NETWORK THEORY 2

Digital Filters

Chapter 2

Design and Realization of non-recursive Filters

UNIVERSITÄT DUISBURG ESSEN

Fachgebiet
Nachrichtentechnische Systeme

2.1 Prefaces to FIR Digital filters

As shown in chapter 1.3, the system function of a not-recursive digital filter looks as follows:

$$H_z(z) = \sum_{m=0}^{M} a_m \, z^{-m}$$

An inverse z-transform gives its impulse response:

$$h(k) = \sum_{m=0}^{M} a_m \, \gamma_0(k-m)$$

Another correspondence is:

$$G_z(z) = H_z(z) \cdot S_z(z)$$

$$\Big\updownarrow \; z$$

$$g(k) = h(k) \; * \; s(k)$$

2.1 Prefaces to FIR Digital filters

Thus it is obtained:

$$G_z(z) = \left(\sum_{m=0}^{M} a_m z^{-m} \right) \cdot S_z(z) = \sum_{m=0}^{M} \left\{ a_m z^{-m} \, S_z(z) \right\}$$

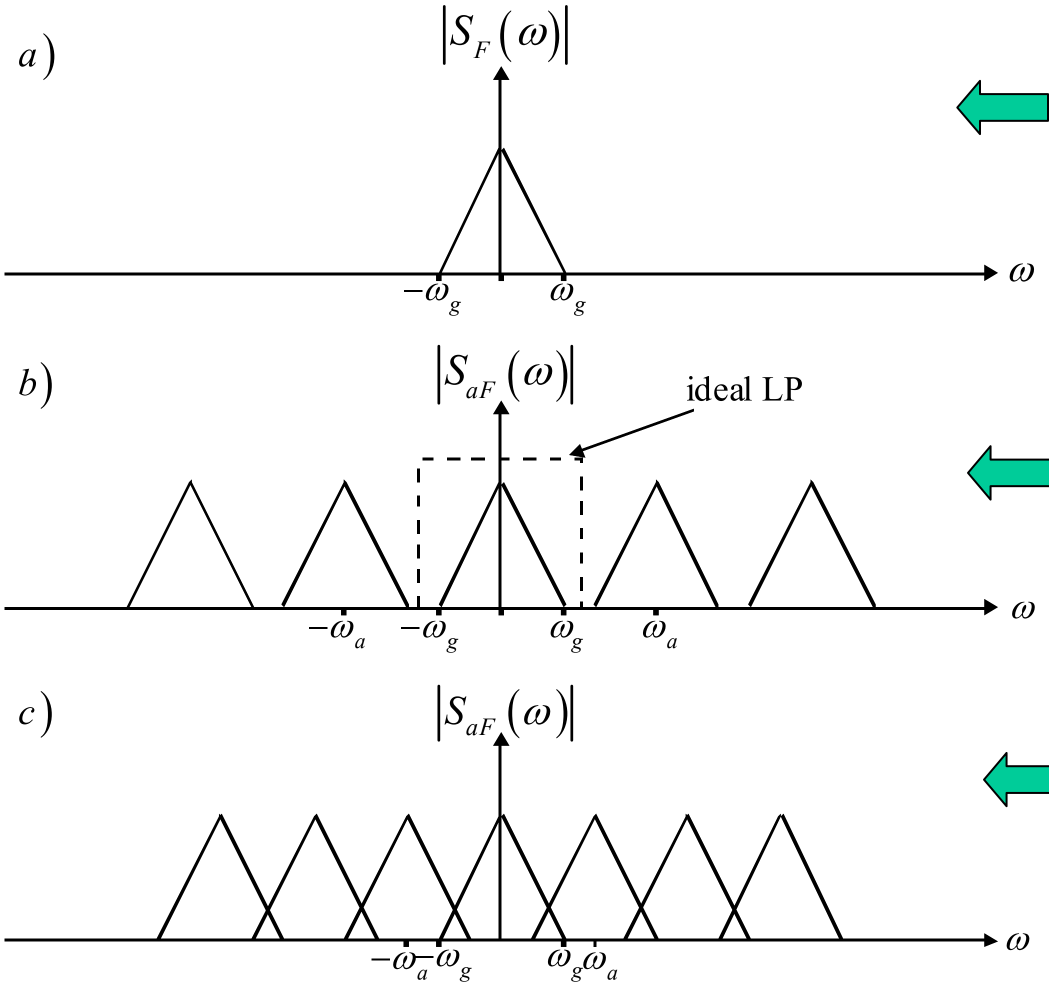
$$\bullet\!\!-\!\!\circ \; z$$

$$g(k) = \left\{ \sum_{m=0}^{M} a_m \gamma_0(k-m) \right\} * s(k) = \sum_{m=0}^{M} a_m s(k-m)$$

From these equations, the following ***advantages*** of FIR digital filters result:

1. FIR digital filters are always stable (no feedback-loop)
2. No group delay distortions

2.2 Design with regulations in the frequency range



Spectrum of band-limited signal *s(t)*

Spectrum of oversampled signal

Spectrum of undersampled signal

2.2 Design with regulations in the frequency range

The design of any FIR filter leads to determining the filter coefficients such that *h(k)* follows and thus the realised system function follows certain demands:

$$h(k) = \sum_{m=0}^{M} h(m) \cdot \gamma_0(k-m) \quad \text{where} \quad h(m) = a_m$$

$$H_{rz}(z) = \sum_{m=0}^{M} h_m \, z^{-m} = \sum_{m=0}^{M} a_m \, z^{-m}$$

So any realizable FIR transfer function must look as follows:

$$H_{rF}(\omega) = H_{rz}(e^{j\omega T_a}) = \sum_{m=0}^{M} a_m \, e^{-j\omega T_a}$$

2.2 Design with regulations in the frequency range

Thus for any type of FIR filter the amplitude characteristics are given by:

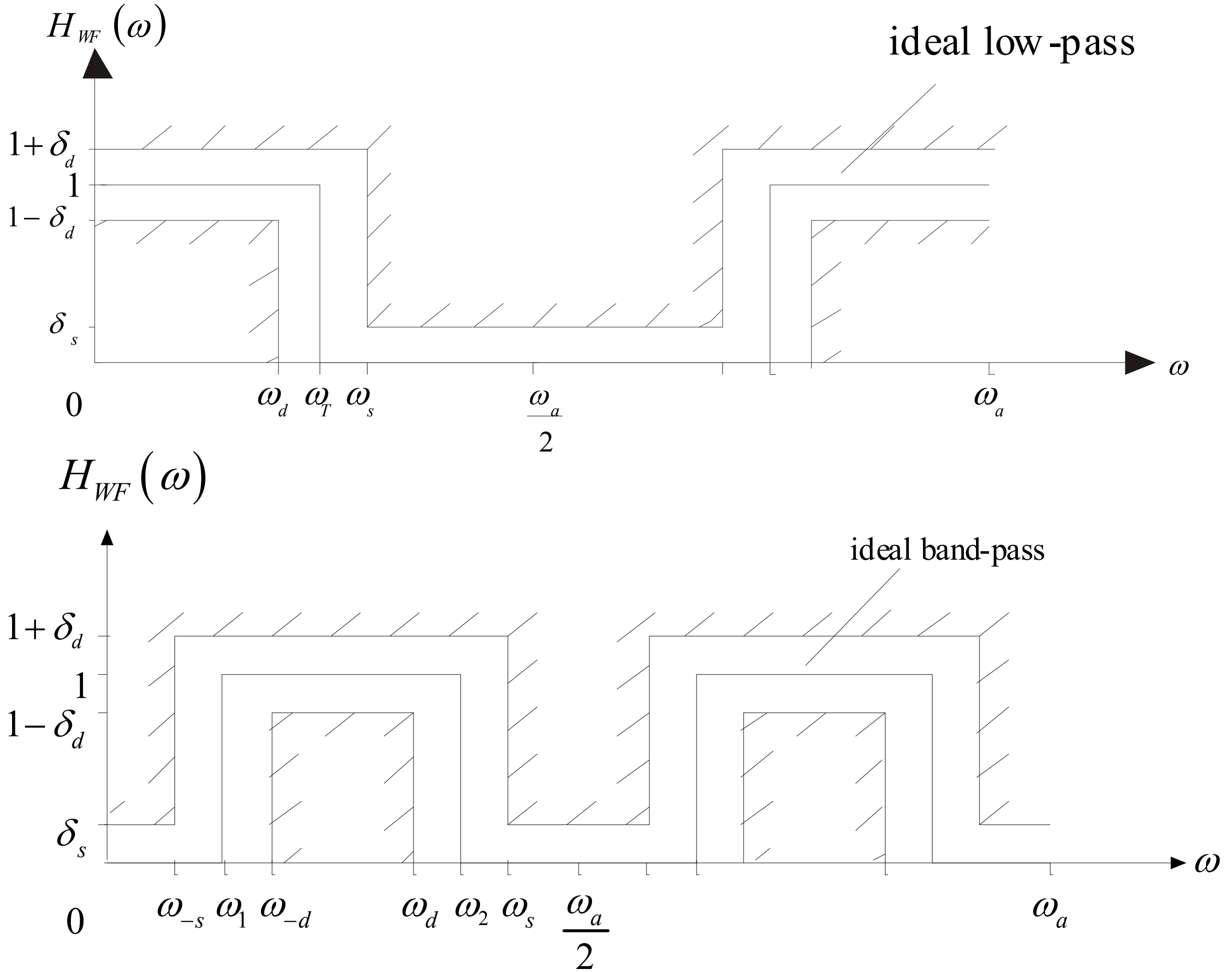
$$\left|H_{rF}(\omega)\right| = \left|H_{rz}(e^{j\omega T_a})\right| = \left|\sum_{m=0}^{M} a_m \, e^{-j\omega T_a}\right|$$

