s}$$

Note:

Small deviation for lower, significant deviation for the higher frequency!

Thus the modified tolerance pattern results according to figure b for the analog reference system.

3.5 Design of IIR Digital filter by means of bi-linear Z-transform

The following system function $H_L(p_k)$ was chosen as it fulfils the tolerance pattern according to figure b:

$$H_L(p_k) = \frac{200\,\omega_d^2}{{p_k}^2 + 20\,\omega_d \cdot p_k + 200\,\omega_d^2}$$

Replacing p_k with $p_k(z)$ according to $p_k(z) = \frac{2}{T_a} \cdot \frac{z-1}{z+1}$

gives the system function $H_z(z)$ of the digital filter:

$$H_z(z) = \frac{200 \cdot \omega_d^2}{\left(\frac{2}{T_a} \cdot \frac{z-1}{z+1}\right)^2 + 20\omega_d \cdot \left(\frac{2}{T_a} \cdot \frac{z-1}{z+1}\right) + 200\omega_d^2}$$

3.5 Design of IIR Digital filter by means of bi-linear Z-transform

After using the numerical values for ω_d and T_a, one obtains:

$$H_z(z) = \frac{0.077z^2 + 0.154z + 0.077}{z^2 - 0.92z + 0.385}$$

$$= \frac{0.077 + 0.154z^{-1} + 0.077z^{-2}}{1 - 0.92z^{-1} + 0.385z^{-2}}$$

Again it is recommended to use a canonical structure for this system according to the descriptions in chapter 1.4.