# Maxwell's Equations / Wave Equation

Maxwell's equations in differential form can be written in phasor notation

$$\begin{aligned} \text{rot}\,\underline{\mathbf{H}} &= \underline{\mathbf{J}} + \text{j}\omega\underline{\mathbf{D}} & (1)\\ \text{rot}\,\underline{\mathbf{E}} &= -\text{j}\omega\underline{\mathbf{B}} & (2)\\ \text{div}\,\underline{\mathbf{D}} &= \rho & (3)\\ \text{div}\,\underline{\mathbf{B}} &= 0\,, & (4)\end{aligned}$$

where $\underline{\mathbf{E}}$ is the electric field, $\underline{\mathbf{H}}$ is the magnetic field, $\underline{\mathbf{D}}$ is the displacement or electric flux density, $\underline{\mathbf{B}}$ is the magnetic flux density, $\underline{\mathbf{J}}$ is the electric current density, and $\rho$ is the electric charge density.

For linear, homogeneous, isotropic and lossy materials holds

$$\begin{aligned} \underline{\mathbf{J}} &= \kappa\underline{\mathbf{E}} & (5)\\ \underline{\mathbf{D}} &= \varepsilon_0\varepsilon_r\underline{\mathbf{E}} & (6)\\ \underline{\mathbf{B}} &= \mu_0\mu_r\underline{\mathbf{H}}\,, & (7)\end{aligned}$$

where $\kappa$, $\varepsilon_0$, $\varepsilon_r$, $\mu_0$, $\mu_r$ are the conductivity, the permittivity of vacuum, the relative permittivity, the permeability of vacuum and the relative permeability, respectively.

## Problem 1

Derive the wave equation for the electric field $\underline{\mathbf{E}}$ based on *Maxwell*'s equations in phasor notation (also referred to as *Helmholz* equation).
**Hint:** rot rot $\mathbf{V}$ = grad div $\mathbf{V} - \Delta\mathbf{V}$, where

$$\begin{aligned} \text{grad}\,S &= \frac{\partial S}{\partial x}\mathbf{e}_x + \frac{\partial S}{\partial y}\mathbf{e}_y + \frac{\partial S}{\partial z}\mathbf{e}_z & (8)\\ \Delta\mathbf{V} &= \Delta V_x\mathbf{e}_x + \Delta V_y\mathbf{e}_y + \Delta V_z\mathbf{e}_z & (9)\\ \Delta S &= \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} + \frac{\partial^2 S}{\partial z^2} & (10)\end{aligned}$$

are defined in Cartesian coordinates.

## Problem 2

Now assume $\kappa = 0$ and $\rho = 0$.

2.1 Determine $\alpha$ and $k$, for which $\underline{\mathbf{E}} = \underline{E}_y\mathbf{e}_y + \underline{E}_z\mathbf{e}_z = \left(\underline{E}_{y+}\mathbf{e}_y + \underline{E}_{z+}\mathbf{e}_z\right)\text{e}^{-(\alpha+\text{j}k)x}$ is a solution of the wave equation.

2.2 Determine the magnetic field $\underline{\mathbf{H}}$.

2.3 Determine the electric field $\mathbf{E}(t)$ in the time domain.

2.4 In which direction does the wave propagate?

## Solution of Problem 1

Inserting Equations (5) – (7) into (1) and (2) one obtains with $\varepsilon = \varepsilon_0\varepsilon_\mathrm{r}$ and $\mu = \mu_0\mu_\mathrm{r}$

$$\mathrm{rot}\ \underline{\mathbf{H}} \overset{(5),(6)}{=} \kappa\underline{\mathbf{E}} + \mathrm{j}\omega\varepsilon\underline{\mathbf{E}} \tag{11}$$

$$\mathrm{rot}\ \underline{\mathbf{E}} \overset{(7)}{=} -\mathrm{j}\omega\mu\underline{\mathbf{H}}\ . \tag{12}$$

Since we are interested in the electric field $\underline{\mathbf{E}}$, we can extract the magnetic field $\underline{\mathbf{H}}$ in Equation (12) with (11)

$$\begin{aligned}
& \mathrm{rot\ rot}\ \underline{\mathbf{E}} \overset{(12)}{=} -\mathrm{j}\omega\mu\ \mathrm{rot}\ \underline{\mathbf{H}} \\
\Longleftrightarrow\quad & \mathrm{rot\ rot}\ \underline{\mathbf{E}} \overset{(11)}{=} -\mathrm{j}\omega\mu\left(\kappa\underline{\mathbf{E}} + \mathrm{j}\omega\varepsilon\underline{\mathbf{E}}\right) \\
\Longleftrightarrow\quad & \mathrm{rot\ rot}\ \underline{\mathbf{E}} = \left(-\mathrm{j}\omega\mu\kappa + \omega^2\mu\varepsilon\right)\underline{\mathbf{E}}\ . \qquad (13)
\end{aligned}$$

