Fachgebiet Nachrichtentechnische Systeme

Network Theory 3
Advanced Digital Filters

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Books

- [1] U. Zölzer, Digitale Audiosignalverarbeitung Teubner Verlag, wiesbaden 2005
- [2] Johnson, J.R.Digitale Signalverarbeitung. (deutsche Version) Carl Hanser Verlag, München 1991
- [3] Introduction to Digital Signal Processing. (engl. version) Prentice-Hall, London 1991
- [4]Grünigen, D.Ch., Digitale Signalverarbeitung.AT Verlag, Berlin 1993

Chapter 1

Introduction

1 Introduction

- Used in growing number of applications
- Found in DSP chips, mobile communication, audio/video processing

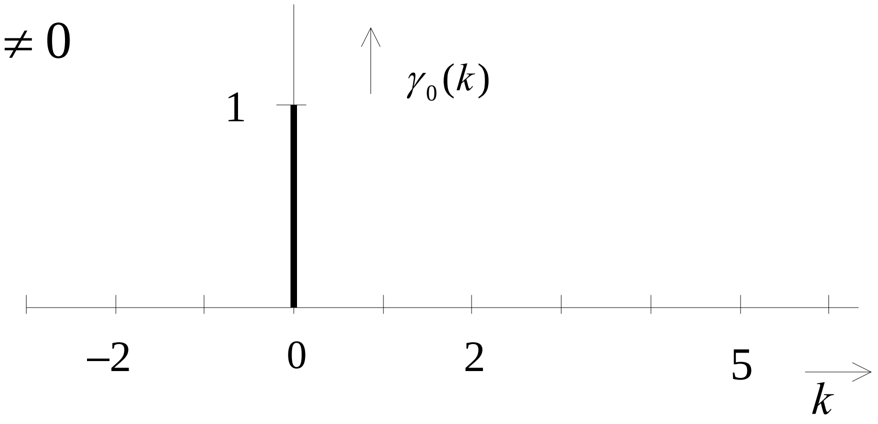
- Enable to set-up high precision filtering/signal processing
- Limited frequency band

1.1 Basic Signals

Additional basic sequences (basic discrete signals)

The unit-impulse:

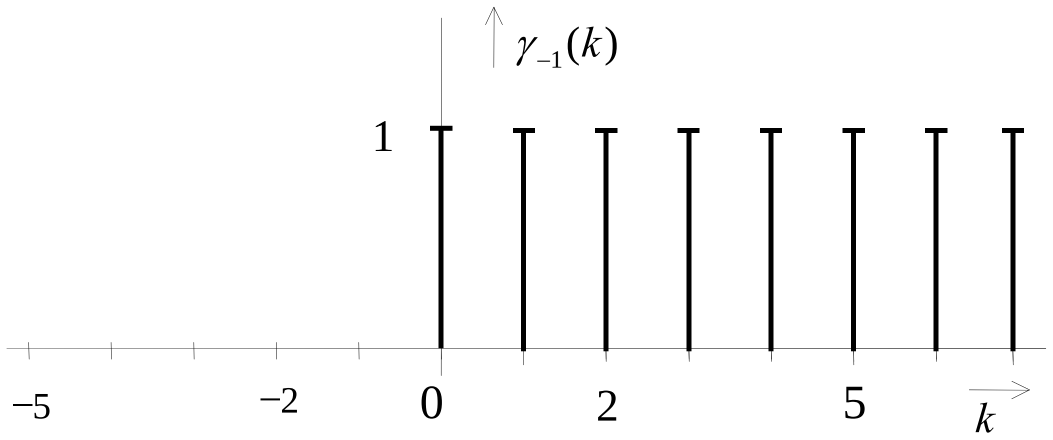
$$\{s(k)\} = \gamma_0(k) = \begin{cases} 1 & \text{for} \quad k = 0 \\ 0 & \text{for} \quad k \neq 0 \end{cases}$$



1.1 Basic Signals

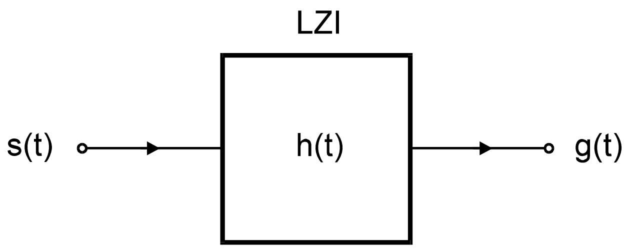
The unit-step sequence:

$$\gamma_{-1}(k) = \begin{cases} 0 & \text{for} \quad k < 0 \\ 1 & \text{for} \quad k \geq 0 \end{cases}$$



1.2 Signals at analog LTI systems

Description of the system properties in the time domain:



$$g(t) = s(t) * h(t) = \int_{-\infty}^{+\infty} s(\tau) \cdot h(t-\tau) d\tau$$

$$= h(t) * s(t) = \int_{-\infty}^{+\infty} h(\tau) \cdot s(t-\tau) d\tau$$

with *h(t)* as the impulse response

1.2 Signals at analog LTI systems

Laplace transform and the system function $H_L(p)$

A basis for the description of signals within the p-domain is:

$$S_L(p) = \int_0^{+\infty} s(t) \cdot e^{-pt}\, dt, \text{ where } s(t) = 0 \text{ for } t < 0 \text{ (causality condition)}$$

The inverse Laplace transform is determined by:

$$s(t) = \frac{1}{2\pi j} \cdot \oint_C S_L(p) \cdot e^{pt}\, dp$$

With the help of the Laplace transform, the **system function** can be defined:

$$g(t) = s(t) * h(t)$$

$$\circ\!\!-\!\!\bullet \; L$$

$$G_L(p) = S_L(p) \cdot H_L(p) \quad \rightarrow \quad H_L(p) = \frac{G_L(p)}{S_L(p)}$$

1.2 Signals at analog LTI systems

The Fourier transform and the transfer function $H_F(\omega)$:

The Fourier transform of the signal *s(t)* and its inverse transform:

$$S_F(\omega) = \int_{-\infty}^{+\infty} s(t) \cdot e^{-j\omega t} dt \qquad s(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_F(\omega) \cdot e^{j\omega t} d\omega$$

Some basic properties of the Laplace transform:

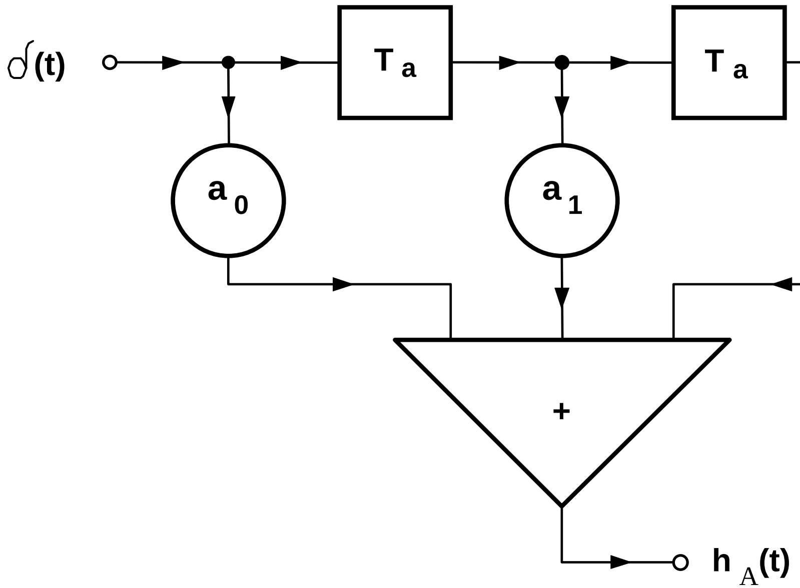
1. Stable systems have all the poles of the system function $H_L(p)$ in the left open p-half plane
2. Transfer function in this case is also given by: $H_F(\omega) = H_L(j\omega)$
3. Under the condition of a causal impulse response with poles only in left open p-plane, one can write:

$$H_L(p) = H_F(\omega)\big|_{j\omega \to p}$$

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

For discrete systems the z-transform is used. The following shows why by means of an example.

Example: An analog system (transverse filters) is given as follows:



1.3 The Z-transform and its relationship with the Laplace and Fourier transform

The impulse response can be determined as follows:

$$h_a(t) = a_0 \cdot \delta(t) + a_1 \cdot \delta(t - T_a) + a_2 \cdot \delta(t - 2T_a)$$

A Laplace transformation gives:

$$\delta(t - kT_a) \;\circ\!\!\stackrel{\mathrm{L}}{-\!\!-\!\!-}\!\!\bullet\; e^{-pkT_a}$$

Thus the system function (a non-rational function in p) gives:

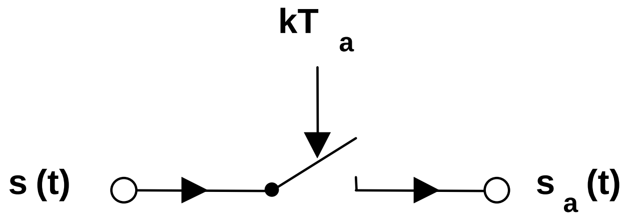
$$H_{aL}(p) = a_0 + a_1 \cdot e^{-pT_a} + a_2 \cdot e^{-2pT_a}$$

A substitution with $z = e^{pT_a}$ gives a rational fractioned system function in z.

$$H_z(z) = a_0 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2} = \frac{a_0 \cdot z^2 + a_1 \cdot z + a_2}{z^2}$$

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

Additional relations of z and Laplace transform are shown using an “**Ideal sampler**“:



Assuming that the input signal *s(t)* is a causal and real signal, one gets:

$$s_a(t) = s(t)\cdot\sum_{k=-\infty}^{+\infty}\delta(t-kT_a) = \sum_{k=-\infty}^{+\infty} s(kT_a)\cdot\delta(t-kT_a)$$

$$= \sum_{k=0}^{+\infty} s(kT_a)\cdot\delta(t-kT_a) \text{ (because of the causality of } s(t))$$

If the Laplace transform is applied, it is obtained:

$$S_{aL}(p) = \sum_{k=0}^{+\infty} s(kT_a)\cdot e^{-pkT_a}$$

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

This equation can be rewritten, by implementing the following steps:

Step 1: Abbreviation $s(kT_a) \rightarrow s(k)$ gives a new discrete function

Step 2: $z = e^{pT_a}$ with the complex frequency $p = \sigma + j\omega$

Step 3: Combination of step 1+2, one gets:

$$S_{aL}(p) = \sum_{k=0}^{+\infty} s(k) \cdot z^{-k} = S_z(z)$$

giving the z-transform of $s(k)$

For example:

$$s(k) = a_1 \cdot s_1(k) + a_2 \cdot s_2(k)$$

$$\circ\!\!-\!\!\bullet \; z$$

$$S_z(z) = a_1 \cdot S_{1z}(z) + a_2 \cdot S_{2z}(z)$$

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

The Laplace transform can always be recovered from the z-transform:

$$S_{aL}(p) = S_z(e^{pT_a})$$

The inverse *z*-transform is defined as follows:

$$s(k) = \frac{1}{2\pi j} \cdot \oint_C S_z(\mathrm{z}) \cdot z^{k-1}\,\mathrm{dz}$$