Such a realizable frequency response in principle always is periodic.

Only the two low-passes (in front of the sampling device and for reconstruction of the output signal) lead to a non-periodic behaviour.

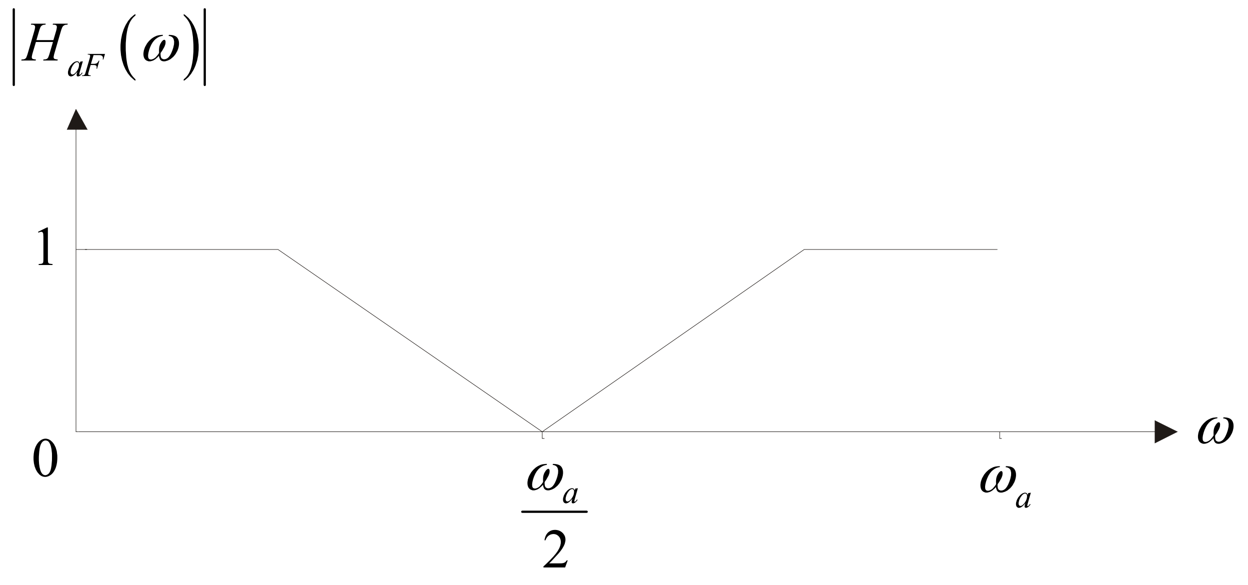
The next slides shows examples of possible (partly ideal) desired frequency responses.

2.2 Design with regulations in the frequency range



Amplitude frequency responses of ideal low-pass and ideal band-pass

2.2 Design with regulations in the frequency range



Example: Linear decaying/rising amplitude frequency response of a band-stop (digital low-pass)

2.2 Design with regulations in the frequency range

The realization of nearly ideal digital low-passes, band-pass filters, high-passes or band-stops is based on the **desired (wanted) transfer function** $H_{wF}(\omega)$ of a digital filter

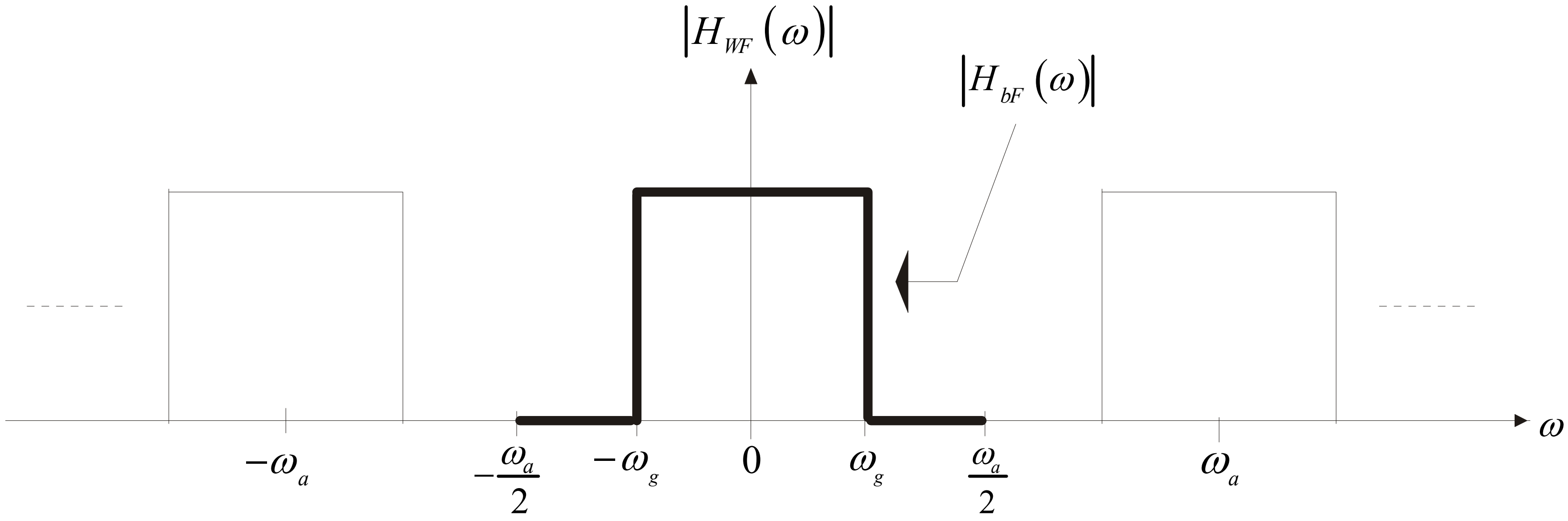
$$H_{wF}(\omega) = \sum_{n=-\infty}^{+\infty} H_{bF}(\omega - n\omega_a) \quad \text{being periodic in} \quad \omega_a = 2\pi / T_a$$

with the **baseband spectral function** according to

$$H_{bF}(\omega) = H_{wF}(\omega) \cdot rect\left(\frac{\omega}{\omega_a}\right)$$

The next slide shows in an example the relations of these functions.

2.2 Design with regulations in the frequency range



Baseband *$H_{BF}(\omega)$* (thick line) and the desired periodic function *$H_{wF}(\omega)$*

2.2 Design with regulations in the frequency range

If one applies the inverse Fourier transform to

$$H_{wF}(\omega) = H_{bF}(\omega) * \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_a)$$

The wanted impulse response is obtained:

$$h_w(t) = 2\pi \cdot h_b(t) \cdot \frac{1}{\omega_a} \sum_{n=-\infty}^{+\infty} \delta(t - nT_a) \quad \text{with} \quad \frac{2\pi}{\omega_a} = T_a$$

and thus:

$$h_w(t) = \sum_{n=-\infty}^{+\infty} T_a \cdot h_b(nT_a) \cdot \delta(t - nT_a)$$

2.2 Design with regulations in the frequency range

This impulse response must be made finite but in addition is not causal in general.

The countermeasure is to shift $h_f(t)$ by the period $N_f T_a$ so that in any case a causal signal is obtained.

Note: This has no effect on the magnitude frequency response (as it is just a delay).