Using the Hint

$$\begin{aligned}
\mathrm{rot\ rot}\ \underline{\mathbf{E}} &= \mathrm{grad\ div}\ \underline{\mathbf{E}} - \Delta\underline{\mathbf{E}} \\
&= \left(-\mathrm{j}\omega\mu\kappa + \omega^2\mu\varepsilon\right)\underline{\mathbf{E}}
\end{aligned}$$

and equation (3) and (6) leads

$$\begin{aligned}
& \mathrm{div}\ \underline{\mathbf{D}} \overset{(6)}{=} \varepsilon\ \mathrm{div}\ \underline{\mathbf{E}} \overset{(3)}{=} \rho \\
\Longleftrightarrow\quad & \mathrm{div}\ \underline{\mathbf{E}} = \frac{\rho}{\varepsilon}\ . \qquad (14)
\end{aligned}$$

Finally we can obtain the inhomogeneous wave equation for the electric field $\underline{\mathbf{E}}$

$$\begin{aligned}
\mathrm{grad\ div}\ \underline{\mathbf{E}} - \Delta\underline{\mathbf{E}} &\overset{(14)}{=} \frac{1}{\varepsilon}\mathrm{grad}\ \rho - \Delta\underline{\mathbf{E}} \\
&= \left(-\mathrm{j}\omega\mu\kappa + \omega^2\mu\varepsilon\right)\underline{\mathbf{E}} \\
\Longleftrightarrow\quad \Delta\underline{\mathbf{E}} + \left(\omega^2\mu\varepsilon - \mathrm{j}\omega\mu\kappa\right)\underline{\mathbf{E}} &= \frac{1}{\varepsilon}\mathrm{grad}\ \rho\ . \qquad (15)
\end{aligned}$$

This wave equation, Equation (15), is of crucial importance because it describes the wave propagation at any point in space and time!

## Solution of Problem 2

2.1 From the result of problem 1 we know that the inhomogeneous wave equation for the electric field $\underline{\mathbf{E}}$ is

$$\Delta\underline{\mathbf{E}} + \left(\omega^2\mu\varepsilon - \mathrm{j}\omega\mu\kappa\right)\underline{\mathbf{E}} = \frac{1}{\varepsilon}\mathrm{grad}\ \rho\ .$$

For $\kappa = 0$ and $\rho = 0$, one obtains this wave equation

$$\Delta\underline{\mathbf{E}} + \omega^2\mu\varepsilon\underline{\mathbf{E}} = 0$$

$$\overset{(9)}{\Longleftrightarrow} \quad \Delta\underline{E}_x\mathbf{e}_x + \Delta\underline{E}_y\mathbf{e}_y + \Delta\underline{E}_z\mathbf{e}_z + \omega^2\mu\varepsilon\left(\underline{E}_x\mathbf{e}_x + \underline{E}_y\mathbf{e}_y + \underline{E}_z\mathbf{e}_z\right) = 0\,. \quad (16)$$

From the electric field $\underline{\mathbf{E}} = \underline{E}_y\mathbf{e}_y + \underline{E}_z\mathbf{e}_z = \left(\underline{E}_{y+}\mathbf{e}_y + \underline{E}_{z+}\mathbf{e}_z\right)\mathrm{e}^{-(\alpha+\mathrm{j}k)x}$ we can know that

$$\begin{aligned}\underline{E}_x &= 0\\ \underline{E}_y &= \underline{E}_{y+}\mathrm{e}^{-(\alpha+\mathrm{j}k)x}\\ \underline{E}_z &= \underline{E}_{z+}\mathrm{e}^{-(\alpha+\mathrm{j}k)x}\end{aligned}$$

and consequently with Equation (10)

$$\Delta\underline{E}_x = \frac{\partial^2\underline{E}_x}{\partial x^2} + \frac{\partial^2\underline{E}_x}{\partial y^2} + \frac{\partial^2\underline{E}_x}{\partial z^2} = 0 \quad (17)$$

$$\Delta\underline{E}_y = \frac{\partial^2\underline{E}_y}{\partial x^2} + \frac{\partial^2\underline{E}_y}{\partial y^2} + \frac{\partial^2\underline{E}_y}{\partial z^2} = \frac{\partial^2\underline{E}_y}{\partial x^2} = \left[-\left(\alpha+\mathrm{j}k\right)\right]^2\underline{E}_y \quad (18)$$

$$\Delta\underline{E}_z = \frac{\partial^2\underline{E}_z}{\partial x^2} + \frac{\partial^2\underline{E}_z}{\partial y^2} + \frac{\partial^2\underline{E}_z}{\partial z^2} = \frac{\partial^2\underline{E}_z}{\partial x^2} = \left[-\left(\alpha+\mathrm{j}k\right)\right]^2\underline{E}_z\,. \quad (19)$$