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

It is also valid: $$S_{aF}(\omega) = S_{aL}(j\omega) = S_z(e^{j\omega T_a})$$

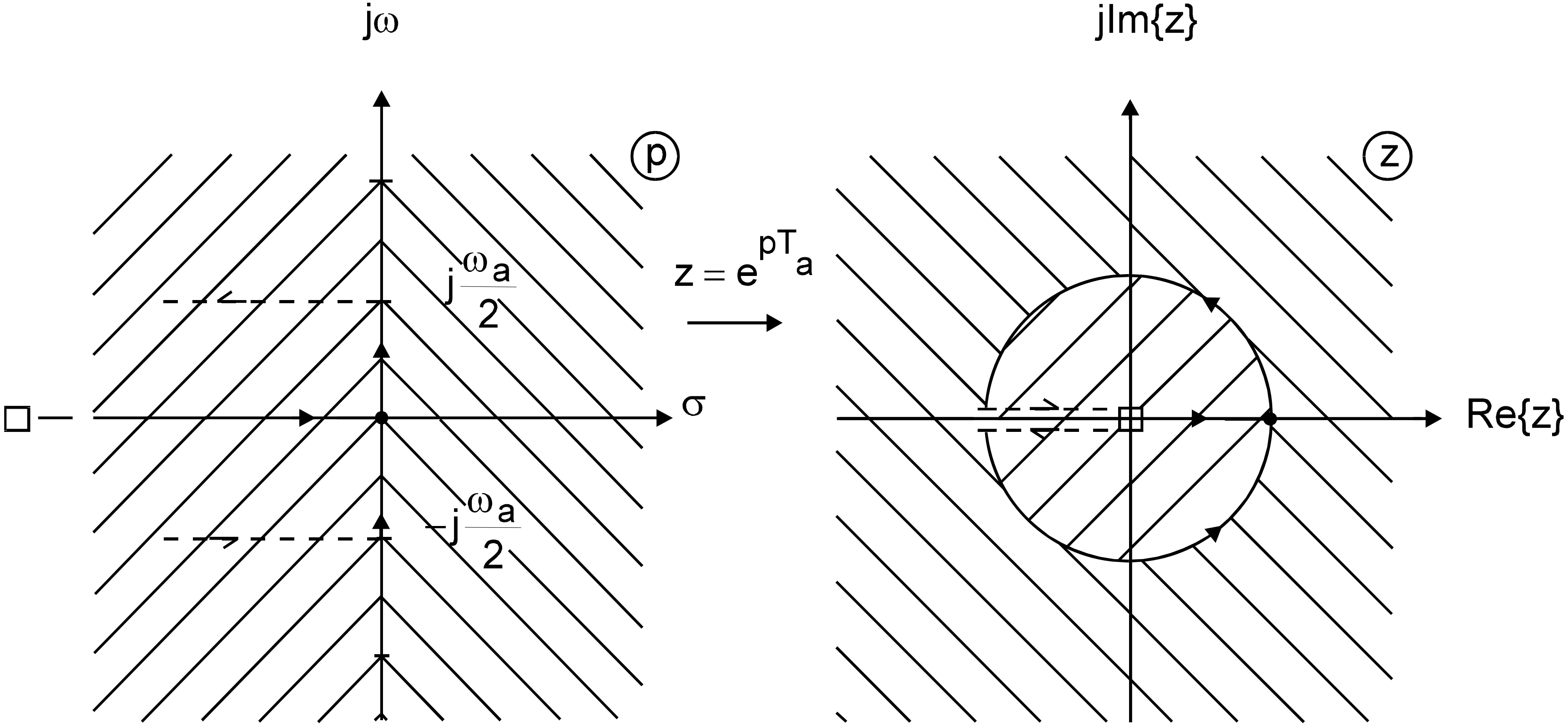
Magnitude spectrum: $$\left|S_{aF}(\omega)\right| = \left|S_{aL}(j\omega)\right| = \left|S_z(e^{j\omega T_a})\right|$$

Phase spectrum: $$\varphi_{aF}(\omega) = \sphericalangle S_{aF}(\omega) = \arctan\left(\frac{\operatorname{Im}\left\{S_z(e^{j\omega T_a})\right\}}{\operatorname{Re}\left\{S_z(e^{j\omega T_a})\right\}}\right)$$

Due to $z = e^{pT_a}$ important mappings for a simplified description of discrete LTI-analog system can be observed:

- Property 1: The $j\omega$-axis of the p-plane $\leftrightarrow$ the unit circle of the z-plane
- Property 2: The left open p-plane $\leftrightarrow$ area inside of the unit circle of z-plane
- Property 3: The right open p-plane $\leftrightarrow$ areas outside of the unit circle of z-plane

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

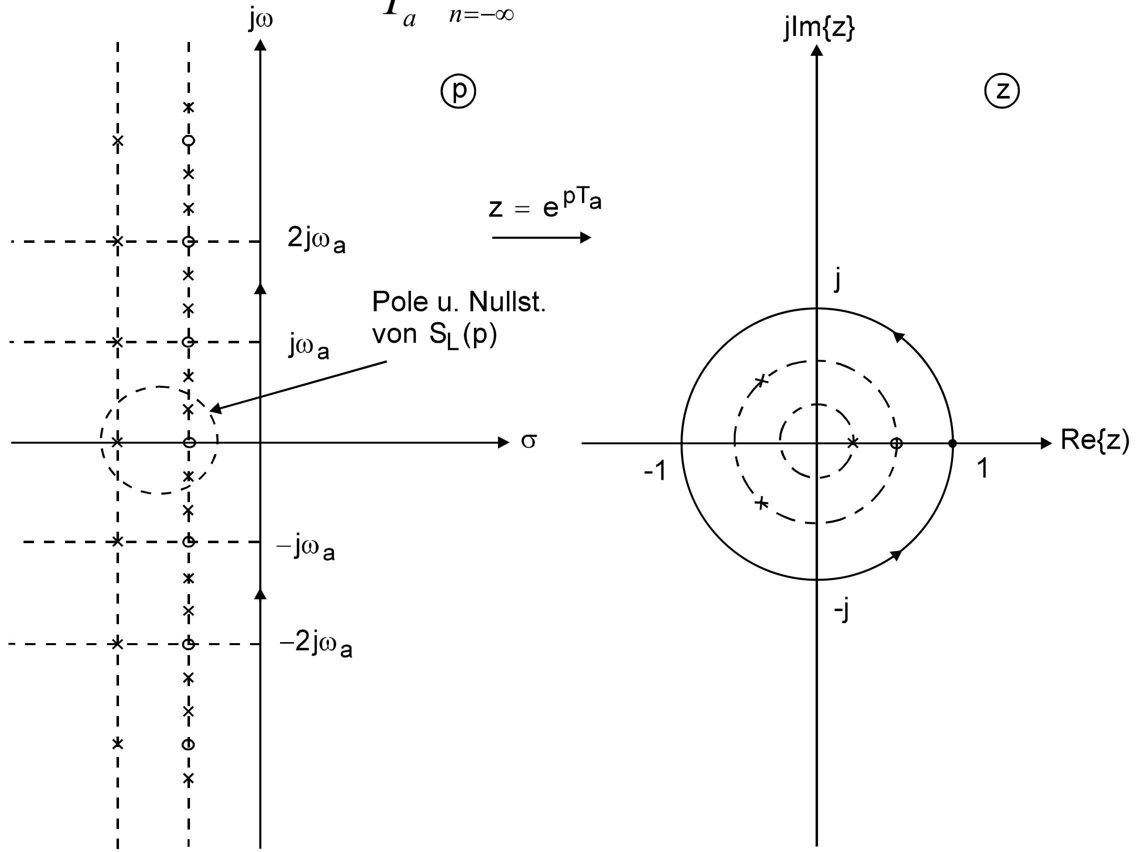


Mapping of the p-plane into the z-plane by the function $z = e^{pT_a}$

1.3 The Z-transform and its relationship with the Laplace and Fourier transform

It can be shown that it holds for the relation of transforms of sampled and normal signals:

$$S_{aL}(p) = S_z(e^{pT_a}) = \frac{1}{T_a} \cdot \sum_{n=-\infty}^{n=\infty} S_L(p - jn\omega_a)$$



1.3 The Z-transform and its relationship with the Laplace and Fourier transform

The figure clarifies:
The sampling operation gives a repeating pattern of the original pole-zero plot with infinite number of poles and zeros (looking like a stripe of poles and zeros).

So the z-plane representation is much more compact compared to the p-plane representation!

The repetition period is ω_a.

The mapping reduces the infinitely large number of poles and zeros to a finitely large number of them.

The mapping of each pole and each zero follows these relations:

$$z_{\infty n} = e^{p_{\infty n} T_a} \quad \text{for poles}$$

$$z_{0m} = e^{p_{0m} T_a} \quad \text{for zeros}$$

1.4 The linear, shift-invariant, causal digital filter

Example (with 3 non-zero samples for both $s_a(t)$ and $h_a(t)$) :

$$s_a(t) = s(0)\delta(t) + s(T_a)\delta(t-T_a) + s(2T_a)\delta(t-2T_a)$$

$$h_a(t) = h(0)\delta(t) + h(T_a)\delta(t-T_a) + h(2T_a)\delta(t-2T_a)$$

Due to $a\cdot\delta(t-c) * b\cdot\delta(t-d) = ab\cdot\delta(t-c-d)$ it follows:

$$g_a(t) = s_a(t) * h_a(t) =$$

$$[h(0)\delta(t) + h(T_a)\delta(t-T_a) + h(2T_a)\delta(t-2T_a)]s(0) * \delta(t) +$$

$$[h(0)\delta(t) + h(T_a)\delta(t-T_a) + h(2T_a)\delta(t-2T_a)]s(T_a) * \delta(t-T_a) +$$

$$[h(0)\delta(t) + h(T_a)\delta(t-T_a) + h(2T_a)\delta(t-2T_a)]s(2T_a) * \delta(t-2T_a)$$

$$\Rightarrow g_a(t) =$$

$$s(0)h(0)\cdot\delta(t) + [s(T_a)h(0) + s(0)h(T_a)]\cdot\delta(t-T_a) +$$

$$[s(0)h(2T_a) + s(T_a)h(T_a) + s(2T_a)h(0)]\cdot\delta(t-2T_a) +$$

$$[s(T_a)h(2T_a) + s(2T_a)h(T_a)]\cdot\delta(t-3T_a) + s(2T_a)h(2T_a)\cdot\delta(t-4T_a)$$

1.4 The linear, shift-invariant, causal digital filter

The last result can be written in shortened form as follows:

$$g(k) =$$

$$s(0)h(0)\cdot\gamma_0(k) + [s(1)h(0) + s(0)h(1)]\cdot\gamma_0(k-1) +$$

$$[s(0)h(2) + s(1)h(1) + s(2)h(0)]\cdot\gamma_0(k-2) +$$

$$[s(1)h(2) + s(2)h(1)]\cdot\gamma_0(k-3) + s(2)h(2)\cdot\gamma_0(k-4)$$

Concernig the result in the rectangular brackets (i.e. a specific value of *g(k)*) the clock shift of the unit-impulses is the same as the sum of the clock shifts of *s(k)* and the clock shifts of *h(k)* according to:

$$g(k) \; = \sum_{\upsilon=0}^{k} s(\upsilon)\cdot h(w)\gamma_0(k-v-w) \text{ with } k = v + w \text{ or } w = k - v$$

$$\Rightarrow g(k) = \sum_{\upsilon=0}^{k} s(\upsilon)\cdot h(k-\upsilon)\gamma_0(k-v-w)$$

Example for k = 4:

$$g(4) = s(0)h(4) + s(1)h(3) + s(2)h(2) + s(3)h(1) + s(4)h(0)$$

$$= s(2)h(2) \text{ due to non-zero values only for arguments in range 0 to 2}$$

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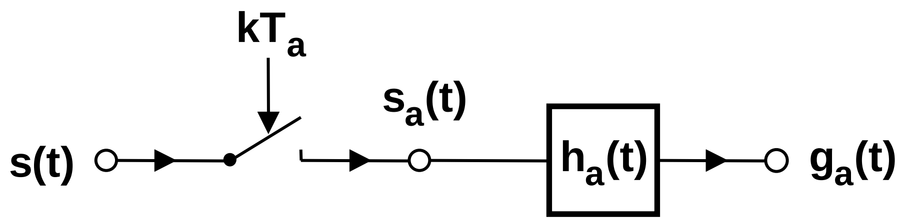
1.4 The linear, shift-invariant, causal digital filter

As shown the system function features a periodic pole zero diagram:

$$H_{aL}(p) = \frac{1}{T_a} \cdot \sum_{n=-\infty}^{+\infty} H_L(p - jn\omega_a) \quad \text{where} \quad T_a = \frac{2\pi}{\omega_a}$$



System consisting of an ideal sampler and an LTI analog filter with time-discrete and causal impulse response

Another interpretation: $H_{aL}(p)$ is the Laplace transform of an impulse response $h(t)$ which is sampled at a rate of $T_a = 2\pi / \omega_a$.

