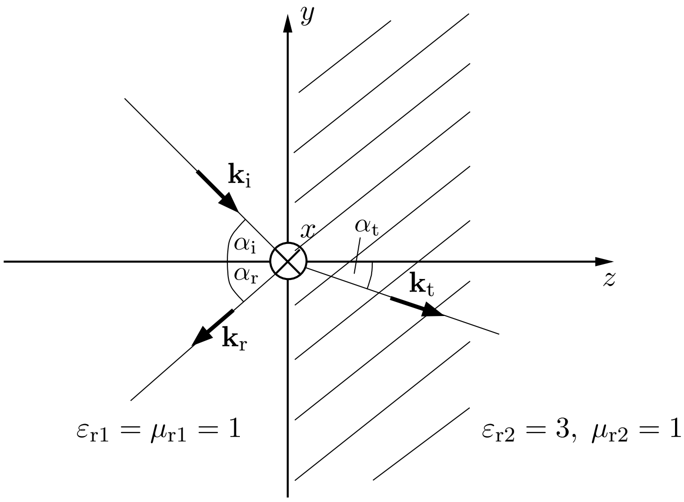
Reflection and Refraction of Electromagnetic Waves

A homogeneous, plane wave specified by its phasor of the electric field

$$\underline{\mathbf{E}}_{\mathrm{i}} = E_0 \left(-\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y + \frac{\sqrt{3}}{2}\mathbf{e}_z \right) \mathrm{e}^{-\mathrm{j}\mathbf{k}_{\mathrm{i}}\cdot\mathbf{r}} \tag{1}$$

is incident at the air-dielectric fluid boundary ($z = 0$) with an angle of incidence α_{i}. Assume that the x-component of the wave vector $\mathbf{k}_{\mathrm{i}}$ is zero.



Problem 1

1.1 Determine the phasor $\underline{\mathbf{E}}_{\mathrm{i}\perp}$ that is perpendicular to the incident plane.

1.2 Determine the phasor $\underline{\mathbf{E}}_{\mathrm{i}\parallel}$ that is parallel to the incident plane.

1.3 Determine the wave vector $\mathbf{k}_{\mathrm{i}}$ of the incident wave.

1.4 Determine the angle of reflection α_{r}.

1.5 Determine the wave vector $\mathbf{k}_{\mathrm{r}}$ of the reflected wave.

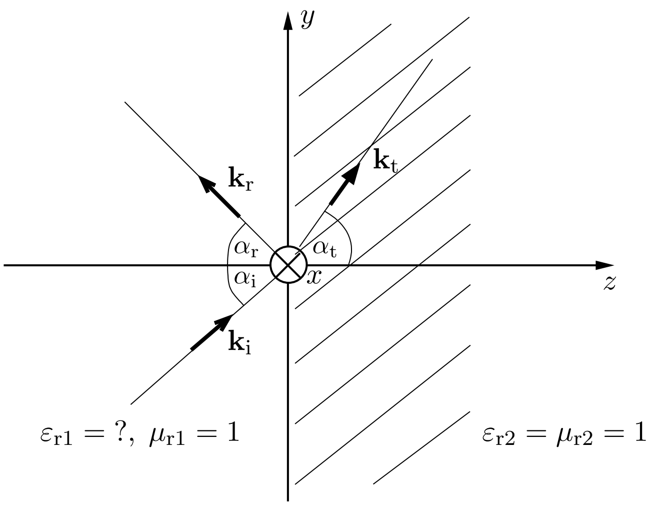
1.6 Determine $\underline{E}_{\mathrm{r}\perp}$ and $\underline{E}_{\mathrm{r}\parallel}$ of the reflected wave.