Inserting Equations (17) – (19) into (16) yields

$$\begin{aligned}\frac{\partial^2\underline{E}_y}{\partial x^2}\mathbf{e}_y + \frac{\partial^2\underline{E}_z}{\partial x^2}\mathbf{e}_z + \omega^2\mu\varepsilon\left(\underline{E}_y\mathbf{e}_y + \underline{E}_z\mathbf{e}_z\right) &= 0\\ \left[-\left(\alpha+\mathrm{j}k\right)\right]^2\left(\underline{E}_y\mathbf{e}_y + \underline{E}_z\mathbf{e}_z\right) + \omega^2\mu\varepsilon\left(\underline{E}_y\mathbf{e}_y + \underline{E}_z\mathbf{e}_z\right) &= 0\\ \Longleftrightarrow \quad \left(\left[-\left(\alpha+\mathrm{j}k\right)\right]^2 + \omega^2\mu\varepsilon\right)\left(\underline{E}_y\mathbf{e}_y + \underline{E}_z\mathbf{e}_z\right) &= 0\,.\end{aligned}$$

Finally, from $\left[-\left(\alpha+\mathrm{j}k\right)\right]^2 + \omega^2\mu\varepsilon = 0$, we can determine

$$\begin{aligned}\alpha &= 0\\ k^2 &= \omega^2\mu\varepsilon \quad \Rightarrow \quad k = \omega\sqrt{\mu\varepsilon} \text{ is the scalar wave number}\,.\end{aligned}$$

2.2 The second Maxwell's equation says

$$\begin{aligned}\mathrm{rot}\,\underline{\mathbf{E}} &\overset{!}{=} \begin{vmatrix}\mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \underline{E}_x & \underline{E}_y & \underline{E}_z\end{vmatrix}\\ &= \left(\frac{\partial\underline{E}_z}{\partial y} - \frac{\partial\underline{E}_y}{\partial z}\right)\mathbf{e}_x + \left(\frac{\partial\underline{E}_x}{\partial z} - \frac{\partial\underline{E}_z}{\partial x}\right)\mathbf{e}_y + \left(\frac{\partial\underline{E}_y}{\partial x} - \frac{\partial\underline{E}_x}{\partial y}\right)\mathbf{e}_z\\ &\overset{(2)}{=} -\mathrm{j}\omega\underline{\mathbf{B}} \overset{(7)}{=} -\mathrm{j}\omega\mu\underline{\mathbf{H}}\\ &= -\mathrm{j}\omega\mu\left(\underline{H}_x\mathbf{e}_x + \underline{H}_y\mathbf{e}_y + \underline{H}_z\mathbf{e}_z\right)\,. \end{aligned} \quad (20)$$

In subproblem 2.1 we have obtained $\alpha = 0$ and $k = \omega\sqrt{\mu\varepsilon}$. Hence, the electric field can be now written as $\underline{\mathbf{E}} = \underline{E}_y \mathbf{e}_y + \underline{E}_z \mathbf{e}_z = \left(\underline{E}_{y+}\mathbf{e}_y + \underline{E}_{z+}\mathbf{e}_z\right) \mathrm{e}^{-\mathrm{j}kx}$ with

$$\begin{aligned} \underline{E}_x &= 0 \\ \underline{E}_y &= \underline{E}_{y+}\mathrm{e}^{-\mathrm{j}kx} \\ \underline{E}_z &= \underline{E}_{z+}\mathrm{e}^{-\mathrm{j}kx} . \end{aligned}$$

Therefore, we can get

$$\frac{\partial \underline{E}_x}{\partial x} = \frac{\partial \underline{E}_x}{\partial y} = \frac{\partial \underline{E}_x}{\partial z} = 0 \tag{21}$$

$$\frac{\partial \underline{E}_y}{\partial x} = -\mathrm{j}k\left(\underline{E}_{y+}\mathrm{e}^{-\mathrm{j}kx}\right) = -\mathrm{j}k\underline{E}_y \quad \text{and} \quad \frac{\partial \underline{E}_y}{\partial y} = \frac{\partial \underline{E}_y}{\partial z} = 0 \tag{22}$$

$$\frac{\partial \underline{E}_z}{\partial x} = -\mathrm{j}k\left(\underline{E}_{z+}\mathrm{e}^{-\mathrm{j}kx}\right) = -\mathrm{j}k\underline{E}_z \quad \text{and} \quad \frac{\partial \underline{E}_z}{\partial y} = \frac{\partial \underline{E}_z}{\partial z} = 0 . \tag{23}$$