The Laplace transform $G_{aL}(p)$ of the output $g_a(t)$ can be determined by means of the 2 following methods:

1.4 The linear, shift-invariant, causal digital filter

Substituting in the previous equations z by e^{pT_A}, one receives so called

$z-transform\ of\ the\ impulse\ response\ h(k):$

$$H_{aL}(p) = \sum_{k=0}^{\infty} h(k) \cdot z^{-k} = H_z(z) \qquad \text{for IIR-filters}$$

$$H_{aL}(p) = \sum_{k=0}^{M} h(k) \cdot z^{-k} = H_z(z) \qquad \text{for FIR filters}$$

$z-transform\ G_z(z)\ of\ the\ sequence\ g(k):$

$$G_{aL}(p) = \sum_{k=0}^{\infty} g(k) \cdot z^{-k} = G_z(z)$$

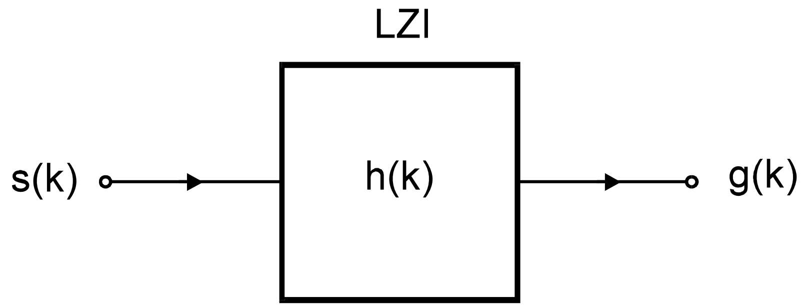
The equation to compute the values of $g(k)$ can be written as:

$$g(k) = \sum_{\nu=0}^{k} s(\nu) \cdot h(k-\nu) = \sum_{\nu=0}^{k} h(\nu) \cdot s(k-\nu)$$

1.4 The linear, shift-invariant, causal digital filter

A shorter writing of the preceding formula leads to **the discrete convolution product of the sequences *s(k)* and *h(k)***:

$$g(k) = s(k) * h(k) = h(k) * s(k)$$



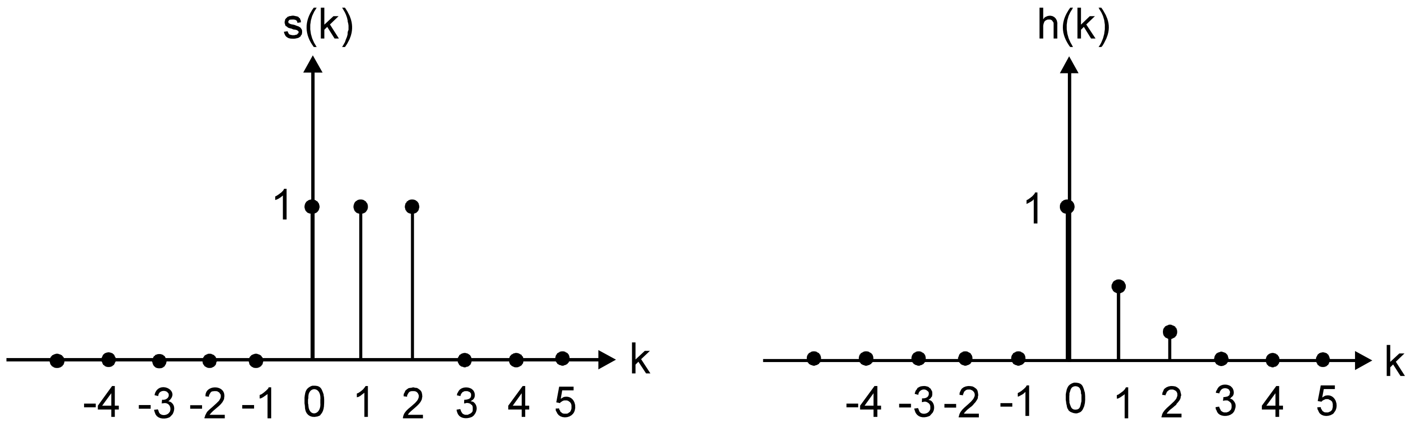
Symbol for a digital filter with the impulse response *h(k)*

1.4 The linear, shift-invariant, causal digital filter

Example: Discrete convolution

Given is a digital filter system with the following properties:

$$s(k) = \sum_{\nu=0}^{2} \gamma_0(k-\nu) \qquad h(k) = \begin{cases} \left(\frac{1}{2}\right)^k & \text{for } 0 \le k \le 2 \\ 0 & \text{else} \end{cases}$$



Input sequence *s(k)* and impulse response *h(k)* of a digital filter

➡ Required: Output sequence *g(k)*

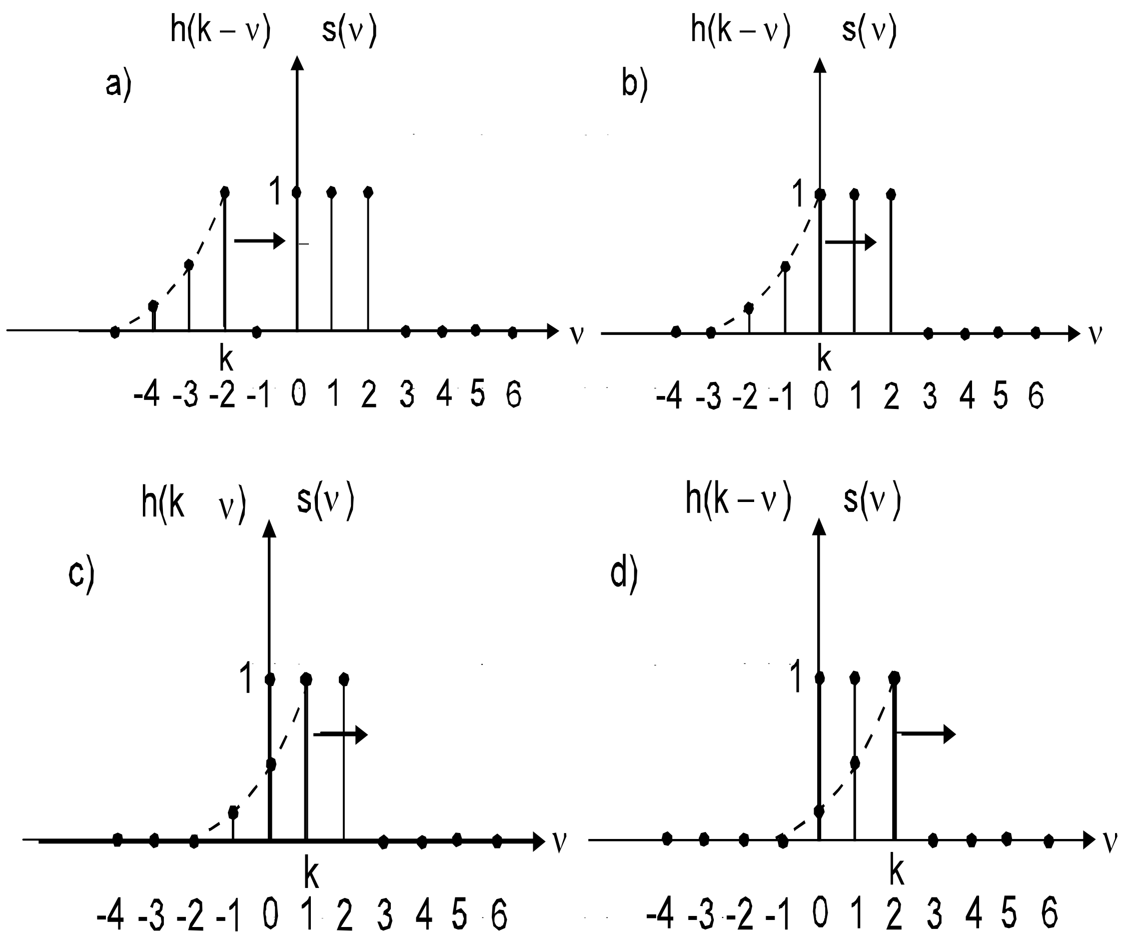
1.4 The linear, shift-invariant, causal digital filter

Using: $g(k)=\sum_{\nu=0}^{k} s(\nu)\cdot h(k-\nu)=\sum_{\nu=0}^{k} h(\nu)\cdot s(k-\nu)$ gives:

1. For $k<0$ $\quad g(k) = 0$
2. For $k=0$ $\quad g(0)=1\cdot 1+1\cdot 0+1\cdot 0=1$
3. For $k=1$ $\quad g(1)=1\cdot 1/2+1\cdot 1+1\cdot 0=3/2$
4. For $k=2$ $\quad g(2) = 1\cdot\frac{1}{4}+1\cdot\frac{1}{2}+1\cdot 1 = \frac{7}{4}$
5. For $k=3$ $\quad g(3)=1\cdot 0+1\cdot 1/4+1\cdot 1/2=3/4$
6. For $k=4$ $\quad g(4)=1\cdot 0+1\cdot 0+1\cdot 1/4=1/4$
7. For $k>4$ $\quad g(k) = 0$

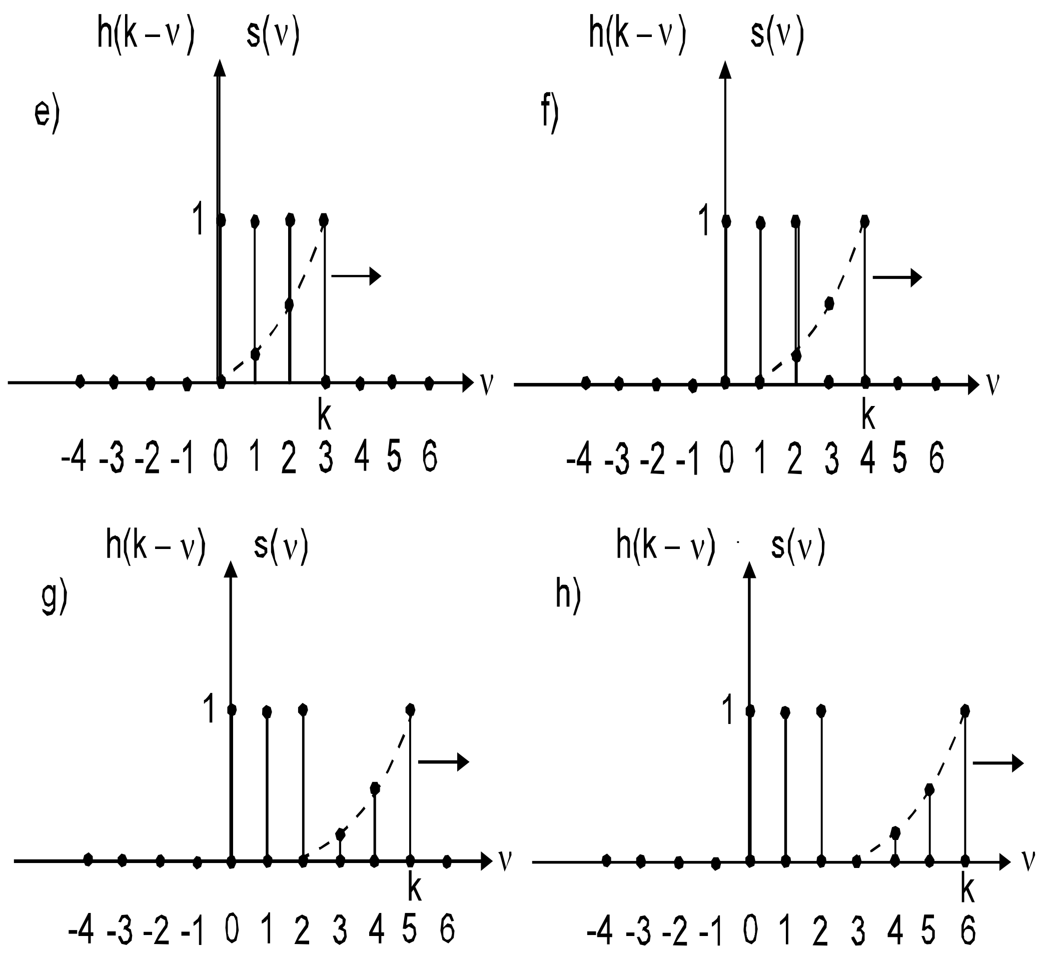
These calculations can also be performed graphically by means of the „foil method“ (comparable to a similar method known from analog convolution operation.

