Polarization / Wave Vector

Assume the following three magnetic fields of homogeneous, plane waves

$$\mathbf{H}_1(t) = H_\mathrm{A}\cos(\omega t - kz)\,\mathbf{e}_x - H_\mathrm{A}\sin(\omega t - kz)\,\mathbf{e}_y \quad (1)$$

$$\mathbf{H}_2(t) = H_\mathrm{A}\cos(\omega t - kz)\,\mathbf{e}_x + H_\mathrm{A}\sin(\omega t - kz)\,\mathbf{e}_y \quad (2)$$

$$\mathbf{H}_3(t) = H_\mathrm{B}\cos(\omega t - kz)\,\mathbf{e}_x + H_\mathrm{B}\sin(\omega t - kz)\,\mathbf{e}_y\ , \quad (3)$$

where $H_\mathrm{A} > 0$, $H_\mathrm{B} > 0$, $H_\mathrm{A} < H_\mathrm{B}$.

Problem 1

1.1 Determine the phasor of the magnetic field $\underline{\mathbf{H}}_1$.

1.2 What is the polarization type of the magnetic fields $\mathbf{H}_1(t)$ and $\mathbf{H}_2(t)$?

1.3 What is the polarization type of the superposition of $\mathbf{H}_1(t)$ and $\mathbf{H}_2(t)$?

1.4 What is the polarization type of the superposition of $\mathbf{H}_1(t)$ and $\mathbf{H}_3(t)$?

Given is the following electric and magnetic field of a homogeneous, plane wave that propagates in a dielectric fluid with $\varepsilon_{\mathrm{r}1} = 3$ and $\mu_{\mathrm{r}1} = 1$

$$\underline{\mathbf{E}} = E_1\left(\frac{1}{2}\mathbf{e}_x - \mathbf{e}_y + 2\mathbf{e}_z\right)\mathrm{e}^{-\mathrm{j}\mathbf{k}_1\cdot\mathbf{r}} \quad (4)$$

$$\underline{\mathbf{H}} = H_1\left(-2\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z\right)\mathrm{e}^{-\mathrm{j}\mathbf{k}_1\cdot\mathbf{r}}\ . \quad (5)$$

Problem 2

2.1 In which direction does the wave propagate? Determine the normalized vector $\mathbf{n}_1$, where $\mathbf{k}_1 = k_1\mathbf{n}_1$ (k_1 is the wave number in the dielectric fluid).

2.2 Determine the wave vector $\mathbf{k}_1$.

2.3 Determine the phase velocity v_ph of a wave propagating in this dielectric fluid.

Solution of Problem 1

1.1 Using $\sin(\omega t) = \cos\left(\omega t - \frac{\pi}{2}\right)$ and $H_\mathrm{A} > 0$, $\mathbf{H}_1(t)$ can be rewritten as

$$\begin{aligned}
\mathbf{H}_1(t) &= H_\mathrm{A}\cos(\omega t - kz)\,\mathbf{e}_x - H_\mathrm{A}\cos\left(\omega t - kz - \frac{\pi}{2}\right)\mathbf{e}_y \\
&= \mathrm{Re}\left\{H_\mathrm{A}\mathrm{e}^{\mathrm{j}(\omega t - kz)}\mathbf{e}_x - H_\mathrm{A}\mathrm{e}^{\mathrm{j}\left(\omega t - kz - \frac{\pi}{2}\right)}\mathbf{e}_y\right\} \\
&= \mathrm{Re}\Big\{\big(H_\mathrm{A}\mathrm{e}^{-\mathrm{j}kz}\mathbf{e}_x - H_\mathrm{A}\underbrace{\mathrm{e}^{-\mathrm{j}\frac{\pi}{2}}}_{=-\mathrm{j}}\mathrm{e}^{-\mathrm{j}kz}\mathbf{e}_y\big)\mathrm{e}^{\mathrm{j}\omega t}\Big\} \\
&= \mathrm{Re}\Big\{\big(H_\mathrm{A}\mathrm{e}^{-\mathrm{j}kz}\mathbf{e}_x + \mathrm{j}H_\mathrm{A}\mathrm{e}^{-\mathrm{j}kz}\mathbf{e}_y\big)\mathrm{e}^{\mathrm{j}\omega t}\Big\} \\
&= \mathrm{Re}\Big\{\underbrace{H_\mathrm{A}(\mathbf{e}_x + \mathrm{j}\mathbf{e}_y)\,\mathrm{e}^{-\mathrm{j}kz}}_{=\underline{\mathbf{H}}_1}\mathrm{e}^{\mathrm{j}\omega t}\Big\}\,.
\end{aligned}$$