So the realizable impulse response follows the equation:

$$h_r(t) = h_f(t - N_f T_a) = \sum_{n=-N_f}^{+N_f} T_a \cdot h_b(nT_a) \cdot h_R(nT_a) \cdot \delta\left(t - \left\{n + N_f\right\} T_a\right)$$

After the substitution $k = n + N_f \Rightarrow n = k - N_f$

the following equation results:

$$h_r(t) = \sum_{k=0}^{2N_f} T_a \cdot h_b\left(\left\{k - N_f\right\} T_a\right) \cdot h_R\left(\left\{k - N_f\right\} T_a\right) \cdot \delta(t - kT_a)$$

2.2 Design with regulations in the frequency range

Now the first expression in the summation is abbreviated using a so-called **baseband sequence**:

$$h_{bk}(k) = T_a \cdot h_b\left(\left\{k - N_f\right\} T_a\right)$$

Furthermore a causal and in k finite **rectangular window sequence** with **the window length** $2N_f$ is defined (it is the second expression in the summation):

$$h_{kR}(k) = \begin{cases} 1 & \text{for } k = 0(1)2N_f \\ 0 & \text{for } k < 0 \text{ and } k > 2N_f \end{cases}$$

2.2 Design with regulations in the frequency range

For rectangular windowing concerning the approximate **impulse response** $h_{rk}(k)$ of the digital filter holds:

$$h_{kr}(k) = h_{bk}(k) \cdot h_{kR}(k) \quad = \begin{cases} h_{bk}(k) & for \quad k = 0(1)2N_f \\ 0 & else \end{cases}$$