1.4 The linear, shift-invariant, causal digital filter



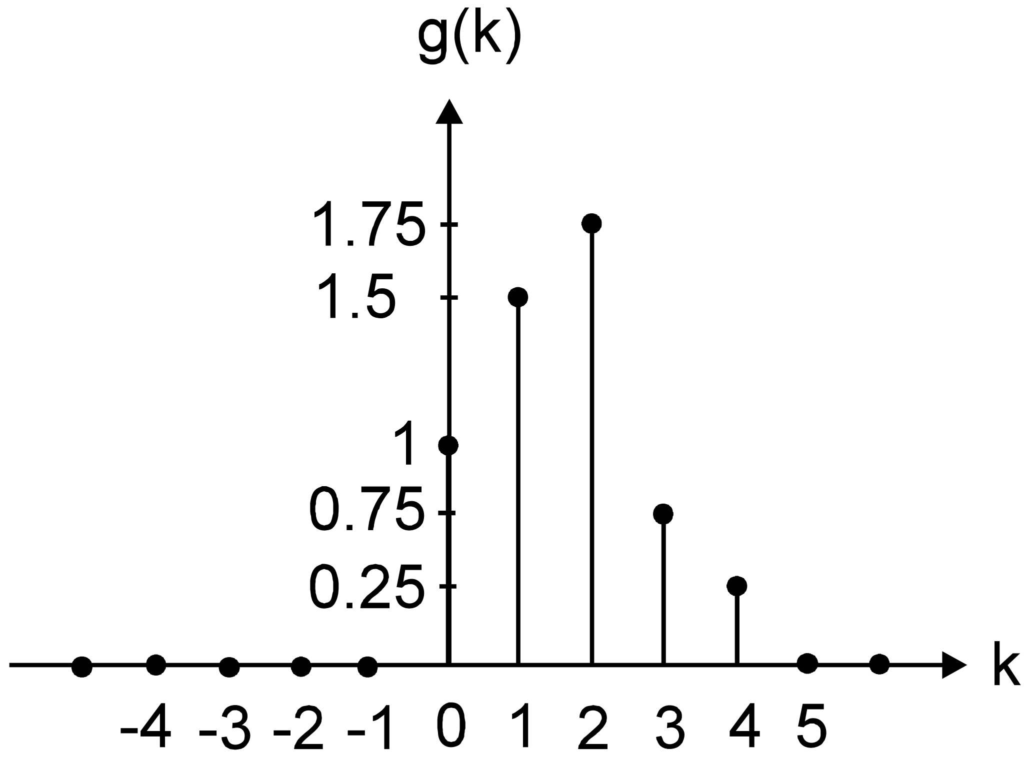
Determination of the output sequence g(k) with the help of the "foil method"

1.4 The linear, shift-invariant, causal digital filter



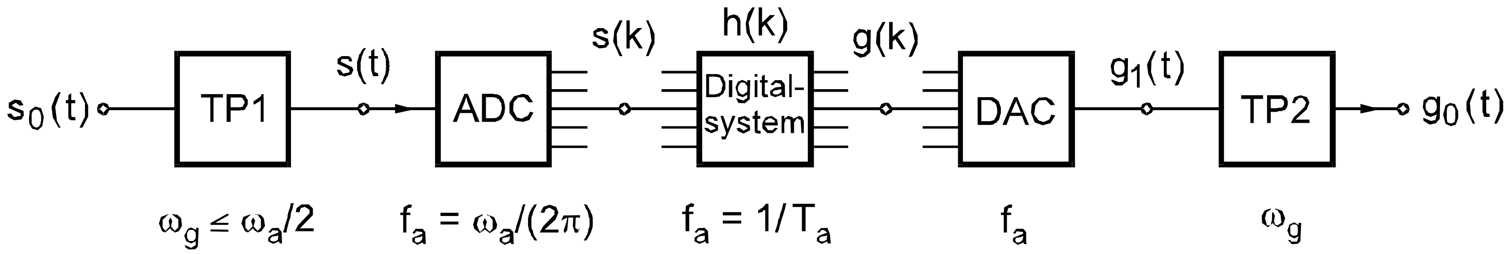
Determination of the output sequence g(k) with the help of the "foil method"

1.4 The linear, shift-invariant, causal digital filter



Output sequence *g(k)*, determined according to the "foil method"

1.4 The linear, shift-invariant, causal digital filter



Block diagram of an arrangement for the digital filtering of analogue signals

TP1,TP2 : Lowpass filter

ADC : Analog-Digital Converter

DAC : Digital-Analog Converter

A certain requirement concerning cut-off frequencies and the sample rate has to be fulfilled:

$$\omega_a > 2\omega_g \text{ (Nyquist condition)} \quad \text{with } 2\omega_g = Nyquist - Rate$$

f_a : sampling or conversion rate

1.4 The linear, shift-invariant, causal digital filter

More details about the Nyquist condition:

- The spectrum $S_{aF}(\omega)$ may not show overlapping (aliasing) with the parts $S_F(\omega - n\omega_a)$
- Otherwise $S_{aF}(\omega)$ will not be identical to $S_F(\omega)$ in the base band.
- No overlapping is given if the Nyquist condition is met.

The essential signal processing of the arrangement represented in the figure above takes place in the digital system describable by the linear transformation:

$$s(k) \rightarrow g(k) = T\left[s(k)\right] = s(k) * h(k)$$

$h(k)$ here represents the systems response to the input stimulation

$$\gamma_0(k) = \begin{cases} 1 \ \forall k = 0 \\ 0 \text{ elsewhere} \end{cases}$$

1.4 The linear, shift-invariant, causal digital filter

The reconstruction low-pass TP2 has to filter out only the base band parts of the spectrum (with *n = 0)* and must compensate the si-expression given above. This can be achieved if the transfer function of TP2 approximately looks as follows:

$$H_{RF}(\omega) = \frac{1}{T_a\, si\left(\omega \frac{T_a}{2}\right)} \cdot rect\left(\frac{\omega}{\omega_a}\right) \cdot e^{j\omega \frac{T_a}{2}}$$

$$G_{oF}(\omega) = H_{RF}(\omega) \cdot G_{1F}(\omega) = \left\{\frac{1}{T_a} \cdot S_F(\omega)\right\} \cdot \left\{\frac{1}{T_a} \sum_{n=-\infty}^{\infty} H_F(\omega - n\omega_a)\right\}$$

$$= \left\{\frac{1}{T_a} \cdot S_F(\omega)\right\} H_{aF}(\omega)$$

$$\Rightarrow g_0(t) = \frac{1}{T_a} \cdot s(t) * h_a(t) \quad \text{with}$$

$$s(t) = \sum_{k=0}^{+\infty} s(k) \cdot si\left(\frac{\omega_a}{2}\{t - kT_a\}\right)$$

1.4 The linear, shift-invariant, causal digital filter

So it is required to fulfil the following conditions:

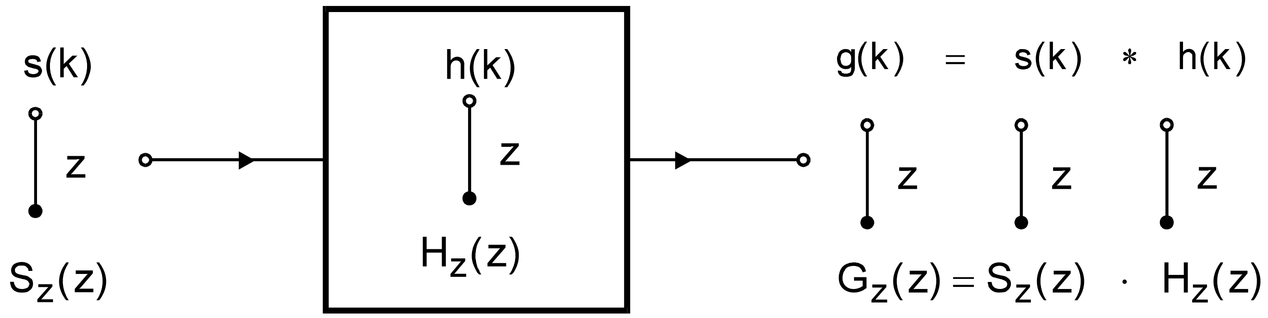
- Perfectly band limited signal *s(t)* exhibiting a cut-off frequency of *fa / 2*
- Infinitely large word length L of digital components
- Suitable transfer function of TP2 according to:

$$H_{RF}(\omega) = \frac{1}{T_a\, si\left(\omega \frac{T_a}{2}\right)} \cdot rect\left(\frac{\omega}{\omega_a}\right) \cdot e^{j\omega \frac{T_a}{2}}$$

Then the digital filter works as an analog filter with the impulse response $h_a(t)$ (apart from the constant $1/T_a$).

1.4 The linear, shift-invariant, causal digital filter

According to comments mentioned above a digital filter is represented by the following symbol:



Ideal linear, shift-invariant and causal digital filter

For easier calculation of z-transforms suitable tables with already determined transforms are used.

1.4 The linear, shift-invariant, causal digital filter

Table of important correspondences for *z*-transform:

Nr	s(k) operation	S(z) operation	Remarks
1	$\gamma_o(k) = \begin{cases} 1 & for \quad k = 0 \\ 0 & for \quad k \neq 0 \end{cases}$	1	Unit Impulse
2	$\gamma_{-1}(k) = \begin{cases} 1 & for \quad k \geq 0 \\ 0 & for \quad k < 0 \end{cases}$	$\frac{z}{z-1}$	Step sequences
3	$e^{p_1 k T_a} \cdot \gamma_{-1}(k) = z_1^k \cdot \gamma_{-1}(k)$	$\frac{z}{z - z_1}$	Exponential Sequences
4	$s(k-\upsilon) \qquad with \qquad \upsilon > 0$	$z^{-\upsilon} \cdot S_z(z)$	Shift
5	$\sum_{\upsilon=0}^{k} h(\upsilon) \cdot s(k-\upsilon) = h(k) * s(k)$	$H_z(z) \cdot S_z(z)$	Convolution ⇕ Multiplication
6	$\sum_{\upsilon} a_\upsilon s_\upsilon(k)$	$\sum_{\upsilon} a_\upsilon S_z(z)$	Linearity

1.4 The linear, shift-invariant, causal digital filter

Provided the zero-state exists, the output sequence *g(k)* can be computed from:

- The present weighted input value *s(k)*
- A linear combination of earlier, weighted input and output values

These circumstances are described directly by the so-called **difference equation:**

$$g(k) = \sum_{m=0}^{M} a_m s(k-m) - \sum_{n=1}^{N} b_n g(k-n)$$

Weighted input values *s(k)*
(m = 0 for the present value*)* and
s(k-m) for earlier input values with

$1 \leq m \leq M$

Earlier values are
denoted by *g(k- n)* with
$1 \leq n \leq N$

1.4 The linear, shift-invariant, causal digital filter

Example3: Computation of the output sequence from a difference equation

In case of $M = 0,\ N = 1,\ a_0 = 1,\ b_1 = 0.5$ one gets:

$$g(k) = s(k) - 0.5 \cdot g(k-1) \quad with \quad k \geq 1,\ g(-1) = 0$$

$s(k) = \gamma_{-1}(k)$; The zero-state condition requires at least $g(-1) = 0$

	s(k)	$1 \cdot s(k)$	*-0.5 g(k- 1)*	*g(k)*
0	1	1	0	1
1	1	1	-0.5	0.5
2	1	1	-0.25	0.75
3	1	1	-0.375	0.625
4	1	1	-0.3125	0.6875
5	1	1	-0.34375	0.65625

1.4 The linear, shift-invariant, causal digital filter

A <u>third method</u> for the description of the characteristics of a digital filters is represented in the form of a "block diagram”.

The block diagram is a graphical description of the difference equation:

$$g(k) = \sum_{m=0}^{M} a_m s(k-m) - \sum_{n=1}^{N} b_n g(k-n) \quad with \quad n, m, M, N \in N_0$$

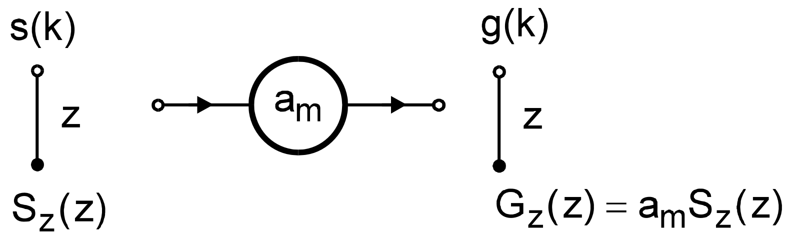
For the execution of the operations "multiplication with a constant", "addition" and "shift" and/or "delay" suitable system components are needed.

The appropriate blocks and the associated input and output relations in k are shown in the following figure.

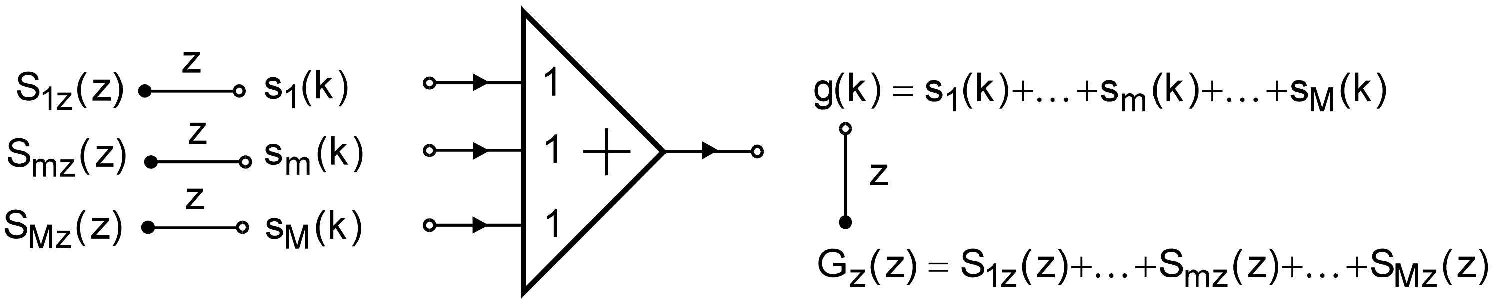
Without loosing generality one could set $N = M$ and sets suitable a_m and b_n values to zero.