Therefore, the phasor of the magnetic field $\underline{\mathbf{H}}_1$ is

$$\underline{\mathbf{H}}_1 = H_\mathrm{A}(\mathbf{e}_x + \mathrm{j}\mathbf{e}_y)\,\mathrm{e}^{-\mathrm{j}kz}\,.$$

1.2 The period of $\mathbf{H}_1(t) = H_\mathrm{A}\cos(\omega t - kz)\,\mathbf{e}_x - H_\mathrm{A}\sin(\omega t - kz)\,\mathbf{e}_y$ is $T = \frac{2\pi}{\omega}$. Now consider the *constant phase plane* $z = 0$ and the different time instants $t = 0$, $\frac{T}{4}$, $\frac{T}{2}$, $\frac{3T}{4}$, we can get with $H_\mathrm{A} > 0$

$$\begin{aligned}
\mathbf{H}_1(t=0,\ z=0) &= H_\mathrm{A}\cos 0\ \mathbf{e}_x - H_\mathrm{A}\sin 0\ \mathbf{e}_y = H_\mathrm{A}\mathbf{e}_x \\
&\Rightarrow \text{ point at the positive } x\text{-axis}\,, \\
\mathbf{H}_1(t=\tfrac{T}{4},\ z=0) &= H_\mathrm{A}\cos\frac{\pi}{2}\ \mathbf{e}_x - H_\mathrm{A}\sin\frac{\pi}{2}\ \mathbf{e}_y = -H_\mathrm{A}\mathbf{e}_y \\
&\Rightarrow \text{ point at the negative } y\text{-axis}\,, \\
\mathbf{H}_1(t=\tfrac{T}{2},\ z=0) &= H_\mathrm{A}\cos\pi\ \mathbf{e}_x - H_\mathrm{A}\sin\pi\ \mathbf{e}_y = -H_\mathrm{A}\mathbf{e}_x \\
&\Rightarrow \text{ point at the negative } x\text{-axis}\,, \\
\mathbf{H}_1(t=\tfrac{3T}{4},\ z=0) &= H_\mathrm{A}\cos\frac{3\pi}{2}\ \mathbf{e}_x - H_\mathrm{A}\sin\frac{3\pi}{2}\ \mathbf{e}_y = H_\mathrm{A}\mathbf{e}_y \\
&\Rightarrow \text{ point at the positive } y\text{-axis}\,.
\end{aligned}$$

Figure 1 illustrates the polarization type of $\mathbf{H}_1(t)$.

Observe the figure and note the *clockwise* rotation direction with the *same* length H_A we can draw the following conclusion that $\mathbf{H}_1(t)$ is *left circularly polarized.*