Inserting (21) – (23) into (20) we can obtain

$$\begin{aligned} -\frac{\partial \underline{E}_z}{\partial x}\mathbf{e}_y + \frac{\partial \underline{E}_y}{\partial x}\mathbf{e}_z &= -\mathrm{j}\omega\mu\underline{\mathbf{H}} \\ \Longleftrightarrow \quad -\left(-\mathrm{j}k\underline{E}_z\right)\mathbf{e}_y + \left(-\mathrm{j}k\underline{E}_y\right)\mathbf{e}_z &= -\mathrm{j}\omega\mu\left(\underline{H}_x\mathbf{e}_x + \underline{H}_y\mathbf{e}_y + \underline{H}_z\mathbf{e}_z\right) . \end{aligned} \tag{24}$$

Compare the coefficients in Equation (24)

$$\begin{aligned} \underline{H}_x &= 0 \\ \underline{H}_y &= \frac{1}{-\mathrm{j}\omega\mu}\left(-\frac{\partial \underline{E}_z}{\partial x}\right) = \frac{1}{-\mathrm{j}\omega\mu}\left[-\left(-\mathrm{j}k\underline{E}_z\right)\right] \\ &= -\frac{k}{\omega\mu}\underline{E}_z = -\sqrt{\frac{\varepsilon}{\mu}}\underline{E}_{z+}\mathrm{e}^{-\mathrm{j}kx} \\ \underline{H}_z &= \frac{1}{-\mathrm{j}\omega\mu}\frac{\partial \underline{E}_y}{\partial x} = \frac{1}{-\mathrm{j}\omega\mu}\left(-\mathrm{j}k\underline{E}_y\right) \\ &= \frac{k}{\omega\mu}\underline{E}_y = \sqrt{\frac{\varepsilon}{\mu}}\underline{E}_{y+}\mathrm{e}^{-\mathrm{j}kx} . \end{aligned}$$

Finally, the magnetic field $\underline{\mathbf{H}}$ can be derivable from the second Maxwell's equation as

$$\begin{aligned} \underline{\mathbf{H}} &= \frac{k}{\omega\mu}\left(-\underline{E}_z\mathbf{e}_y + \underline{E}_y\mathbf{e}_z\right) \\ &= \sqrt{\frac{\varepsilon}{\mu}}\left(-\underline{E}_{z+}\mathbf{e}_y + \underline{E}_{y+}\mathbf{e}_z\right)\mathrm{e}^{-\mathrm{j}kx} . \end{aligned}$$

2.3 With $\alpha = 0$ from subproblem 2.1 and $\underline{E}_{y+} = \hat{E}_{y+}\mathrm{e}^{\mathrm{j}\varphi_y}$ and $\underline{E}_{z+} = \hat{E}_{z+}\mathrm{e}^{\mathrm{j}\varphi_z}$

$$\begin{aligned}
\mathbf{E}(t) &= \mathrm{Re}\left\{\underline{\mathbf{E}}\,\mathrm{e}^{\mathrm{j}\omega t}\right\} \\
&= \mathrm{Re}\left\{\left(\underline{E}_y\mathbf{e}_y + \underline{E}_z\mathbf{e}_z\right)\mathrm{e}^{\mathrm{j}\omega t}\right\} \\
&= \mathrm{Re}\left\{\left(\underline{E}_{y+}\mathrm{e}^{-\mathrm{j}kx}\mathbf{e}_y + \underline{E}_{z+}\mathrm{e}^{-\mathrm{j}kx}\mathbf{e}_z\right)\mathrm{e}^{\mathrm{j}\omega t}\right\} \\
&= \mathrm{Re}\left\{\hat{E}_{y+}\mathrm{e}^{\mathrm{j}\omega t}\mathrm{e}^{-\mathrm{j}kx}\mathrm{e}^{\mathrm{j}\varphi_y}\mathbf{e}_y + \hat{E}_{z+}\mathrm{e}^{\mathrm{j}\omega t}\mathrm{e}^{-\mathrm{j}kx}\mathrm{e}^{\mathrm{j}\varphi_z}\mathbf{e}_z\right\} \\
&= \mathrm{Re}\left\{\hat{E}_{y+}\mathrm{e}^{\mathrm{j}(\omega t - kx + \varphi_y)}\right\}\mathbf{e}_y + \mathrm{Re}\left\{\hat{E}_{z+}\mathrm{e}^{\mathrm{j}(\omega t - kx + \varphi_z)}\right\}\mathbf{e}_z \\
&= \hat{E}_{y+}\cos\left(\omega t - kx + \varphi_y\right)\mathbf{e}_y + \hat{E}_{z+}\cos\left(\omega t - kx + \varphi_z\right)\mathbf{e}_z\,.
\end{aligned} \tag{25}$$

2.4 Sine $\mathbf{k}\cdot\mathbf{r} = kx$, the wave propagates along the positive $x$-axis.