1.4 The linear, shift-invariant, causal digital filter

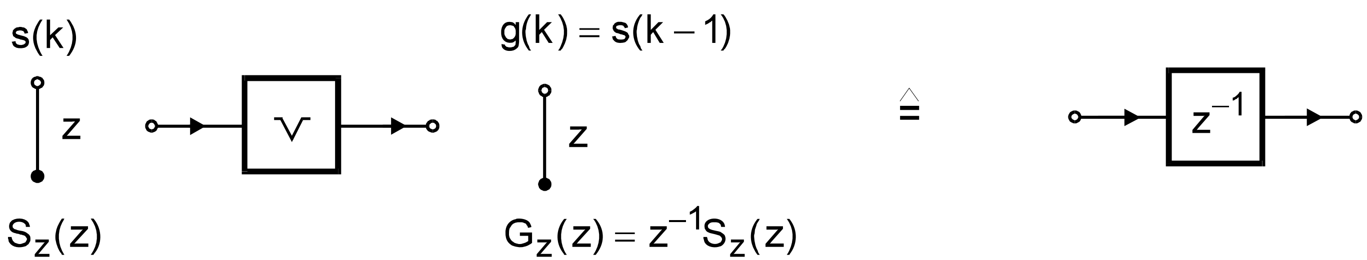
Multiplizierer :



Addierer :

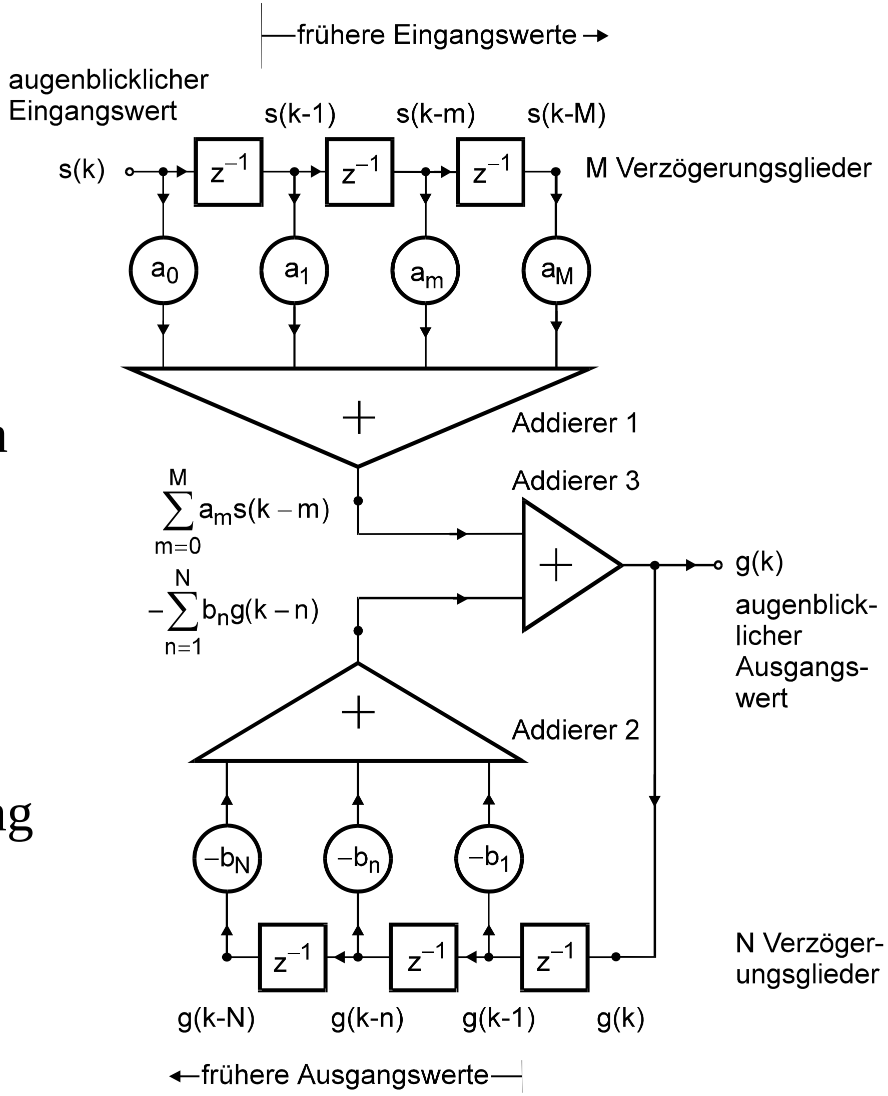


Verzögerungsglied :



1.4 The linear, shift-invariant, causal digital filter

Accordingly these steps result in the so-called ***"direct structure"***



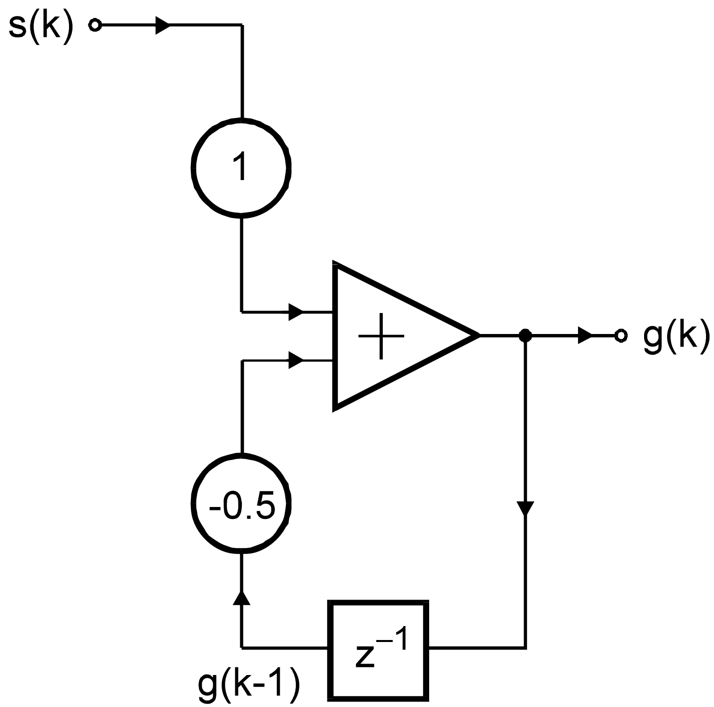
Here $N+M$ delays are needed.

It can be shown that a lower number is sufficient for realising the same filter properties. This leads to so-called canonical structures.

1.4 The linear, shift-invariant, causal digital filter

<u>Example</u>

It shows the conversion of the difference equation presented above into the block diagram.



"Block diagram" for the difference equation $g(k) = s(k) - 0.5g(k-1)$

1.4 The linear, shift-invariant, causal digital filter

The system function of **a recursive digital filter** is gained by z-transforming the differential equation:

$$g(k) = \sum_{m=0}^{M} a_m \, s(k-m) - \sum_{n=1}^{N} b_n \, g(k-n)$$

Thus it follows:

$$G_z(z) = \sum_{m=0}^{M} a_m \, z^{-m} \cdot S_z(z) - \sum_{n=1}^{N} b_n \, z^{-n} \cdot G_z(z)$$

1.4 The linear, shift-invariant, causal digital filter

Rewriting this equation gives:

$$G_z(z)\cdot\left\{1+\sum_{n=1}^{N} b_n\, z^{-n}\right\} = \sum_{m=0}^{M} a_m\, z^{-m}\cdot S_z(z)$$

$$\Leftrightarrow\; G_z(z) = \frac{\sum_{m=0}^{M} a_m\, z^{-m}}{1+\sum_{n=1}^{N} b_n\, z^{-n}}\cdot S_z(z) = H_z(z)\cdot S_z(z)$$

Thus the **system function** of the general, recursive, linear, shift-invariant and causal digital filter is gained.

$$H_z(z) = \frac{\sum_{m=0}^{M} a_m\, z^{-m}}{1+\sum_{n=1}^{N} b_n\, z^{-n}}$$

1.4 The linear, shift-invariant, causal digital filter

If one sets all constants b_n to zero, one obtain the non-recursive filter:

$$H_z(z) = \sum_{m=0}^{M} a_m z^{-m}$$

1.4 The linear, shift-invariant, causal digital filter

The impulse response *h(k)* can be determined by means of three different procedures:

Procedure 1: Application of the inverse z-transform to $H_Z(z)$

Procedure 2:

1. Rewrite $H_z(z)$ so that only positive exponents result
2. Develope $H_z(z)$ into a series by continued division of $\dfrac{\text{Nominator}\{H_z(z)\}}{\text{Denominator}\{H_z(z)\}}$
3. Apply inverse transform to each of the simple addends.

Procedure 3:

Develope $H_z(z)$ in a sum of partial fractions and apply the z-inverse transform to all addends.

1.4 The linear, shift-invariant, causal digital filter

Example: Determination of the impulse response *h(k)* by Procedure 2

A system function is given
$$H_z(z) = \frac{\sum_{m=0}^{M} a_m z^{-m}}{1+\sum_{n=1}^{N} b_n z^{-n}}$$

with $M = 0,\ a_1 = 1,\ N = 1\ and\ b_1 = 0.8$; Required is $h(k)$.

Solution:

With the given information, one directly gets: $H_z(z) = \frac{1}{1+0.8\ z^{-1}}$

As we see a negative exponent of -1 , nominator and denominator are multipled with z giving:

$$H_z(z) = \frac{z}{z+0.8}$$

1.4 The linear, shift-invariant, causal digital filter

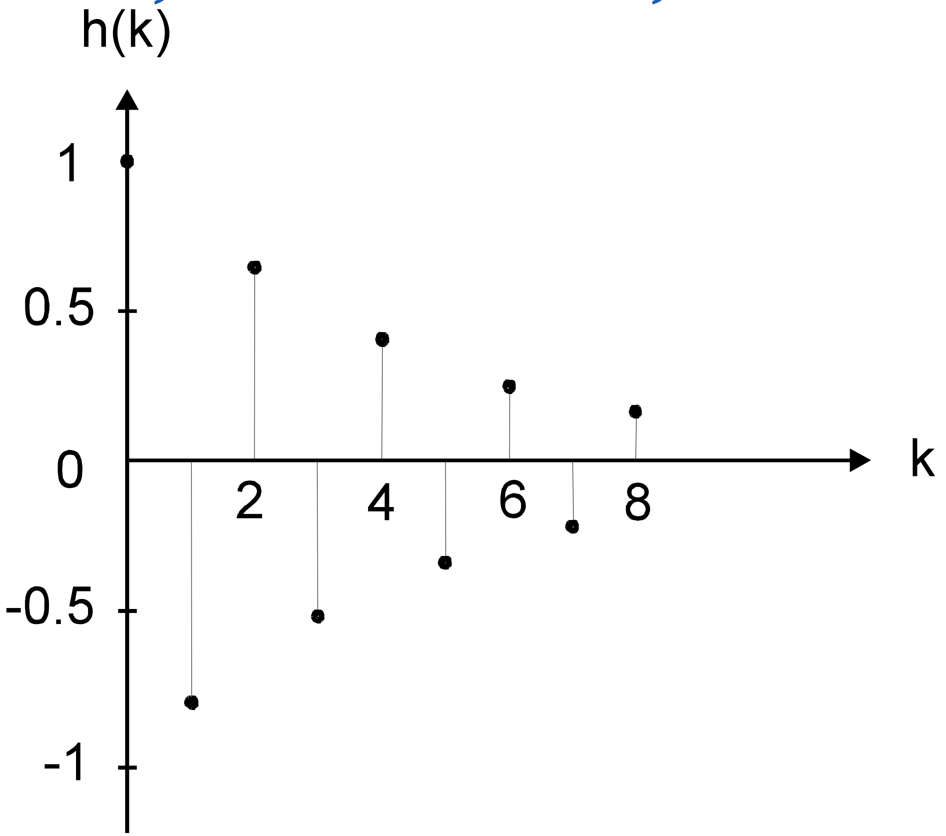
Continued (polynomial) division resultS in

$$H_z(z) = 1 - 0.8\,z^{-1} + 0.64\,z^{-2} - 0.51\,z^{-3} + 0.41\,z^{-4} - 0.33\,z^{-5} + \ldots$$

applying the correspondences Nr.4, one gets the inverse transform:

$$h(k) = \gamma_0(k) - 0.8\cdot\gamma_0(k-1) + 0.64\cdot\gamma_0(k-2) - 0.51\cdot\gamma_0(k-3)$$
$$+ 0.41\cdot\gamma_0(k-4) - \ldots$$

1.4 The linear, shift-invariant, causal digital filter



Impulse response of a recursive filter with the system function
Note: Decaying absolute coefficient values (stable system)

$$H_z(z) = 1 - 0.8\,z^{-1} + 0.64\,z^{-2} - 0.51z^{-3} + 0.41z^{-4} - 0.33\,z^{-5} + \ldots$$