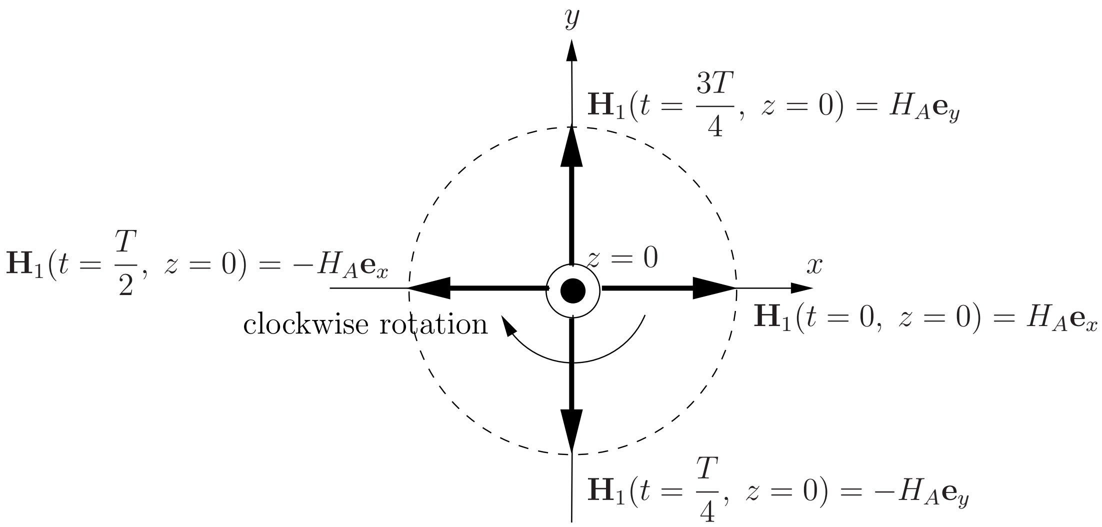

Figure 1: Polarization type of $\mathbf{H}_1(t)$.

For magnetic field $\mathbf{H}_2(t) = H_\mathrm{A}\cos(\omega t - kz)\,\mathbf{e}_x + H_\mathrm{A}\sin(\omega t - kz)\,\mathbf{e}_y$, analogously at the constant phase plane $z = 0$ and $t = 0,\ \frac{T}{4},\ \frac{T}{2},\ \frac{3T}{4}$, we can get with $H_\mathrm{A} > 0$

$$\begin{aligned}
\mathbf{H}_2(t=0,\ z=0) &= H_\mathrm{A}\cos 0\,\mathbf{e}_x + H_\mathrm{A}\sin 0\,\mathbf{e}_y = H_\mathrm{A}\mathbf{e}_x \\
&\Rightarrow \text{point at the positive } x\text{-axis}, \\
\mathbf{H}_2(t=\tfrac{T}{4},\ z=0) &= H_\mathrm{A}\cos\frac{\pi}{2}\,\mathbf{e}_x + H_\mathrm{A}\sin\frac{\pi}{2}\,\mathbf{e}_y = H_\mathrm{A}\mathbf{e}_y \\
&\Rightarrow \text{point at the positive } y\text{-axis}, \\
\mathbf{H}_2(t=\tfrac{T}{2},\ z=0) &= H_\mathrm{A}\cos\pi\,\mathbf{e}_x + H_\mathrm{A}\sin\pi\,\mathbf{e}_y = -H_\mathrm{A}\mathbf{e}_x \\
&\Rightarrow \text{point at the negative } x\text{-axis}, \\
\mathbf{H}_2(t=\tfrac{3T}{4},\ z=0) &= H_\mathrm{A}\cos\frac{3\pi}{2}\,\mathbf{e}_x + H_\mathrm{A}\sin\frac{3\pi}{2}\,\mathbf{e}_y = -H_\mathrm{A}\mathbf{e}_y \\
&\Rightarrow \text{point at the negative } y\text{-axis}.
\end{aligned}$$

Note the *anticlockwise* rotation direction with the *same* length H_A we can draw the following conclusion that $\mathbf{H}_2(t)$ is *right circularly polarized.*

1.3 $\mathbf{H}_1(t) + \mathbf{H}_2(t) = 2H_\mathrm{A}\cos(\omega t - kz)\,\mathbf{e}_x$ is also a solution of the wave equation.