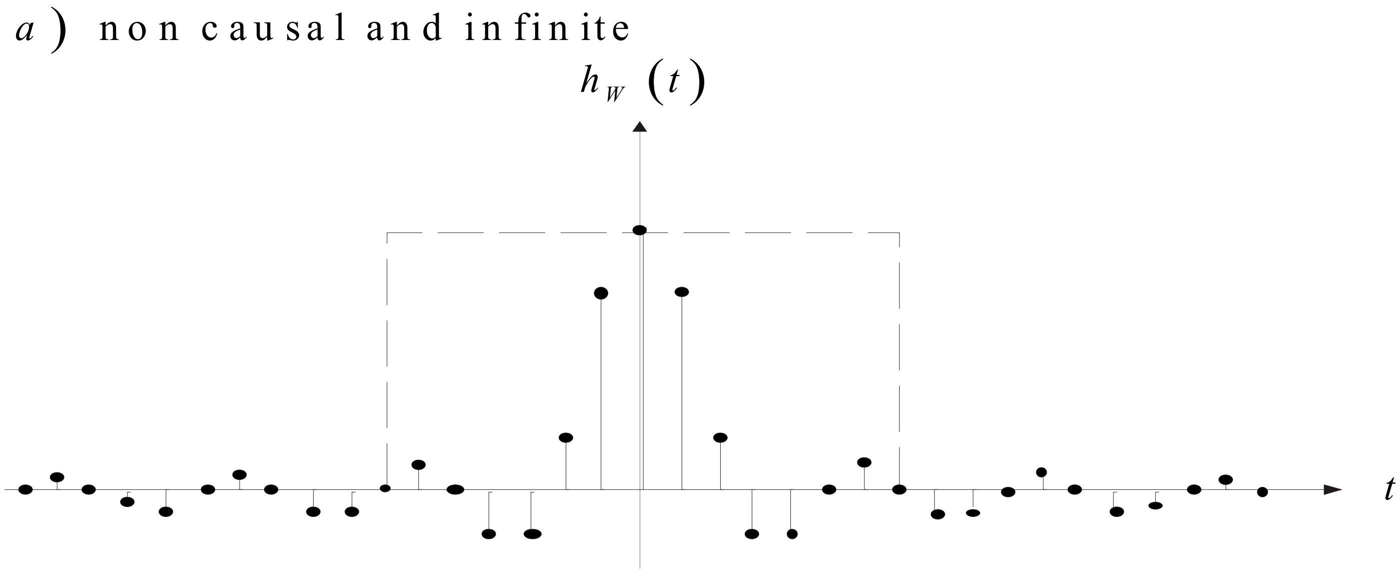


Illustration of sampled functions for the realization of an FIR low-pass filter

2.2 Design with regulations in the frequency range

$b)$ non causal but finite

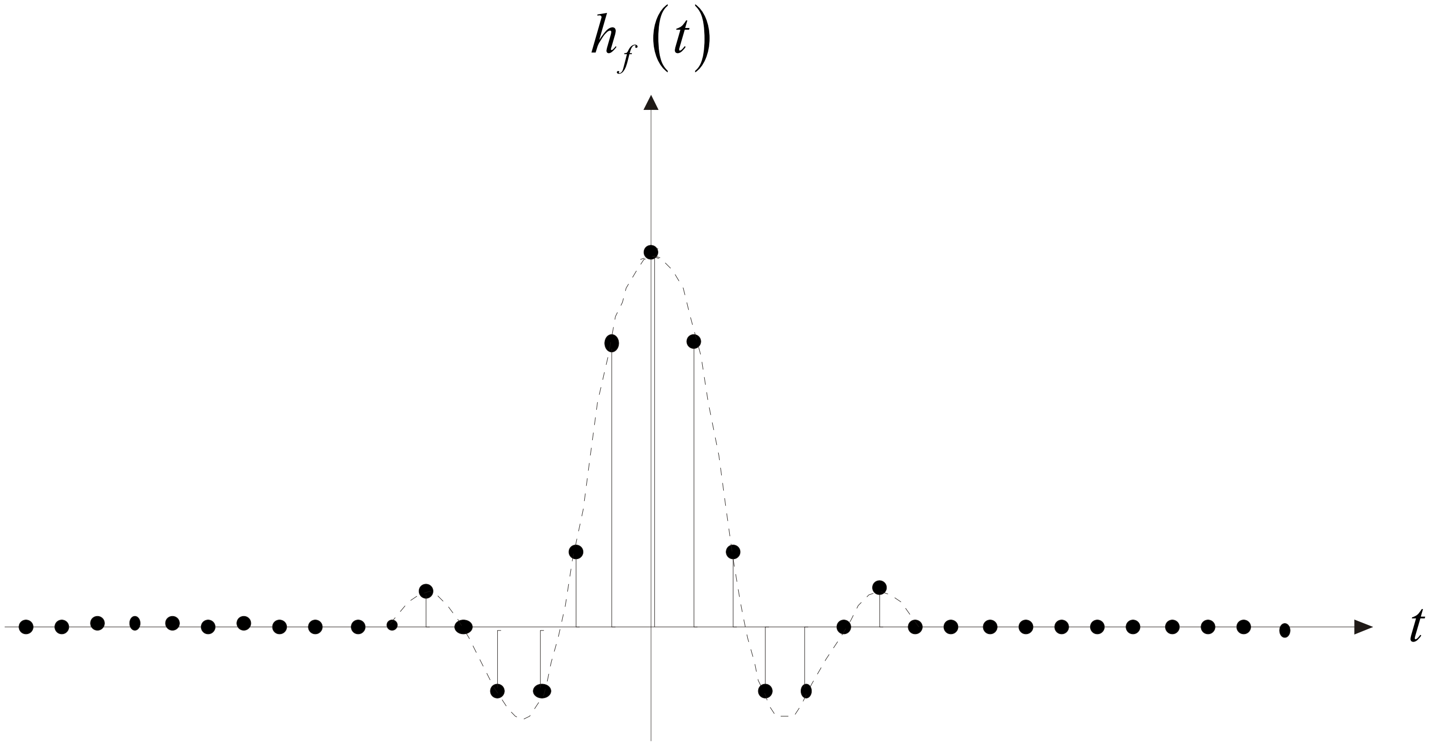


Illustration of sampled functions for the realization of an FIR low-pass filter

2.2 Design with regulations in the frequency range

$c)$ causal and finite

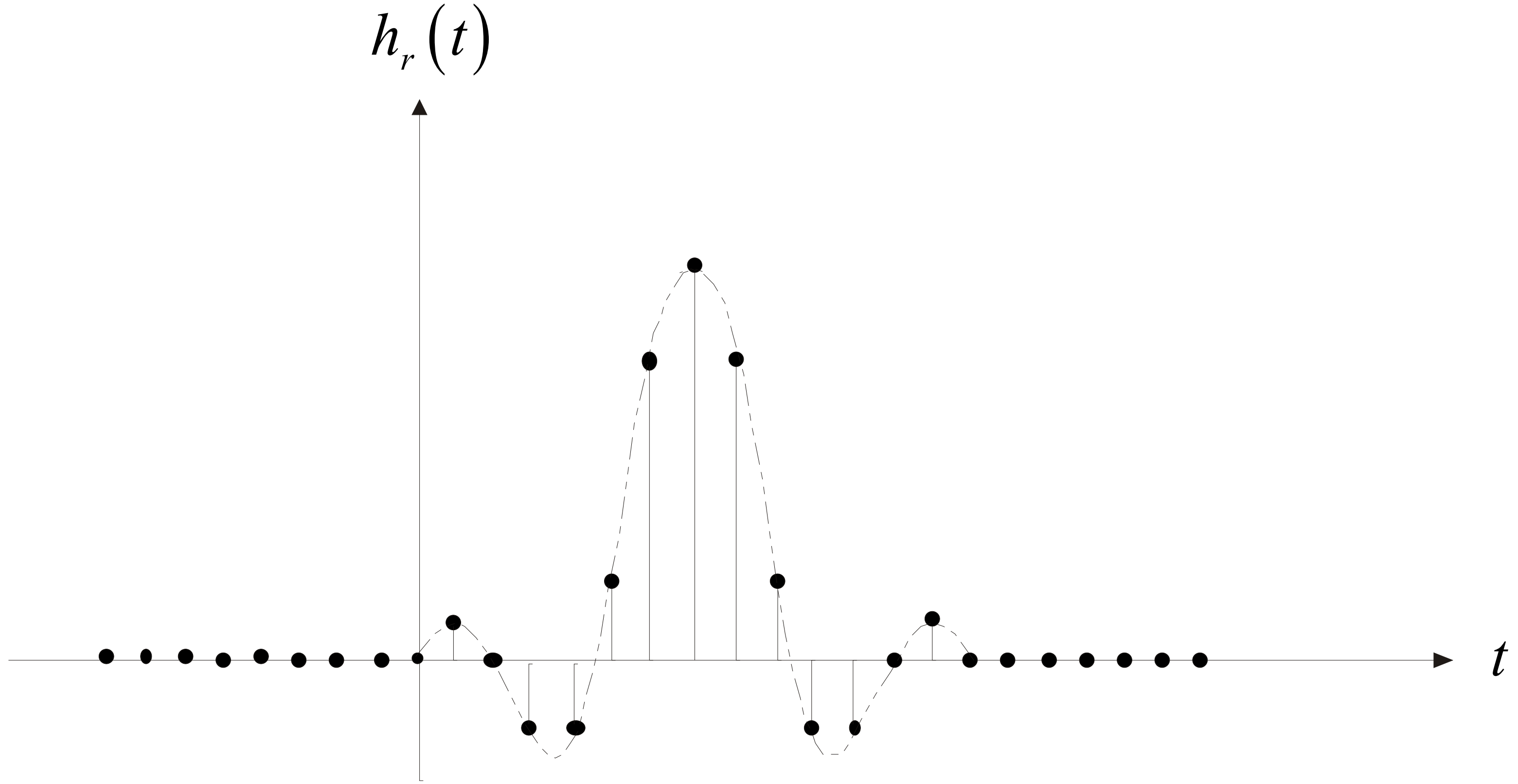
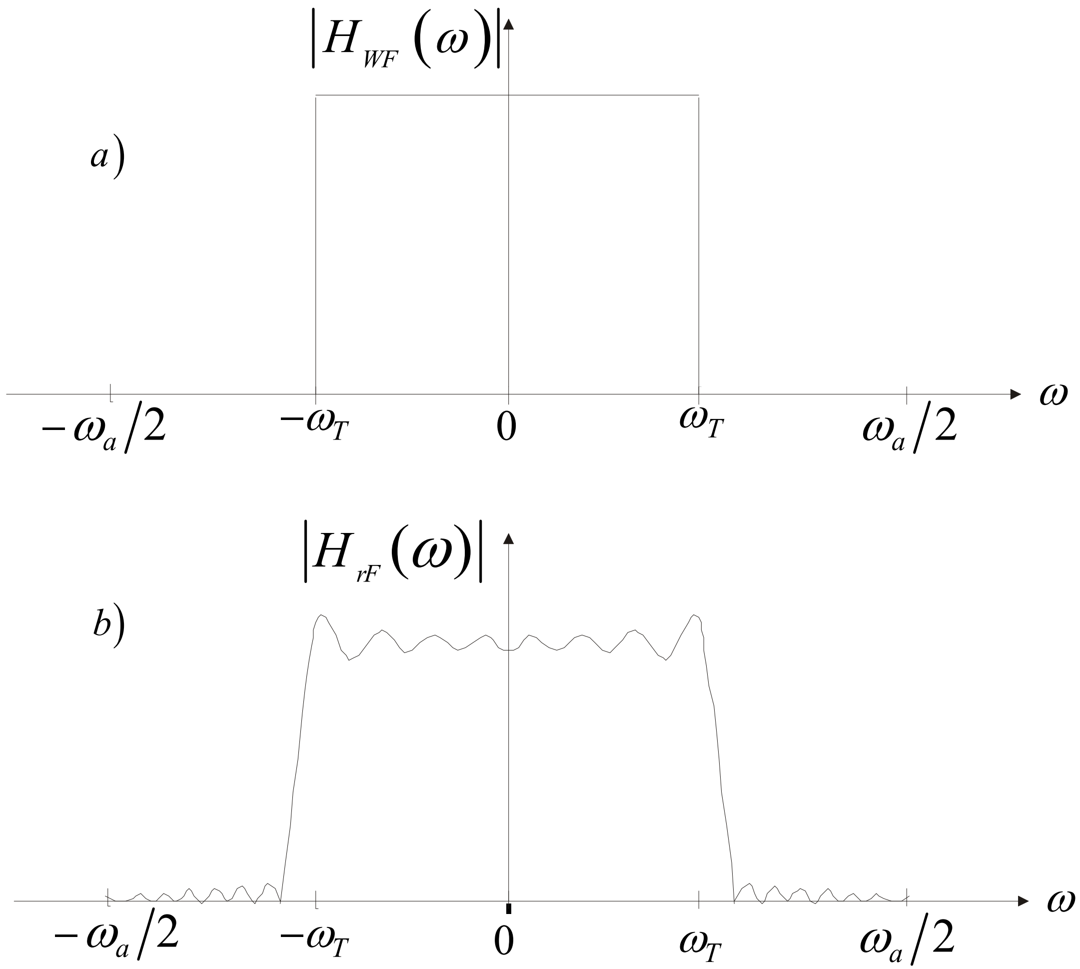


Illustration of sampled functions for the realization of an FIR low-pass filter

2.2 Design with regulations in the frequency range



Spectrum of the ideal and of the approximated digital low-pass

2.2 Design with regulations in the frequency range

So the computation of the values of the baseband sequence

$$h_{bk}(k) = T_a \cdot h_b\left(\left\{k - N_f\right\} T_a\right) \quad \text{for } k = -\infty \ (1) + \infty$$

takes place in several steps:

First from a given baseband spectral function $H_{bF}(\omega)$, the <u>impulse response $h_b(t)$ is computed</u>:

$$h_b(t) = \frac{1}{2\pi} \int_{-\omega_a/2}^{+\omega_a/2} H_{bF}(\omega) \cdot e^{j\omega t} d\omega$$

Sampling of this function at $t = \left\{k - N_f\right\} T_a$ gives:

2.2 Design with regulations in the frequency range

Finally the baseband sequences $h_{bk}(k)$ can be computed as:

$$h_{bk}(k) = T_a \cdot h_b\left(\left\{k - N_f\right\} T_a\right)$$

From the time-discrete equation of the impulse response

$$h_r(t) = \sum_{k=0}^{2N_f = M} h_{bk}(k) \cdot \delta(t - kT_a)$$

the system function of the assigned analog filter follows:

$$H_{rL}(p) = \sum_{k=0}^{M} h_{kb}(k) \cdot e^{-pkT_a}$$

Thus the formula for the computation of the system function of the realizable FIR digital filter is:

$$H_{rL}(p) = \sum_{k=0}^{M} h_{kb}(k) \cdot z^{-k} = H_{rz}(z)\Big|_{z=e^{pT_a}}$$

2.2 Design with regulations in the frequency range

- Here the important relations for a rectangular window are summarized:

$$h_w(t) = \sum_{n=-\infty}^{+\infty} T_a \cdot h_b(nT_a) \cdot \delta(t - nT_a) \qquad H_{wF}(\omega) = H_{bF}(\omega) * \sum_{n=-\infty}^{+\infty} \delta(\omega - n\omega_a)$$

$$h_f(t) = h_w(t)\, rect(\frac{t}{2N_f T_a}) \qquad H_{fF}(\omega) = \frac{N_f T_a}{\pi} \cdot H_{wF}(\omega) * si(\omega N_f T_a)$$

$$h_r(t) = h_f(t - N_f T_a) \qquad H_{rF}(\omega) = H_{fF}(\omega) \cdot e^{-j\omega N_f T_a}$$

$$h_{bk}(k) = T_a \cdot h_b\left(\left\{k - N_f\right\} T_a\right) \qquad H_{rL}(p) = \sum_{k=0}^{M} h_{kb}(k) \cdot z^{-k} = H_{rz}(z)\Big|_{z=e^{pT_a}}$$

- In the following improved windows are presented with superior properties compared to the rectangular window.