1.4 The linear, shift-invariant, causal digital filter

Procedure §: For the case of $N = M+1$ the following equation holds:

$$H_z(z) = \frac{\sum_{m=0}^{M} a_m z^{-m}}{1+\sum_{n=1}^{N} b_n z^{-n}} \overset{N=M+1}{=} \frac{\sum_{m=0}^{M} a_m z^{N-m}}{z^N + \sum_{n=1}^{N} b_n z^{N-n}}$$

Assuming that all poles are single, it can be deduced (with real K and $N = M + 1$):

$$H_z(z) = K \cdot \frac{\sum_{m=0}^{M} a_m z^{N-m}}{\prod_{\upsilon=1}^{N} (z - z_{\infty\upsilon})} = K \cdot \frac{a_0 z \cdot \sum_{m=0}^{M} \frac{a_m}{a_0} z^{N-1-m}}{\prod_{\upsilon=1}^{N} (z - z_{\infty\upsilon})}$$

1.4 The linear, shift-invariant, causal digital filter

Example: Determination of the impulse response using Procedure 3

Given is the system function $$H_z(z) = \frac{\sum_{m=0}^{M} a_m z^{-m}}{1+\sum_{n=1}^{N} b_n z^{-n}}$$

with $M = 0,\ a_0 = 1,\ N = 2,\ b_1 = -0,2 \ and \ b_2 = -0.8$

1.4 The linear, shift-invariant, causal digital filter

With the given information it directly results:

$$H_z(z) = \frac{1}{1 - 0.2 \cdot z^{-1} - 0.8 \cdot z^{-2}}$$

Multiplication of the nominator and denominator by $z^N = z^2$ gives only positive exponent values:

$$H_z(z) = \frac{z^2}{z^2 - 0.2 \cdot z - 0.8}$$

Excluding $a_0 \cdot z = z$ from the numerator, it results:

$$H_z(z) = z \cdot \left(\frac{z}{z^2 - 0.2 \cdot z - 0.8} \right)$$

A partial fractions development now leads to following calculations:

1.4 The linear, shift-invariant, causal digital filter

$$\frac{z}{z^2-0.2z-0.8}$$ gives the roots: $z_{\infty 1,2}=0.1\pm\sqrt{0.01+0.8}=0.1\pm 0.9$

$$\Rightarrow \quad \frac{z}{z^2-0.2z-0.8}=\frac{A}{z-1}+\frac{B}{z+0.8}$$

$$\Rightarrow \quad A=\left.\frac{z}{z+0.8}\right|_{z=1}=\frac{1}{9/5}=\frac{5}{9}$$

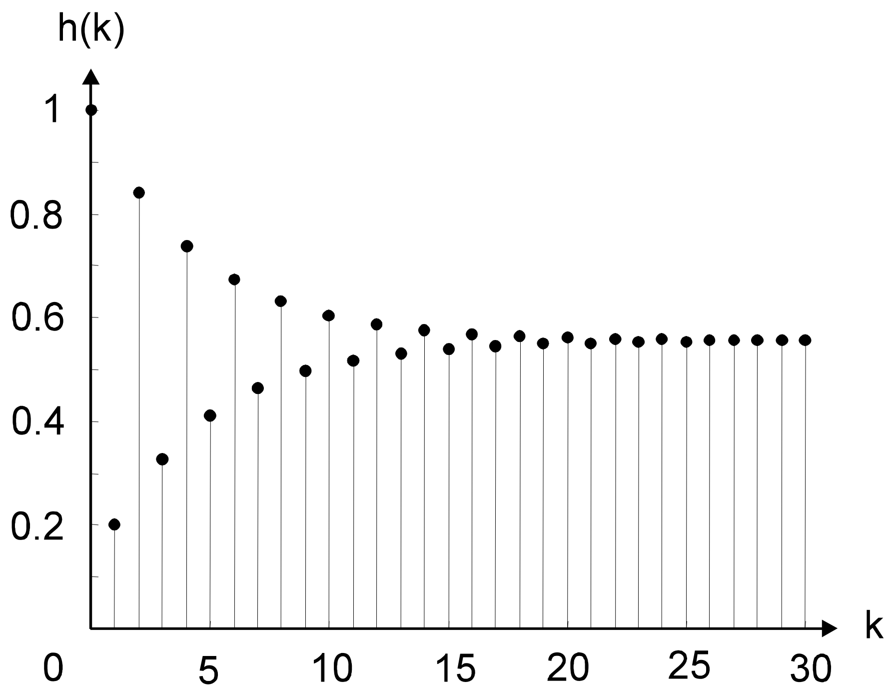
$$B=\left.\frac{z}{z-1}\right|_{z=-0.8}=\frac{-0.8}{-0.8-1}=\frac{-4/5}{-9/5}=\frac{4}{9}$$

Thus we obtain: $$H_z(z)=z\cdot\left(\frac{\frac{5}{9}}{z-1}+\frac{\frac{4}{9}}{z+0.8}\right)=\frac{\frac{5}{9}z}{z-1}+\frac{\frac{4}{9}z}{z-(-0.8)}$$

1.4 The linear, shift-invariant, causal digital filter

Finally an inverse transform leads (with $1^k = 1$) to:

$$h(k) = \frac{5}{9}\,\gamma_{-1}(k) + \frac{4}{9}(-0.8)^k \cdot \gamma_{-1}(k)$$



Impulse response of the recursive filter.

1.5 Canonical digital filter circuits

With the following relations

$G_z(z) \bullet\!\!\overset{z}{-\!\!-\!\!-}\!\!\circ\ g(k)$ output sequence (reaction)

$S_z(z) \bullet\!\!\overset{z}{-\!\!-\!\!-}\!\!\circ\ s(k)$ input sequence (excitation)

the system function can be presented as:

$$H_z(z) = \frac{G_z(z)}{S_z(z)} = \frac{\sum_{m=0}^{M} a_m z^{-m}}{1+\sum_{n=1}^{N} b_n z^{-n}}$$

Formally extending the system function, one gets:

$$H_z(z) = \frac{\sum_{m=0}^{M} a_m z^{-m}}{1+\sum_{n=1}^{N} b_n z^{-n}} = \frac{G_z(z)}{S_z(z)} = \frac{G_z(z)}{X_z(z)} \cdot \frac{X_z(z)}{S_z(z)}$$

1.5 Canonical digital filter circuits

One sets now $\frac{G_z(z)}{X_z(z)}$ to the numerator of $H_z(z)$:
$$\frac{G_z(z)}{X_z(z)} = \sum_{m=0}^{M} a_m z^{-m}$$

Solving for $G_z(z)$ gives:

$$G_z(z) = \left[\sum_{m=0}^{M} a_m z^{-m}\right] \cdot X_z(z) = \sum_{m=0}^{M} a_m z^{-m} X_z(z)$$

This leads to the description of a non-recursive subsystem:

$$g(k) = \sum_{m=0}^{M} a_m x(k-m)$$

1.5 Canonical digital filter circuits

Similiarly, one sets
$$\frac{X_z(z)}{S_z(z)} = \frac{1}{1+\sum_{n=1}^{N} b_n\, z^{-n}}$$

to the denominator which gives:
$$X_z(z)\cdot(1+\sum_{n=1}^{N} b_n\, z^{-n}) \;=\; S_z(z)$$

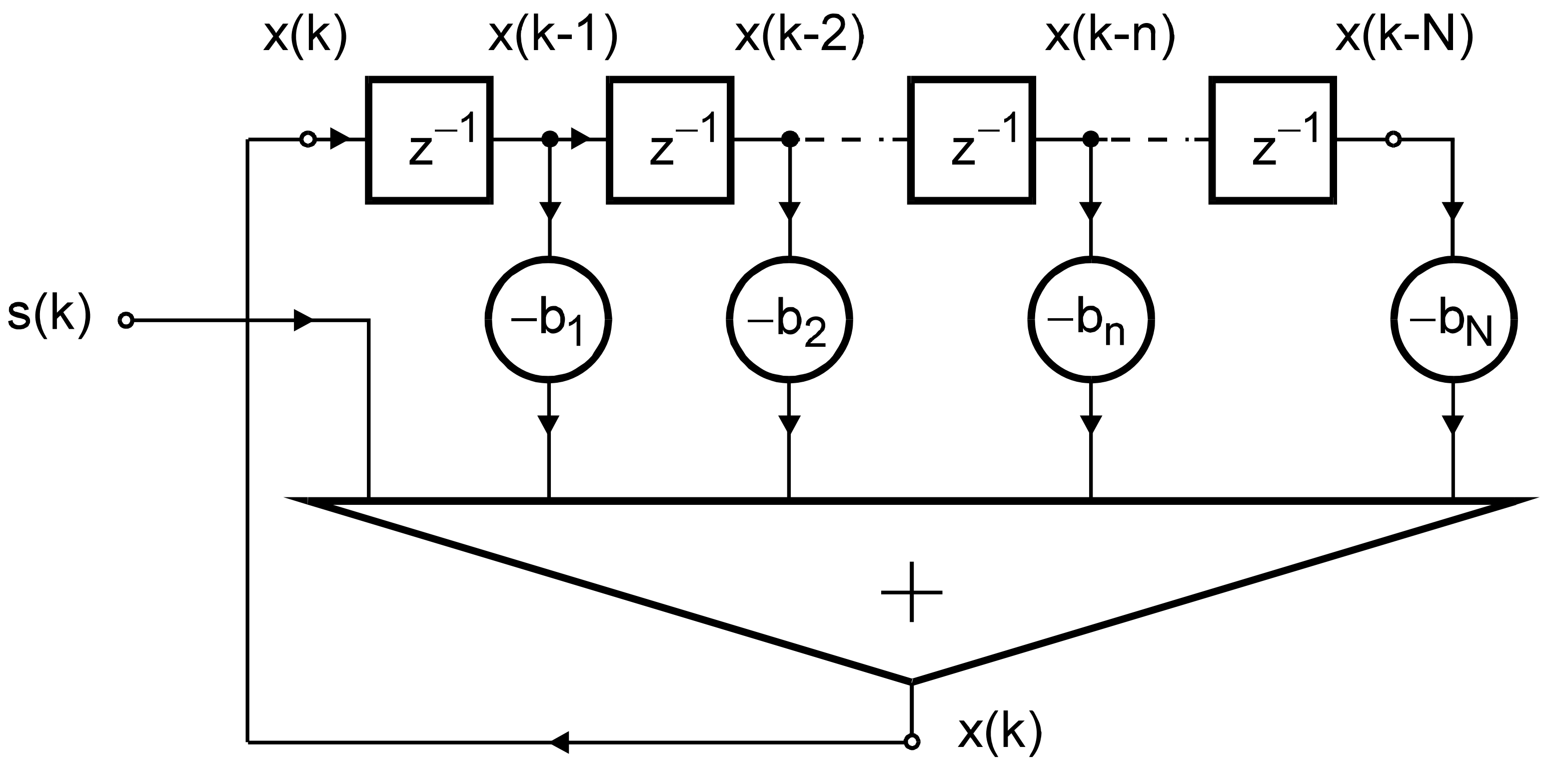
$$\Rightarrow X_z(z) \;=\; S_z(z) - \sum_{n=1}^{N} b_n\, z^{-n} X_z(z)$$

Finally it is obtained:
$$x(k) \;=\; s(k) - \sum_{n=1}^{N} b_n\, x(k-n)$$

Now *x(k)* describes the reaction of another recursive subsystem (as part of the whole digital filter).

The realisation follows the procedures described above.