Consider the equiphase plane $z = 0$ and $t = 0,\ \frac{T}{4},\ \frac{T}{2},\ \frac{3T}{4}$, we can get with $H_\mathrm{A} > 0$

$$\begin{aligned}
\mathbf{H}_1(t=0,\ z=0) + \mathbf{H}_2(t=0,\ z=0) &= 2H_\mathrm{A}\cos 0\,\mathbf{e}_x = 2H_\mathrm{A}\mathbf{e}_x \\
\mathbf{H}_1(t=\tfrac{T}{4},\ z=0) + \mathbf{H}_2(t=\tfrac{T}{4},\ z=0) &= 2H_\mathrm{A}\cos\frac{\pi}{2}\,\mathbf{e}_x = 0 \\
\mathbf{H}_1(t=\tfrac{T}{2},\ z=0) + \mathbf{H}_2(t=\tfrac{T}{2},\ z=0) &= 2H_\mathrm{A}\cos\pi\,\mathbf{e}_x = -2H_\mathrm{A}\mathbf{e}_x
\end{aligned}$$

$$\mathbf{H}_1(t=\frac{3T}{2},\ z=0)+\mathbf{H}_2(t=\frac{3T}{2},\ z=0) \ = \ 2H_\mathrm{A}\cos\frac{3\pi}{2}\,\mathbf{e}_x \ = \ 0\ .$$

Vectors always lie on the same line (x-axis) $\Rightarrow$ *linearly polarized.*

1.4 $\mathbf{H}_1(t)+\mathbf{H}_3(t)=(H_\mathrm{A}+H_\mathrm{B})\cos(\omega t-kz)\,\mathbf{e}_x+(H_\mathrm{B}-H_\mathrm{A})\sin(\omega t-kz)\,\mathbf{e}_y$. Consider the equiphase plane $z=0$ and time instants $t=0,\ \frac{T}{4},\ \frac{T}{2},\ \frac{3T}{4}$, we can get with $H_\mathrm{A}>0,\ H_\mathrm{B}>0,\ H_\mathrm{A}<H_\mathrm{B}\Rightarrow H_\mathrm{B}-H_\mathrm{A}>0$

$$\begin{aligned}
\mathbf{H}_1(t=0,\ z=0)+\mathbf{H}_3(t=0,\ z=0) &= (H_\mathrm{A}+H_\mathrm{B})\,\mathbf{e}_x\\
\mathbf{H}_1(t=\frac{T}{4},\ z=0)+\mathbf{H}_3(t=\frac{T}{4},\ z=0) &= (H_\mathrm{B}-H_\mathrm{A})\,\mathbf{e}_y\\
\mathbf{H}_1(t=\frac{T}{2},\ z=0)+\mathbf{H}_3(t=\frac{T}{2},\ z=0) &= -(H_\mathrm{A}+H_\mathrm{B})\,\mathbf{e}_x\\
\mathbf{H}_1(t=\frac{3T}{4},\ z=0)+\mathbf{H}_3(t=\frac{3T}{4},\ z=0) &= -(H_\mathrm{B}-H_\mathrm{A})\,\mathbf{e}_y\ .
\end{aligned}$$

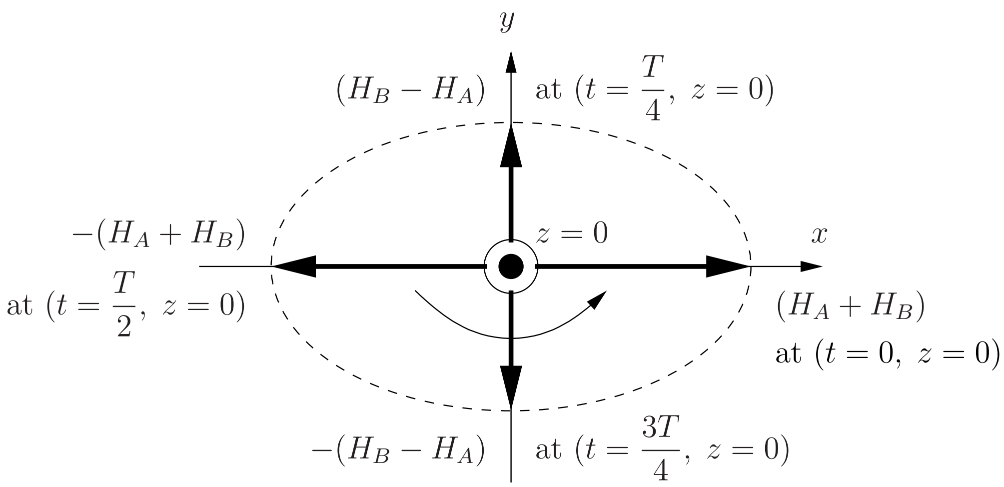

Figure 2: Polarization type of $\mathbf{H}_1(t)+\mathbf{H}_3(t)$.