2.2 Design with regulations in the frequency range

Instead of simple rectangular windows other window functions are known which very much reduce the overshoots due to the Gibb's effect!

The **HANN-window function** is one of these preferred functions:

$$h_{HN}(t) = \begin{cases} 0.5 + 0.5 \cdot \cos\left(\dfrac{\pi}{N_f T_a} t\right) & \mathit{for} \quad -N_f T_a \le t \le +N_f T_a \\ 0 & \mathit{else} \end{cases}$$

2.2 Design with regulations in the frequency range

By multiplying the signal $h_w(t)$ with the Hann window and by shifting one receives:

$$h_r(t) = \sum_{n=-N_f}^{+N_f} h_{bk}(k) \cdot \left\{ 0.5 + 0.5 \cdot \cos\left(\frac{\pi (k - N_f)}{N_f} \right) \right\} \cdot \delta\left(t - kT_a\right)$$

2.2 Design with regulations in the frequency range

Using the **HANN-discrete sequences:**

$$h_{kHN}(k) = \begin{cases} 0.5 + 0.5 \cdot \cos\left(\dfrac{\pi\{k - N_f\}}{N_f}\right) & with \;\; k = 0(1)2N_f \\ 0 & \text{else} \end{cases}$$

the realizable impulse response for the analog filter can be described as:

$$h_r(t) = \sum_{k=0}^{M} h_{bk}(k) \cdot h_{HNk}(k) \cdot \delta\left(t - kT_a\right)$$

$$h_{HMk}(k) = 0.54 + 0.46 \cdot \cos\left(\frac{\pi\{k - N_f\}}{N_f}\right) \quad with \;\; k = 0(1)2N_f$$

2.2 Design with regulations in the frequency range

Another related windowing method is based on **the HAMMING function**:

$$h_{HM}(t) = \begin{cases} 0.54 + 0.46 \cdot \cos(\frac{\pi}{N_f T_a} t) & for \ \ -N_f T_a \le t \le +N_f T_a \\ 0 & else \end{cases}$$

Here the impulse response of the assigned FIR Digitalfilters is as follows:

Using $h_{HMk}(k) = h_{HM}\left(\left\{k - N_f\right\} T_a\right)$

the assigned causal HAMMING discrete sequence is obtained :

$$h_{HM}(k) = \begin{cases} 0.54 + 0.46 \cos\left(\frac{\pi \left\{k - N_f\right\}}{N_f}\right) & with\, k = 0(1)N_f \\ 0 & else \end{cases}$$

2.2 Design with regulations in the frequency range

The BLACKMAN window function

$$h_B(t) = \begin{cases} 0.42 + 0.5 \cdot \cos\left(\dfrac{\pi}{N_f T_a} t\right) + 0.08 \cdot \cos\left(\dfrac{2\pi}{N_f T_a} t\right) & \text{for } -N_f T_a \le t \le +N_f T_a \\ 0 & \text{else} \end{cases}$$

With the approriate $h_{Bk}(k) = h_B\left(\left\{k - N_f\right\} T_a\right)$, the BLACKMAN discrete sequence follows:

$$h_B(k) = \begin{cases} 0.42 + 0.5 \cos\left(\dfrac{\pi\left\{k - N_f\right\}}{N_f}\right) + 0.08 \cdot \cos\left(\dfrac{2\pi\left\{k - N_f\right\}}{N_f}\right) & \text{with } k = 0(1)2N_f \\ 0 & \text{else} \end{cases}$$

2.2 Design with regulations in the frequency range

Compared to the previous windows **the Kaiser window function** offers an additional parameter α for controlling the approximation:

$$h_K(t) = \begin{cases} \dfrac{I_0\left(\alpha \cdot \sqrt{1-(\frac{t}{N_f T_a})^2}\right)}{I_0(\alpha)} & for \;\; -N_f T_a \le t \le +N_f T_a \\ 0 & else \end{cases}$$

Here $I_0(x)$ represents the modified Bessel function which is defined as:

$$I_0(x) = 1 + \sum_{m=1}^{\infty} \left(\frac{(x/2)^m}{m!} \right)^2$$

In practice one stops the summation when the elements of the sum show a value < 10^{-8}.

2.2 Design with regulations in the frequency range

The substitution $h_{Kk}(k) = h_K\left(\left\{k - N_f\right\} T_a\right)$ leads to the KAISER discrete sequence:

$$h_{kK}(k) = \begin{cases} \dfrac{I_0\left(\alpha \cdot \sqrt{1 - \left(\dfrac{k - N_f}{N_f}\right)^2}\right)}{I_0(\alpha)} & \text{for } k = 0(1)2N_f \\ 0 & \text{else} \end{cases}$$

2.2 Design with regulations in the frequency range

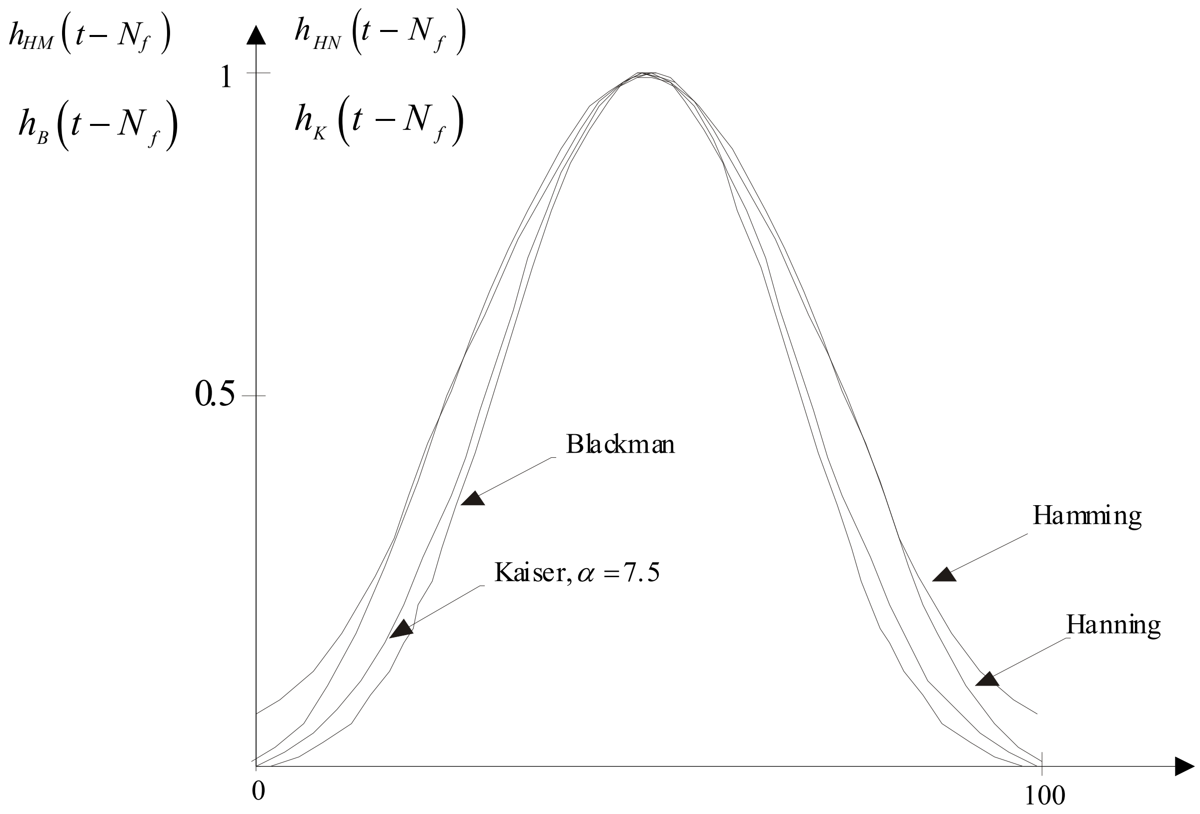
The parameter α is typically set in the range of 4 to 9.