1.5 Canonical digital filter circuits



Recursive subsystem

1.5 Canonical digital filter circuits

The realisation of g(k) follows the relation described above: $g(k) = \sum_{m=0}^{M} a_m \, x(k-m)$

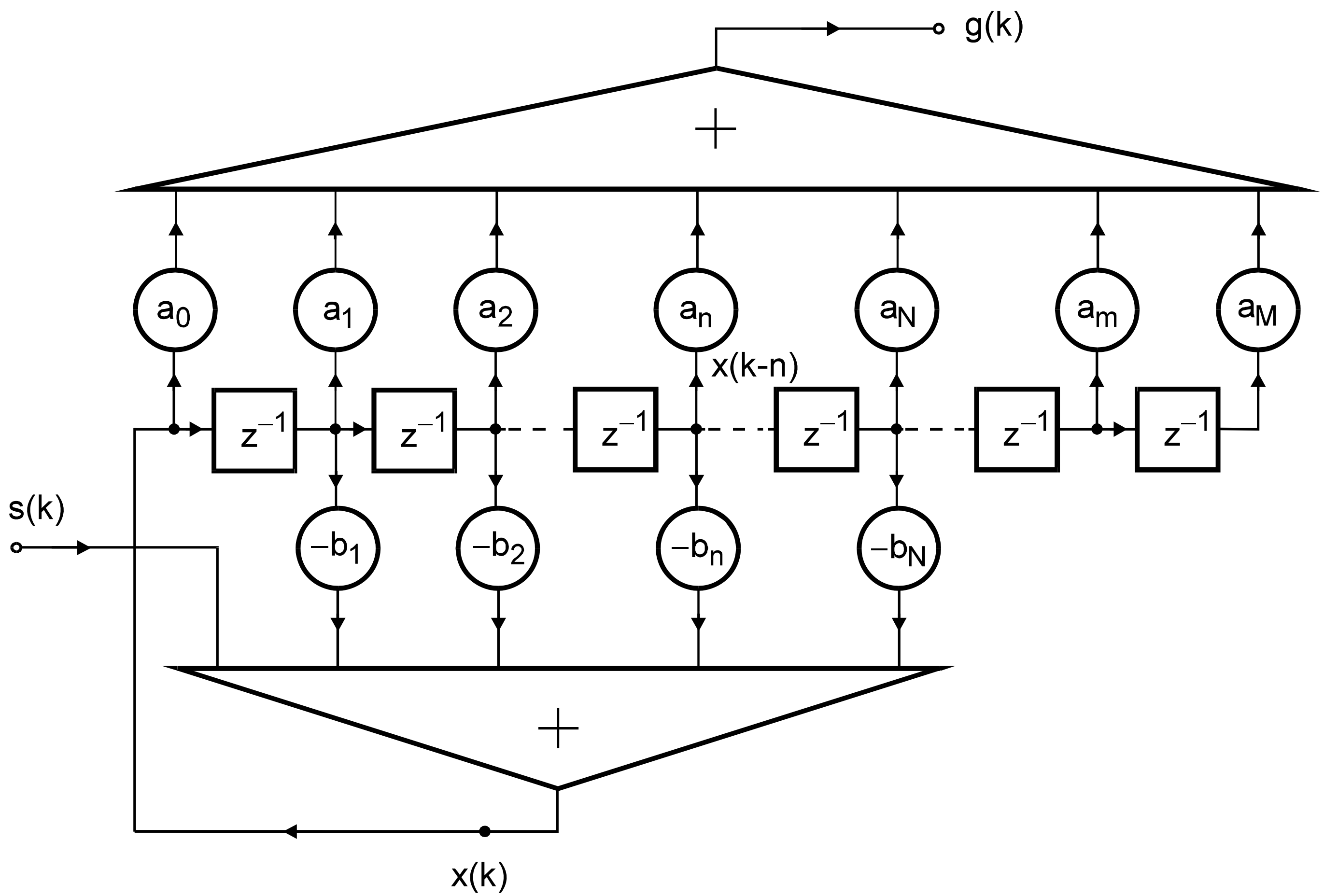
Doing so from the recursive subsystem the sequences $x(k-m)$
are used and multiplied by a_m.
Thus the following diagram on the next slide results.

Moreover: The amount of delays is reduced to a minimum.

Please note:

There are additional methods for finding other canonical structures.

1.5 Canonical digital filter circuits



Canonical realization of a general recursive digital filter for the case M > N

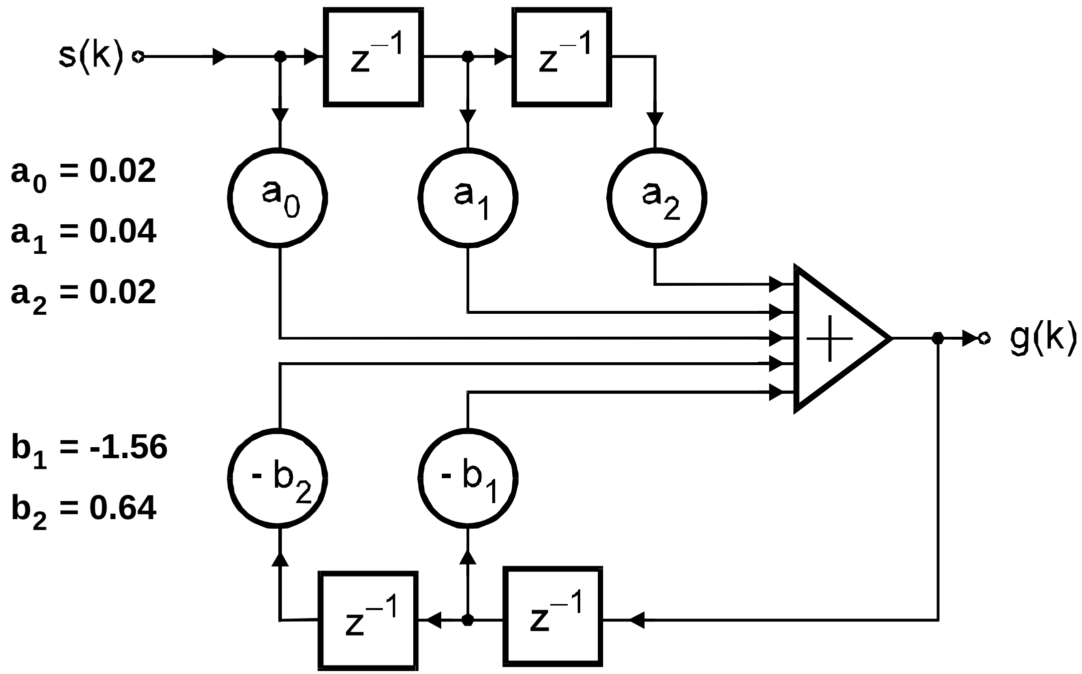
1.5 Canonical digital filter circuits

Example: Derivation of the "direct structure" and of the "canonical structure"

Given is the differential equation:

$$g(k) = 1.56 \cdot g(k-1) - 0.64 \cdot g(k-2) + 0.02 \cdot s(k) + 0.04 \cdot s(k-1) + 0.02 \cdot s(k-2)$$

Direct structure resulting from this:



1.5 Canonical digital filter circuits

"**Canonical filter structure**"

The first step is to z-transform the difference equation:

$$G_z(z) = 1.56\cdot z^{-1}G_z(z) - 0.64\cdot z^{-2}G_z(z) + 0.02\cdot S_z(z) + 0.04\cdot z^{-1}S_z(z) + 0.02\cdot z^{-2}S_z(z)$$

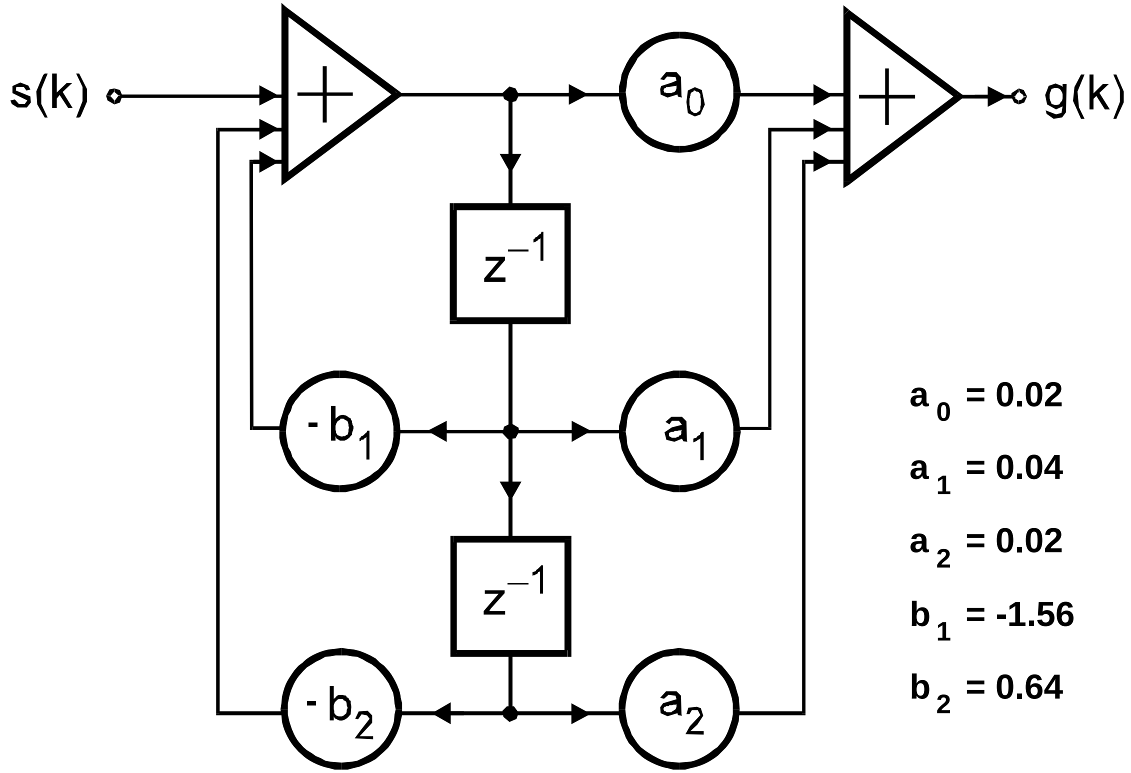
$$\Rightarrow \quad G_z(z)\cdot(1-1.56\cdot z^{-1} + 0.64\cdot z^{-2}) = S_z(z)\cdot(0.02 + 0.04\cdot z^{-1} + 0.02\cdot z^{-2})$$

Thus It applies:

$$H_z(z) = \frac{G_z(z)}{S_z(z)} = \frac{0.02 + 0.04\cdot z^{-1} + 0.02\cdot z^{-2}}{1-1.56\cdot z^{-1} + 0.64\cdot z^{-2}}$$

By a comparison of filter coefficients with the general canonical filter structure (see slide 92) the following diagram is obtained.

1.5 Canonical digital filter circuits



$a_0 = 0.02$

$a_1 = 0.04$

$a_2 = 0.02$

$b_1 = -1.56$

$b_2 = 0.64$

“Canonical structure" for the realization of the system function

1.5 Canonical digital filter circuits

- There is one disadvantage of this canonical structure: A significant sensitivity to parameter values exists.

- Small deviations of coefficients a_m and b_n lead to large deviations in the position of poles and zeros in the z-plane (and to corresponding changes in filter properties).

- Therefore more robust structures are often desired.

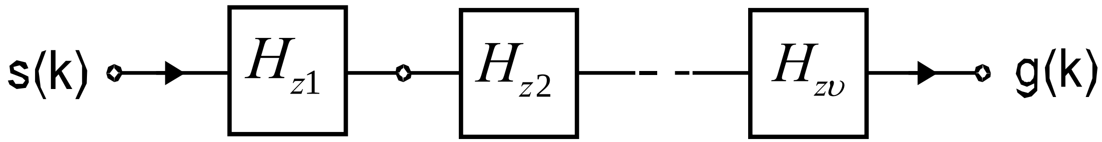
1.5 Canonical digital filter circuits

a) Robust filters using a cascade structure

Using a cascade one divides the system function into a product of appropriate product terms of first and second order subsystems.

The following equation describes this onset:

$$H_z(z) = \frac{\prod_{m=1}^{M_1}(z^2 + c_m z + d_m)}{\prod_{n=1}^{N_1}(z^2 + u_n z + v_n)} \bullet \frac{\prod_{m=1}^{M_2}(z + e_m)}{\prod_{n=1}^{N_2}(z + l_n)} \bullet \ldots = H_{z1} \cdot H_{z2} \cdots H_{z\upsilon}$$