Figure 2 illustrates the polarization type of $\mathbf{H}_1(t)+\mathbf{H}_3(t)$.

Observe the figure and note the *anticlockwise* rotation direction with the *different* length we can draw the following conclusion that $\mathbf{H}_1(t)+\mathbf{H}_3(t)$ is *right elliptically polarized.*

Some Remarks on Planar Wave in an Arbitrary Direction

From the lecture we know that a generalized planar harmonic wave propagating in an arbitrary direction, which is specified by its electric field, can be represented as

$$\underline{\mathbf{E}} = (\underline{E}_1\mathbf{e}_1 + \underline{E}_2\mathbf{e}_2)\,\mathrm{e}^{-\mathrm{j}\mathbf{k}\cdot\mathbf{r}} \quad (6)$$

and a time-dependent electric wave propagating along the direction of $\mathbf{k}$ can be expressed as

$$\begin{aligned}\mathbf{E}(t) &= \mathrm{Re}\left\{\underline{\mathbf{E}}\,\mathrm{e}^{\mathrm{j}\omega t}\right\}\\ &= \Big[E_1(t)\mathbf{e}_1 + E_2(t)\mathbf{e}_2\Big]\mathrm{e}^{-\mathrm{j}\mathbf{k}\cdot\mathbf{r}}\\ &= \Big[\hat{E}_1\cos(\omega t+\varphi_1)\,\mathbf{e}_1 + \hat{E}_2\cos(\omega t+\varphi_2)\,\mathbf{e}_2\Big]\,\mathrm{e}^{-\mathrm{j}\mathbf{k}\cdot\mathbf{r}} \quad (7)\end{aligned}$$

with $\mathbf{e}_1\cdot\mathbf{k}=0$, $\mathbf{e}_2\cdot\mathbf{k}=0$ and $\mathbf{e}_1\cdot\mathbf{e}_2=0$, i.e., they follow the *right hand orthogonal rule*, see Figure 3.

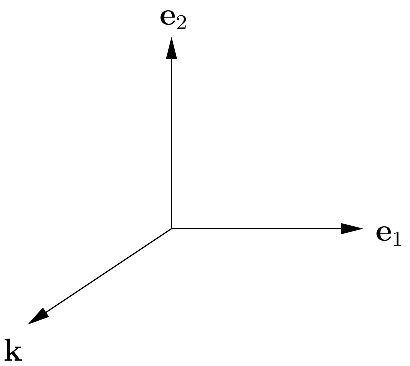

Figure 3: Right hand orthogonal rule of $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{k}$.

In the cartesian coordinate system, a *location vector* $\mathbf{r}$ can be represented as

$$\mathbf{r} = x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z\ .$$

$\mathbf{k}$ is *vector wave number* and defined as

$$\mathbf{k} = k_x\mathbf{e}_x + k_y\mathbf{e}_y + k_z\mathbf{e}_z$$

with relationship to *scalar wave number* k

$$\mathbf{k}\cdot\mathbf{k} = k^2 \quad\Rightarrow\quad k = \sqrt{k_x^2+k_y^2+k_z^2}\ .$$

We have found that the fields of the electromagnetic wave are *perpendicular to each other*, and that they are also *perpendicular* (or *transverse*) *to* the direction of propagation $\mathbf{k}$.

Electromagnetic power flows with the wave along the direction of propagation and it is also constant on the equiphase planes. The power density is described by the time dependent Poynting vector

$$\mathbf{P}(t) \;=\; \mathbf{E}(t) \times \mathbf{H}(t)\ .$$

The Poynting vector is *perpendicular to both field components*, and is *parallel to* the direction of wave propagation. It means that the following relationships

$$\mathbf{k} \perp \underline{\mathbf{E}},\ \mathbf{k} \perp \underline{\mathbf{H}},\ \underline{\mathbf{E}} \perp \underline{\mathbf{H}}\ \text{ and }\ \mathbf{k} \parallel (\underline{\mathbf{E}} \times \underline{\mathbf{H}} \;=\; \underline{\mathbf{P}})$$

hold true.