For a given window width of $2N_f$ an additional possibility is introduced for further modification of the amplitude frequency response $H_{rF}(\omega)$.

Thus a compromise between window width and acceptable overshoots often can be found.

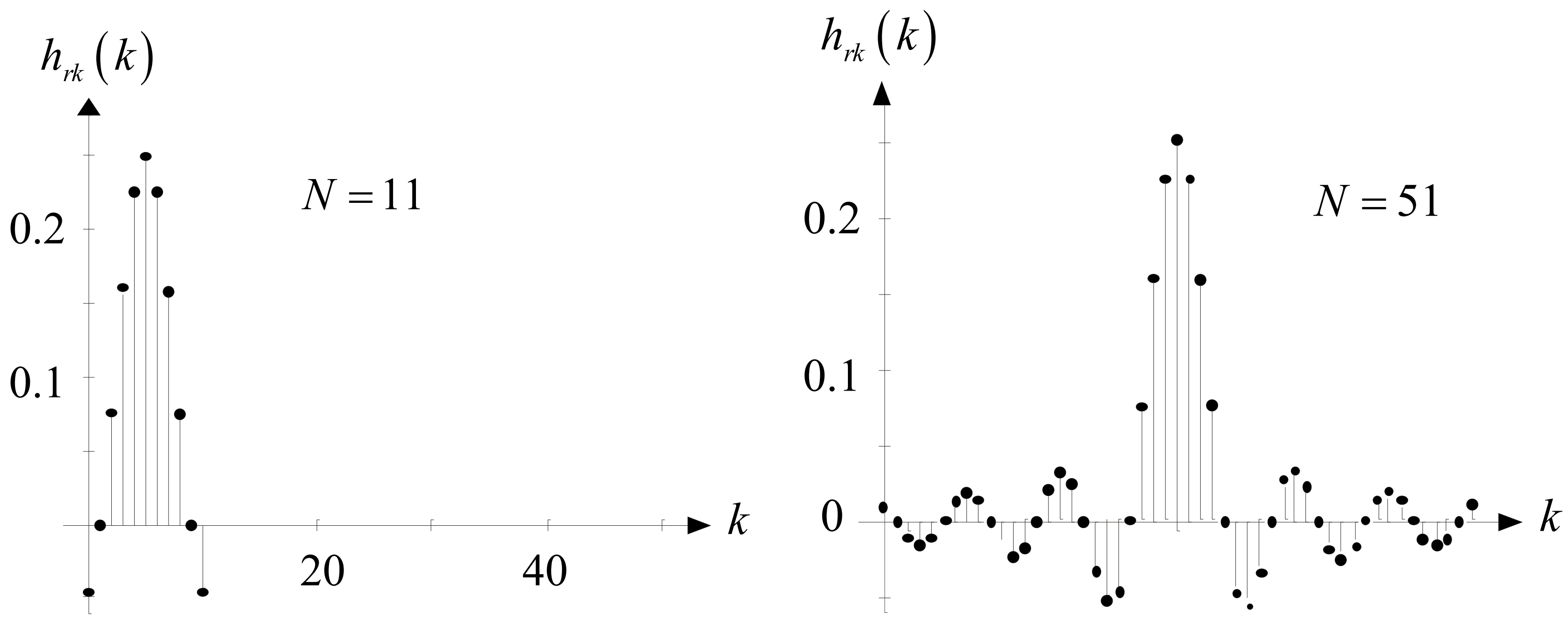
The next figure shows the important window functions in a comparison.

2.2 Design with regulations in the frequency range



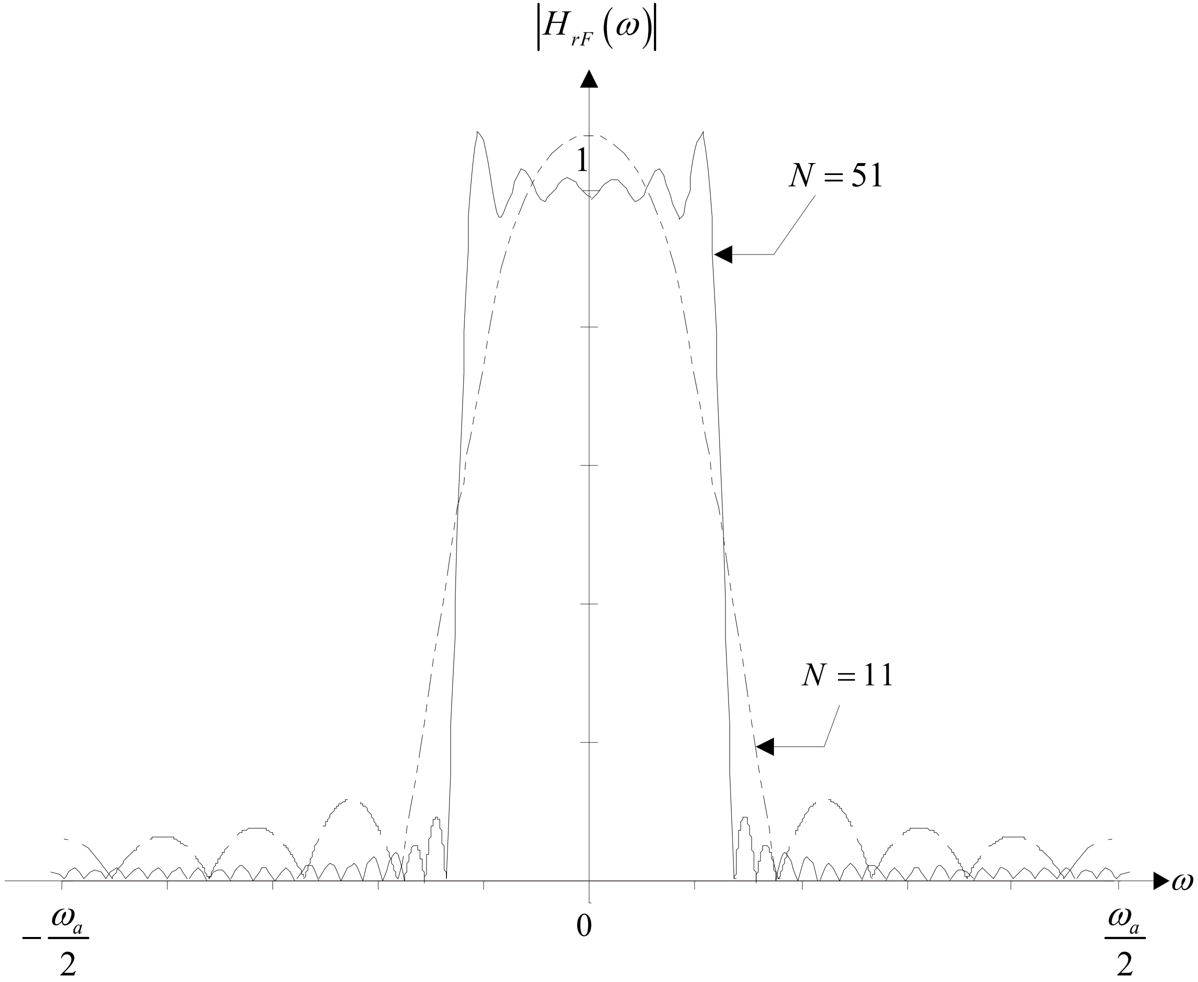
Envelope of the window sequences after Hamming, Hanning, Blackman and Kaiser for a window length of ***$2N_f=100$***

2.2 Design with regulations in the frequency range



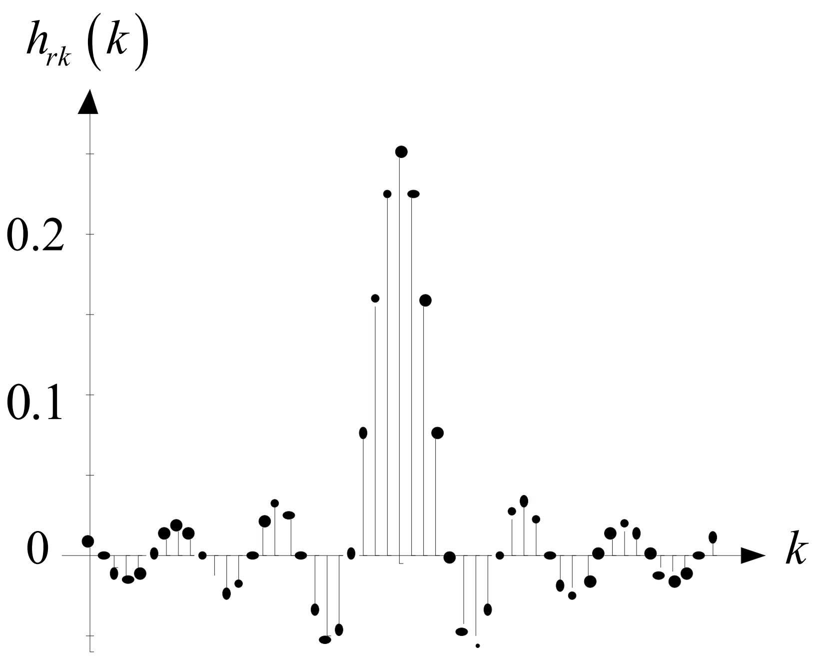
Influence of the number $N = 2N_f+1 = M+1$ of filter coefficients on the impulse response $h_{rk}(k)$ and on the amplitude characteristic of a FIR digital low-pass with $\omega_c = \omega_a / 8$