Realization of a digital filter as a cascade structure

1.5 Canonical digital filter circuits

Example: Realization of a digital filter in a cascade structure

Given is the system function

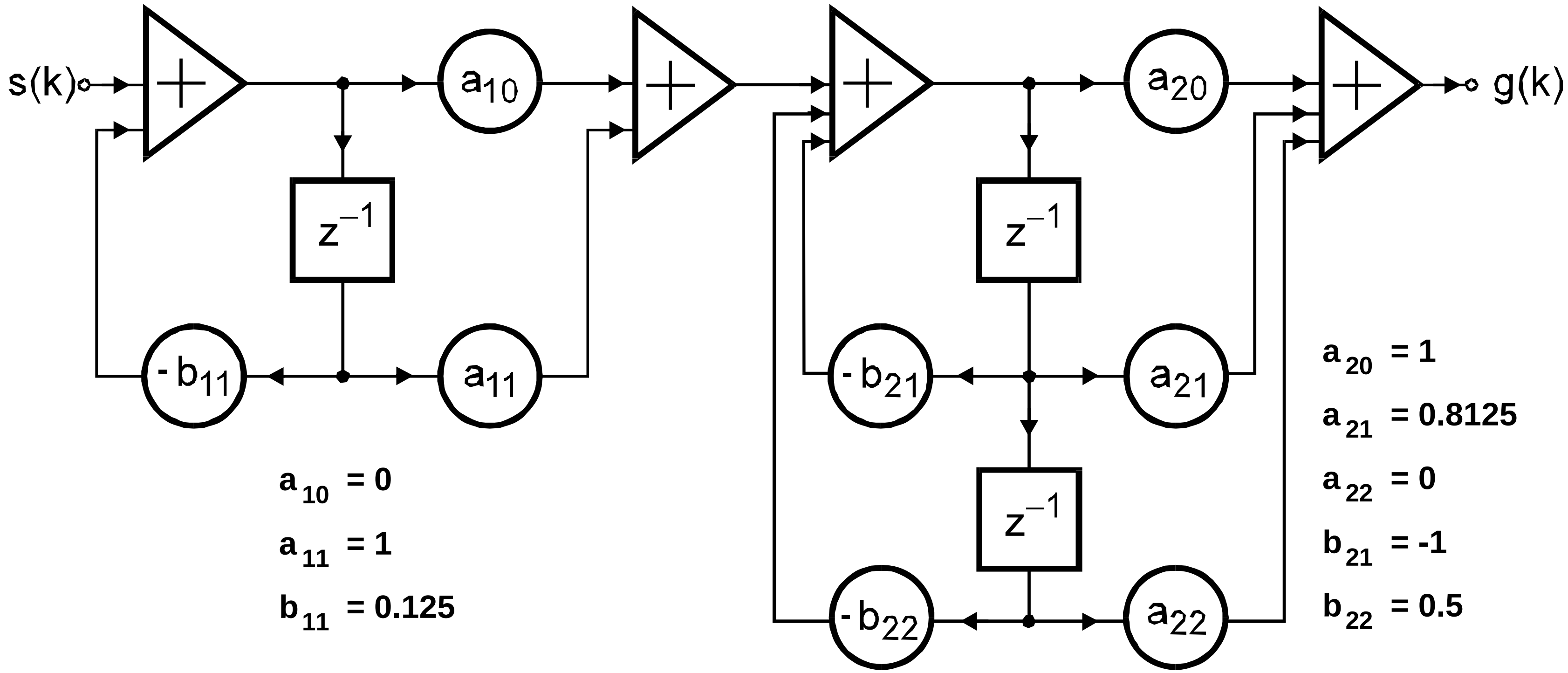
$$H_z(z) = \frac{z^{-1} + 0.8125 \cdot z^{-2}}{1 - 0.875 \cdot z^{-1} + 0.375 \cdot z^{-2} + 0.0625 \cdot z^{-3}}$$

A separation of $H_z(z)$ of the degree of 3 into a product of two subsystems of the order of 1 and 2 gives:

$$H_z(z) = \frac{z^{-1}}{1 + 0.125 \cdot z^{-1}} \cdot \frac{1 + 0.8125 \cdot z^{-1}}{1 - z^{-1} + 0.5 \cdot z^{-2}}$$

The following diagram shows the corresponding filter structure.

1.5 Canonical digital filter circuits



$a_{10} = 0$

$a_{11} = 1$

$b_{11} = 0.125$

$a_{20} = 1$

$a_{21} = 0.8125$

$a_{22} = 0$

$b_{21} = -1$

$b_{22} = 0.5$

Cascade structure of digital filters of 1st and 2nd Order (with canonical structures also for subsystems)

1.5 Canonical digital filter circuits

b) Robust filters with a parallel structure:

Another onset is to develop the system function $H_z(z)$ into a sum of partial fractions (assuming $N \geq M$). Each fraction then represents a subsystem of lower order (1 or 2).

For the case of simple poles this sum can be written as follows:

$$H_z(z) = K_0 + \sum_{\nu=1}^{N_1} \frac{K_\nu}{z - z_{\infty\nu}}$$

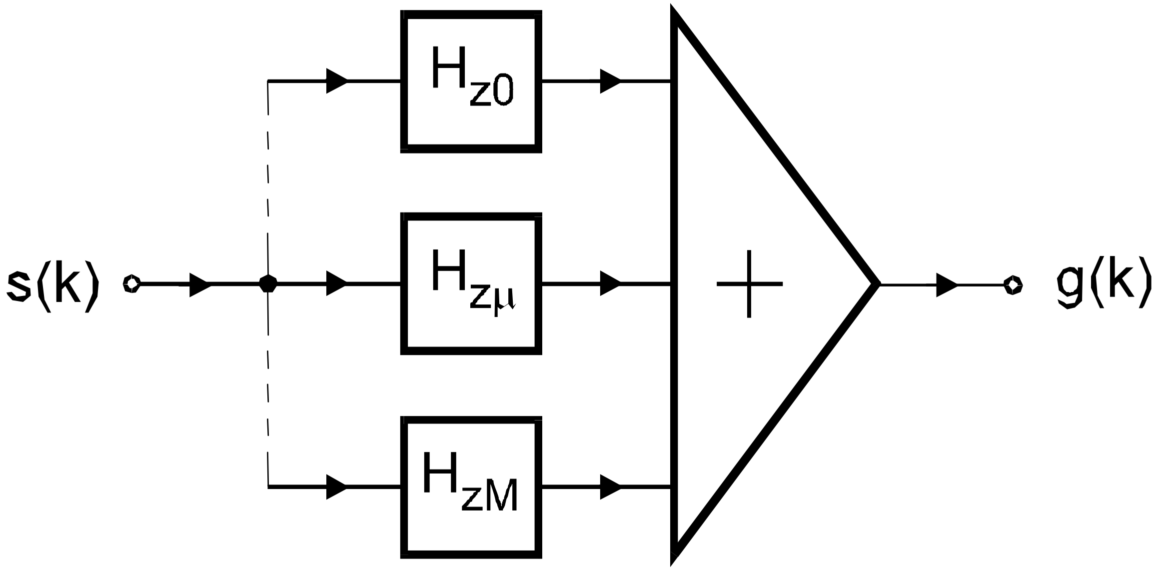
Each of these addends corresponds to a pole. In case of two conjugated complex poles such a pair has to be converted into a real fraction of second order with:

$\dfrac{K_\nu}{z - z_{\infty\nu}}$ and $\dfrac{K_{\nu+1}}{z - z_{\infty\nu+1}}$ with $K_{\nu+1} = K_\nu^*$ and $z_{\infty\nu+1} = z_{\infty\nu}^*$ leading to

$$H_z(z) = K_0 + \sum_{\nu=1}^{N_1} \frac{K_\nu}{z - z_{\infty\nu}} = H_{z0}(z) + H_{z1}(z) + H_{z2}(z) + \cdots + H_{z\mu}(z) + \ldots + H_{zM}(z)$$

1.5 Canonical digital filter circuits

The following diagram shows the general filter structure:



Realization of a digital filter using parallel connections

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Fachgebiet
Nachrichtentechnische Systeme

NTS

1.5 Canonical digital filter circuits

Example: Realization of a digital filter using parallel connections

Given is one known root of the denominator (*z = -1/8*) and the system function:

$$H_z(z) = \frac{z^{-1} + 0.8125 \cdot z^{-2}}{1 - 0.875 \cdot z^{-1} + 0.375 \cdot z^{-2} + 0.0625 \cdot z^{-3}} = \frac{z^2 + 0.8125 \cdot z}{z^3 - 0.875 \cdot z^2 + 0.375 \cdot z + 0.0625}$$

A decomposition into partial fractions starts with:

$$\begin{array}{l} z^3 - 0.875 \cdot z^2 + 0.375 \cdot z + 0.0625 \quad : \quad z + 1/8 \quad = z^2 - z + 0.5 \\ -\;\underline{z^3 + 0.125 \cdot z^2} \\ \qquad\qquad -z^2 + 0.375 \cdot z + 0.0625 \\ \qquad\quad -\;\underline{-z^2 - 0.125 \cdot z} \\ \qquad\qquad\qquad 0.5 \cdot z + 0.0625 \\ \qquad\qquad -\;\underline{0.5 \cdot z + 0.0625} \\ \qquad\qquad\qquad\qquad 0 \end{array}$$

1.5 Canonical digital filter circuits

Thus it holds: $H_z(z) = \dfrac{z^2 + 0.8125z}{\left(z + 1/8\right)\cdot\left(z^2 - z + 1/2\right)}$

$$H_z(z) \overset{!}{=} \frac{A}{\left(z + 1/8\right)} + \frac{B \cdot z + D}{\left(z^2 - z + 1/2\right)}$$

with $A = \left.\dfrac{z^2 + 0.8125z}{z^2 - z + 0.5}\right|_{z=-1/8} = \dfrac{0.016 - 0.102}{0.016 + 0.125 + 0.5} = -0.134$ and

$$\left.H_z(z)\right|_{z=0} = \frac{-0.134}{0.125} + \frac{D}{1/2} = 0 \Rightarrow -1.073 + 2D = 0 \quad \Rightarrow D = 0.537$$

$$\left.H_z(z)\right|_{z=1} = \frac{-0.134}{1.125} + \frac{B + 0.537}{1/2} = \frac{1.8125}{1 - 0.875 + 0.375 + 0.0625}$$

$$\Rightarrow -0.1191 + 2(B + 0.537) = 3.222 \quad \Rightarrow \mathrm{B} = 1.1337$$

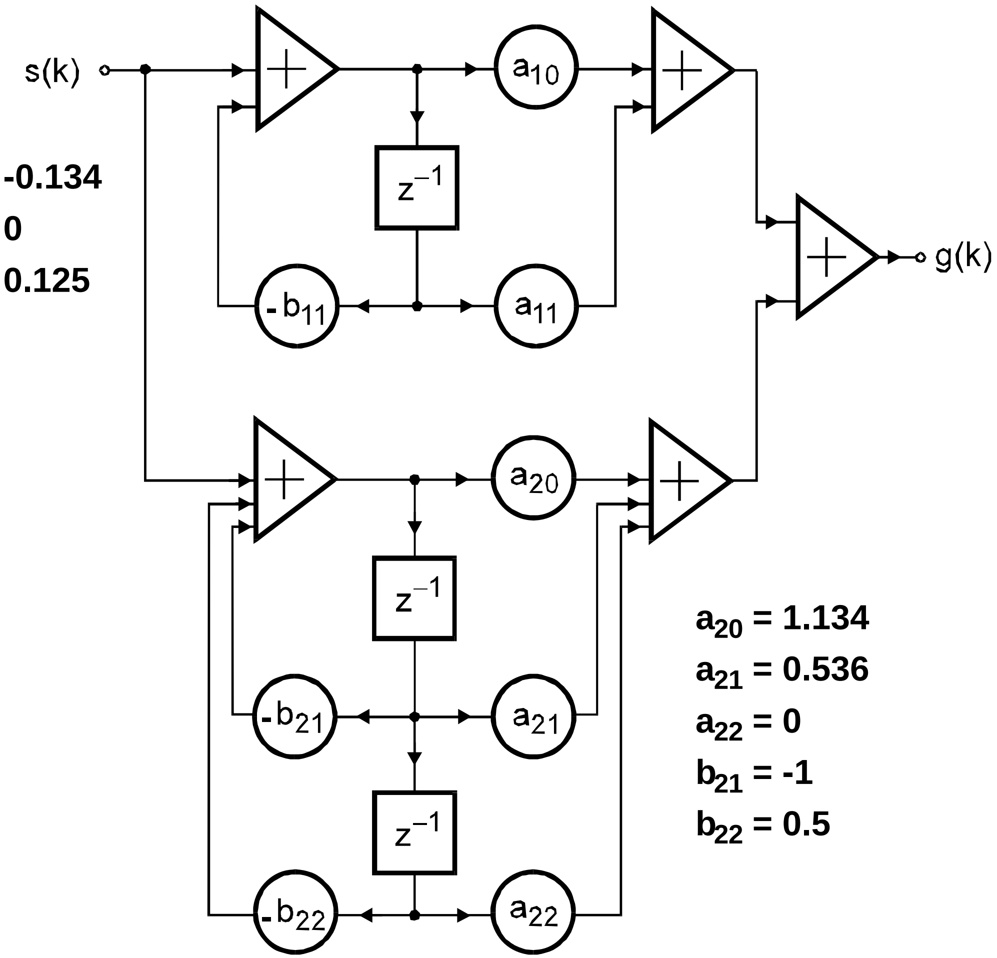
1.5 Canonical digital filter circuits

So finally the partial fractions result in:

$$H_z(z) = \frac{-0.134}{1+0.125\cdot z^{-1}} + \frac{1.134+0.536\cdot z^{-1}}{1-z^{-1}+0.5\cdot z^{-2}}$$

The coefficients of the polynoms of nominator and denominator polynoms then turn into the values of the filter coefficients.

1.5 Canonical digital filter circuits



$a_{10} = -0.134$
$a_{11} = 0$
$b_{11} = 0.125$

$a_{20} = 1.134$
$a_{21} = 0.536$
$a_{22} = 0$
$b_{21} = -1$
$b_{22} = 0.5$

Digital filters for parallel connections of systems of 1st and 2nd order