Solution of Problem 2

2.1 A homogeneous planar wave one holds

$$\mathbf{n}_1 \perp \underline{\mathbf{E}},\quad \mathbf{n}_1 \perp \underline{\mathbf{H}},\quad \underline{\mathbf{E}} \perp \underline{\mathbf{H}} \quad\Rightarrow\quad \mathbf{n}_1 \parallel (\underline{\mathbf{E}} \times \underline{\mathbf{H}})$$

as illustrated in Figure 4.

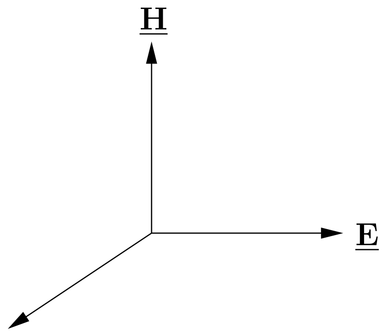

Figure 4: Right hand orthogonal rule of $\underline{\mathbf{E}}$, $\underline{\mathbf{H}}$ and $\mathbf{k}_1 = k_1\mathbf{n}_1$.

Since E_1 and H_1 are constants and $\mathrm{e}^{-\mathbf{j}\mathbf{k}\cdot\mathbf{r}}$ denotes a certain phase, we can obtain

$$(\underline{\mathbf{E}} \times \underline{\mathbf{H}}) \;\parallel\; \left(\frac{1}{2}\mathbf{e}_x - \mathbf{e}_y + 2\mathbf{e}_z\right) \times (-2\mathbf{e}_x + \mathbf{e}_y + \mathbf{e}_z)$$

$$\Rightarrow\quad (\underline{\mathbf{E}} \times \underline{\mathbf{H}}) \;\parallel\; \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ \frac{1}{2} & -1 & 2 \\ -2 & 1 & 1 \end{vmatrix} \;=\; -\left(3\mathbf{e}_x + \frac{9}{2}\mathbf{e}_y + \frac{3}{2}\mathbf{e}_z\right)\ .$$

Hence,

$$\mathbf{n}_1 \;\parallel\; -\left(3\mathbf{e}_x + \frac{9}{2}\mathbf{e}_y + \frac{3}{2}\mathbf{e}_z\right)\ .$$

Normalization $|\mathbf{n}_1| = 1$ yields (*both propagation directions are possible*!)

$$\mathbf{n}_1 = \pm\frac{1}{\sqrt{3^2+\left(\frac{9}{2}\right)^2+\left(\frac{3}{2}\right)^2}}\left(3\mathbf{e}_x+\frac{9}{2}\mathbf{e}_y+\frac{3}{2}\mathbf{e}_z\right)$$

$$\Rightarrow \quad \mathbf{n}_1 = \pm\frac{1}{\sqrt{14}}\left(2\mathbf{e}_x+3\mathbf{e}_y+\mathbf{e}_z\right) .$$

2.2

$$k_1 = \omega\sqrt{\varepsilon_1\mu_1} = \omega\sqrt{\varepsilon_{r1}\varepsilon_0\mu_{r1}\mu_0} = \underbrace{\sqrt{\varepsilon_{r1}\mu_{r1}}}_{=\sqrt{3}}\underbrace{\omega\sqrt{\varepsilon_0\mu_0}}_{=k_0} = \sqrt{3}k_0 ,$$

where k_0 is the scalar wave number of vacuum. Therefore,

$$\mathbf{k}_1 = k_1\mathbf{n}_1 = \sqrt{3}k_0\mathbf{n}_1 = \pm\sqrt{\frac{3}{14}}\,k_0\left(2\mathbf{e}_x+3\mathbf{e}_y+\mathbf{e}_z\right) .$$