2.2 Design with regulations in the frequency range

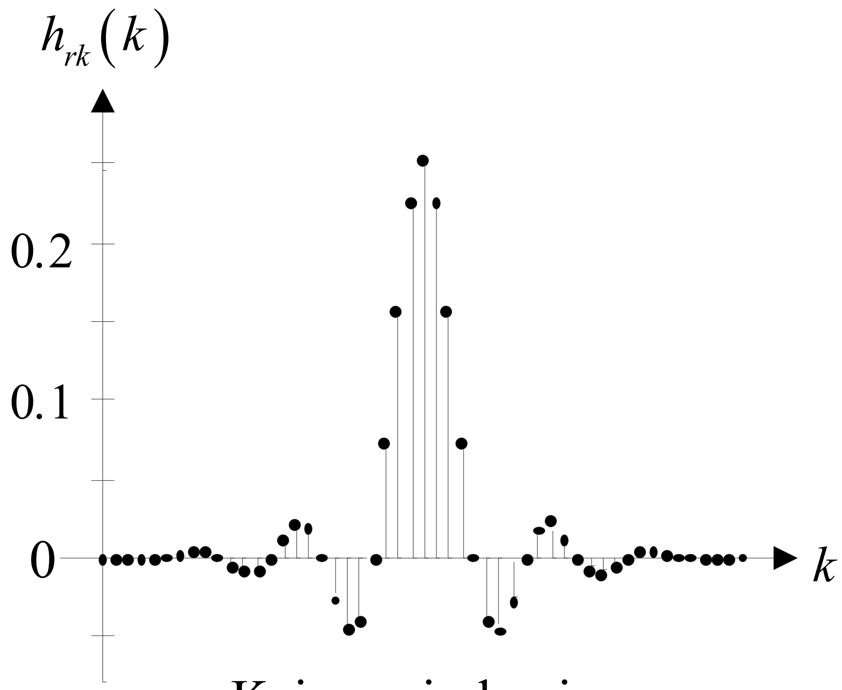


Influence of different values for the parameter N (or different window width) for an approximation of an ideal low-pass with $\omega_c = \omega_a / 8$

2.2 Design with regulations in the frequency range



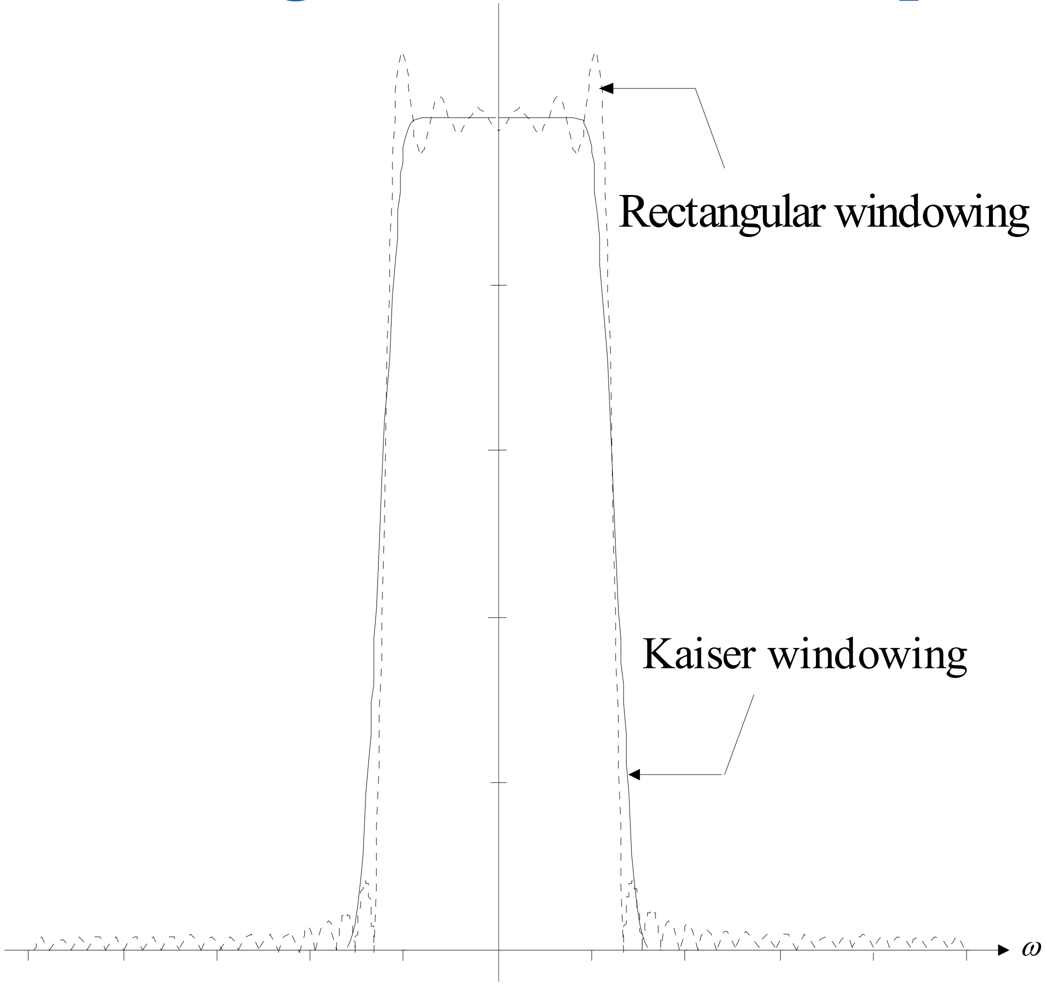
Rectangular windowing



Kaiser windowing

Influence of the rectangular and of the Kaiser windows on impulse response and on amplitude characteristic of an ideal low-pass filter

2.2 Design with regulations in the frequency range



Frequency response for rectangular and Kaiser windowing

2.2 Design with regulations in the frequency range

Example 1: **Low-pass baseband sequence**

The assigned baseband spectral function for the ideal low-pass is:

$$H_{bF}(\omega) = rect\left(\frac{\omega}{2\omega_T}\right) with \quad 0 < \omega_T < \frac{\omega_a}{2}$$

and $h_b(t) = \frac{\omega_T}{\pi} si(\omega_T t)$

The appropriate non-causal, infinite low-pass baseband sequence can be computed as follows:

$$h_{bk}(k) = T_a h_b((k - N_f)T_a) = \frac{2\pi}{\omega_a} h_b((k - N_f)\frac{2\pi}{\omega_a})$$

$$= 2\frac{\omega_T}{\omega_a} \cdot si\left(2\pi\frac{\omega_T}{\omega_a}(k - N_f)\right)$$

2.2 Design with regulations in the frequency range

Example 2: **Band-pass filter baseband sequence**

The baseband spectrum of an ideal band-pass filter can be written as:

$$H_{bF}(\omega) = rect\left(\frac{\omega}{\Delta\omega}\right) * \{\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\}$$

with $\omega_0 = \frac{\omega_2+\omega_1}{2}$ as the centre frequency

and $\Delta\omega = \omega_2 - \omega_1$ as the width of the pass band

and $h_b(t) = 2\pi\frac{\Delta\omega}{2\pi} si(\frac{\Delta\omega t}{2}) \cdot \frac{1}{\pi}\cos(\omega_0 t)$

$$= \Delta\omega\, si(\frac{\Delta\omega t}{2}) \cdot \frac{1}{\pi}\cos(\omega_0 t)$$

2.2 Design with regulations in the frequency range

The assigned infinite band-pass filter baseband sequence then results as follows:

$$h_{kb}(k) = T_a h_b((k-N_f)T_a) = \frac{2\pi}{\omega_a} h_b((k-N_f)\frac{2\pi}{\omega_a})$$

$$= \frac{2\pi}{\omega_a} \cdot \Delta\omega si(\frac{\Delta\omega t}{2}) \cdot \frac{1}{\pi} \cos(\omega_0 t) \bigg|_{t=(k-N_f)\frac{2\pi}{\omega_a}}$$

$$= \frac{2\Delta\omega}{\omega_a} \cdot si\left(\frac{\pi\Delta\omega}{\omega_a}\left\{k-N_f\right\}\right) \cdot \cos\left(\frac{2\pi\omega_0}{\omega_a}\left\{k-N_f\right\}\right)$$

2.2 Design with regulations in the frequency range

In the following **two special cases** are examined regarding characteristic of $H_{wF}(\omega)$ and the characteristics of the realizable system function of a digital filter :

Case 1: $\mathrm{Re}\{H_{wF}(\omega)\}$ real valued and even in ω

$$\mathrm{Im}\{H_{wF}(\omega)\} = 0 \qquad \forall \omega$$

Under these conditions also the corresponding baseband spectrum is even and real valued. From this it is clear that its inverse Fourier transform $h_b(t)$ is also real valued and even.

If also the window function $h_{wi}(t)$ is real and even in t, then (without proof) it holds:

$$h_{rk}(k) = h_{kb}(k) \cdot h_{kwi}(k)$$

and moreover a **point symmetry relationship** is given:

$$h_{rk}(k) = h_{rk}(2N_f - k) \quad \text{for} \quad k = 0(1)2N_f$$

2.2 Design with regulations in the frequency range

Another consideration:

For the transfer function holds:

$$H_{rF}(\omega) = H_{rz}(e^{j\omega T_a}) = \sum_{k=0}^{2N_f} h_r(k)\cdot e^{-j\omega k T_a} = e^{-j\omega N_f T_a}\cdot\sum_{k=0}^{2N_f} h_r(k)\cdot e^{-j\omega(k-N_f)T_a}$$

$$= e^{-j\omega N_f T_a}\left(\sum_{k=0}^{N_f-1} h_r(k)\cdot e^{-j\omega(k-N_f)T_a} + h_r(N_f)\cdot 1 + \sum_{k=N_f+1}^{2N_f} h_r(k)\cdot e^{-j\omega(k-N_f)T_a}\right)$$

$$= e^{-j\omega N_f T_a}\left(h_r(N_f) + \sum_{k=0}^{N_f-1} h_r(k)\cdot e^{-j\omega(k-N_f)T_a} + \sum_{k=0}^{N_f-1} h_r(k)\cdot e^{+j\omega(k-N_f)T_a}\right)$$

In accordance with the EULER' formula it results:

$$H_{rF}(\omega) = \left[h_r\left(N_f\right) + 2\cdot\sum_{m=0}^{N_f-1} h_r(m)\cdot\cos\left(\left\{N_f - m\right\}\omega T_a\right)\right]\cdot e^{-j\omega N_f T_a}$$

with $\left|H_{rF}(\omega)\right| = [\ldots]$ and $\sphericalangle H_{rF}(\omega) = -\omega N_f T_a$

2.2 Design with regulations in the frequency range

From this equation some remarkable characteristics of the realizable digital filter transfer function can be read off:

1. The expression in rectangular brackets

$$h_{rk}\left(N_f\right)+2\cdot\sum_{m=0}^{N_f-1} h_r(m)\cdot\cos\left(\left\{N_f-m\right\}\omega T_a\right)$$

is real and even in ω.

2. The phase angle $\sphericalangle H_{rF}(\omega)=\varphi_r(\omega)$ is linear depending on ω:

$$\varphi_r(\omega) = -\omega N_f T_a$$

and thus the group envelope delay of the digital filter is constant all over:

$$\tau_{gr}(\omega) = -\frac{d\varphi_r(\omega)}{d\omega} = N_f T_a$$

2.2 Design with regulations in the frequency range

Case 2:

$$\mathrm{Re}\{H_{wF}(\omega)\} = 0 \qquad \forall \omega$$

$$\mathrm{Im}\{H_{wF}(\omega)\} \qquad \text{real and odd in } \omega$$

Similiar to case 1 one gets the so called **antisymmetry** relationship:

$$h_r(k) = -h_r(2N_f - k) \qquad \text{for } k = 0(1)2N_f \text{ with } k \neq N_f$$

$$h_r(k) = 0 \qquad \text{for } k = N_f$$

Here the phase delay and group angles have the same properties as in case1.

$$H_{rF}(\omega) = H_{rz}(e^{j\omega T_a}) = \sum_{k=0}^{2N_f} h_r(k) \cdot e^{-j\omega k T_a}$$

$$= e^{-j\omega N_f T_a} \cdot \sum_{k=0}^{2N_f} h_r(k) \cdot e^{-j\omega (k - N_f) T_a}$$

2.2 Design with regulations in the frequency range

Example: Construction of a FIR low-pass with $\omega_T = \omega_a / 4$ and $N_f = 4$ and an ideal low-pass as reference filter

The following is unknown:

- the filters coefficients a_k,
- the phase function $\varphi_r(\omega)$,
- the envelope delay $\tau_{gr}(\omega)$

Step 1: $$a_k = h_{rk}(k) = h_{kbTP}(k) \cdot h_{kHM}(k)$$

Step 2: The desired transfer function of the reference digital low-pass in accordance with

$$H_{bF}(\omega) = H_{wF}(\omega) \cdot rect\left(\frac{\omega}{\omega_a}\right) = rect\left(\frac{\omega}{2\omega_T}\right)$$

2.2 Design with regulations in the frequency range

Step 3: The last equation is inversely Fourier transformed and thus gives $h_b(t)$
This intermediate result is then used in the next equation

$$h_{bk}(k) = T_a \cdot h_b((k - N_f)T_a)$$

which then leads to:

$$h_{kbTP}(k) = \frac{2\omega_T}{\omega_a} \cdot si\left(\frac{2\pi\omega_T}{\omega_a}\left\{k - N_f\right\}\right) \quad \text{for } k = -\infty(1)\infty$$

Step 4: Now the HAMMING window sequence is computed:

$$h_{kHM}(k) = 0.54 + 0.46 \cdot \cos\left(\frac{\pi\left\{k - N_f\right\}}{N_f}\right) \quad \text{for} \quad k = 0(1)2N_f$$

2.2 Design with regulations in the frequency range

Step 5:

Using $\omega_T = \omega_a / 4$ and $N_f = 4$, the coefficients a_k can be computed as follows:

$$a_k = h_{rk}(k) = h_{bkTP}(k) \cdot h_{HMk}(k)$$

$$= \frac{1}{2} \mathrm{si}\left(0.5\pi \{k-4\}\right) \cdot \left(0.54 + 0.46 \cdot \cos\left(\frac{\pi\{k-4\}}{4}\right)\right), \quad k = 0(1)8$$

2.2 Design with regulations in the frequency range

Step 6: Final determination of filter coefficients uses the following table:

k	$h_{bkTP}(k)$	$h_{HMk}(k)$	$a_k = h_{rk}(k) = h_{bkTP}(k) \cdot h_{HMk}(k)$
0	0	0.54	0
1	-0.1061	0.215	-0.0228
2	0	0.08	0
3	0.3183	0.865	0.2754
4	0.5	1	0.5
5	0.3183	0.865	0.2754
6	0	0.08	0
7	-0.1061	0.215	-0.0228
8	0	0.54	0

Table of the values for the filter coefficients

2.2 Design with regulations in the frequency range

On the basis the table one recognizes that the impulse response of the realizable digital low-pass **is real** and is **symmetrical** regarding the point $k = Nf = 4$.

So the symmetry condition is fulfilled:

$$h_{rk}(k) = h_{rk}(2N_f - k) = h_{rk}(8-k)$$

Step 7: One determines ***the phase function***

$$\varphi_r(\omega) = -\omega N_f T_a = -\omega \cdot 4T_a \quad \text{with} \quad T_a = \frac{2\pi}{\omega_a}$$

and ***the envelope delay***

$$\tau_{gr}(\omega) = 4T_a$$

2.2 Design with regulations in the frequency range

Important note:

A formal conversion of the filter coefficients represented in the table of the values of a_k to the assigned FIR Digitalfilter structure would lead to a filter circuit with 9 constant multipliers and 8 delay units.

However as a_0, a_2, a_6 *and* a_8 show the value of zero, the number of constant multipliers can be reduced to 5.

Moreover 2 delay units are not used here.