2.3 A time dependent electric field $\mathbf{E}_1(t)$ can be described as

$$\begin{aligned}\mathbf{E}_1(t) &= \mathrm{Re}\left\{\underline{\mathbf{E}}_1 \mathrm{e}^{\mathrm{j}\omega t}\right\}\\ &= \mathrm{Re}\left\{\mathbf{E}_1 \mathrm{e}^{-\mathrm{j}\mathbf{k}_1\cdot\mathbf{r}}\mathrm{e}^{\mathrm{j}\omega t}\right\}\\ &= \mathrm{Re}\left\{\mathbf{E}_1 \mathrm{e}^{\mathrm{j}(\omega t-\mathbf{k}_1\cdot\mathbf{r})}\right\}\\ &= \mathbf{E}_1\cos\left(\omega t-\mathbf{k}_1\cdot\mathbf{r}\right) ,\end{aligned}$$

where $\mathbf{E}_1 = E_1\left(\frac{1}{2}\mathbf{e}_x-\mathbf{e}_y+2\mathbf{e}_z\right)$ and the planar electromagnetic wave propagates along the direction of $\mathbf{k}_1$.

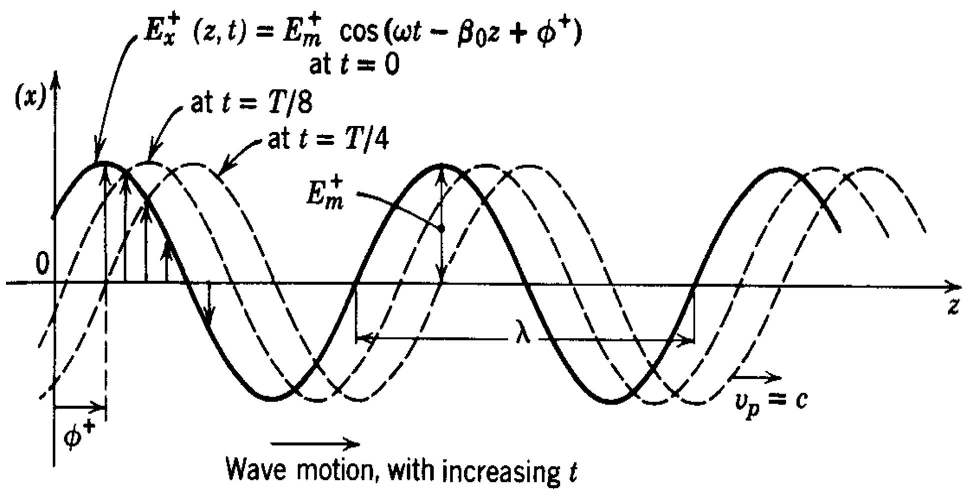

Figure 5: Example of a homogeneous planar wave $E_x^+(z,t)$, which is propagating along the positive z-axis in vacuum. Here, β_0 is the wave number of vacuum.

For reasons of simplicity, we assume the wave with the electric field

$$\mathbf{E}_1(t) = \mathbf{E}_1 \cos\left(\omega t - |\mathbf{k}_1|\, z\right) = \mathbf{E}_1 \cos\left(\omega t - k_1 z\right) ,$$

which propagates along the positive z-axis.

For the isotropic medium one can assume that the wave does not change behavior with its direction. See Figure 5.

Looking at equiphase planes, one obtains

$$\omega t - |\mathbf{k}_1|\, z = \omega t - k_1 z = \text{constant} .$$

Thus, we can get

$$v_{\mathrm{ph}} = \frac{\mathrm{d}z}{\mathrm{d}t} = \frac{\omega}{|\mathbf{k}_1|} = \frac{\omega}{\omega\sqrt{\varepsilon_1 \mu_1}} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}\sqrt{\varepsilon_{\mathrm{r}1}\mu_{\mathrm{r}1}}} = \frac{c}{\sqrt{\varepsilon_{\mathrm{r}1}\mu_{\mathrm{r}1}}} = \frac{c}{\sqrt{3}} .$$