A homogeneous, plane wave with the electric field

$$\underline{\mathbf{E}}_\mathrm{i} \;=\; E_0\mathbf{e}_{E_\mathrm{i}}\mathrm{e}^{-\mathrm{j}\,k_0\left(\frac{\sqrt{3}}{2}\mathbf{e}_y+\frac{3}{2}\mathbf{e}_z\right)\cdot\mathbf{r}} \tag{2}$$

is incident at the dielectric fluid – air boundary ($z = 0$) with an angle of incidence α_i. Following assumptions are made

$$|E_0\mathbf{e}_{E_i}| \;=\; \sqrt{2}\,\mathrm{V/m}, \quad E_{\mathrm{i}x} > 0,\; E_{\mathrm{i}y} > 0,\; |E_{\mathrm{i}x}| \;=\; 2\,|E_{\mathrm{i}z}|\;. \tag{3}$$



Problem 2

2.1 Determine the permittivity ε_r1.

2.2 Determine the amplitude vector $E_0\mathbf{e}_{E_i}$.

2.3 Determine the wave vector $\mathbf{k}_\mathrm{r}$ of the reflected wave.

2.4 Determine the wave vector $\mathbf{k}_\mathrm{t}$ of the transmitted wave.

2.5 Determine the transmission coefficients $\underline{t}_\perp$ and $\underline{t}_\parallel$.

2.6 Determine the incident angle $\alpha_\mathrm{i,tot}$, so that total reflection would occur.

A homogeneous, plane wave with the electric field $\underline{\mathbf{E}}_2$ propagates in medium 2

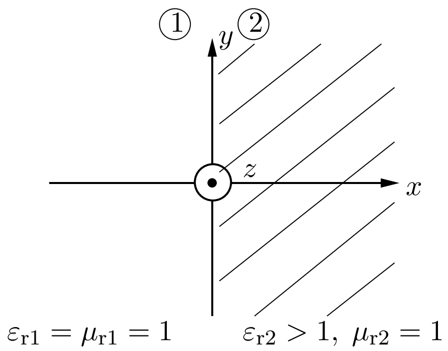
$$\underline{\mathbf{E}}_2 = E_0 \left(\mathbf{e}_x - 3\mathbf{e}_y\right) \mathrm{e}^{-\mathrm{j}\mathbf{k}_2\cdot\mathbf{r}} \,. \tag{4}$$

The phasor of the electric field $\underline{\mathbf{E}}_1$ in medium 1 is given by

$$\underline{\mathbf{E}}_1 = E_0 \mathbf{e}_{E_1} \mathrm{e}^{-\mathrm{j}\mathbf{k}_1\cdot\mathbf{r}} \,. \tag{5}$$

Furthermore, assume that there is no reflected wave and

$$E_{1z} = E_{2z} = 0, \quad k_{1z} = k_{2z} = 0, \quad k_{2x} > 0,\ k_{1x} > 0 \,. \tag{6}$$



Problem 3

3.1 Determine the normalized direction vector $\mathbf{n}_2$ ($\mathbf{k}_2 = k_2\mathbf{n}_2$).

3.2 Determine the normalized direction vector $\mathbf{n}_1$ ($\mathbf{k}_1 = k_1\mathbf{n}_1$).

3.3 Determine the permittivity ε_{r2}.

3.4 Determine the electric field $\underline{\mathbf{E}}_1$.

3.5 Determine the magnetic field $\underline{\mathbf{H}}_1$.

Solution of Problem 1

From the wave vector

$$\mathbf{k}_\mathrm{i} \;=\; k_{\mathrm{i}x}\mathbf{e}_x + k_{\mathrm{i}y}\mathbf{e}_y + k_{\mathrm{i}z}\mathbf{e}_z$$

of the incident electric wave

$$\underline{\mathbf{E}}_\mathrm{i} \;=\; E_0\left(-\mathbf{e}_x + \frac{1}{2}\mathbf{e}_y + \frac{\sqrt{3}}{2}\mathbf{e}_z\right)\mathrm{e}^{-\mathrm{j}\mathbf{k}_\mathrm{i}\cdot\mathbf{r}}$$

having zero x-component, i.e., $k_{\mathrm{i}x} = 0$, we can obtain that

$$\mathbf{k}_\mathrm{i} \;=\; k_{\mathrm{i}y}\mathbf{e}_y + k_{\mathrm{i}z}\mathbf{e}_z$$

and the *incident plane* is *yoz-plane*. See the following figure.

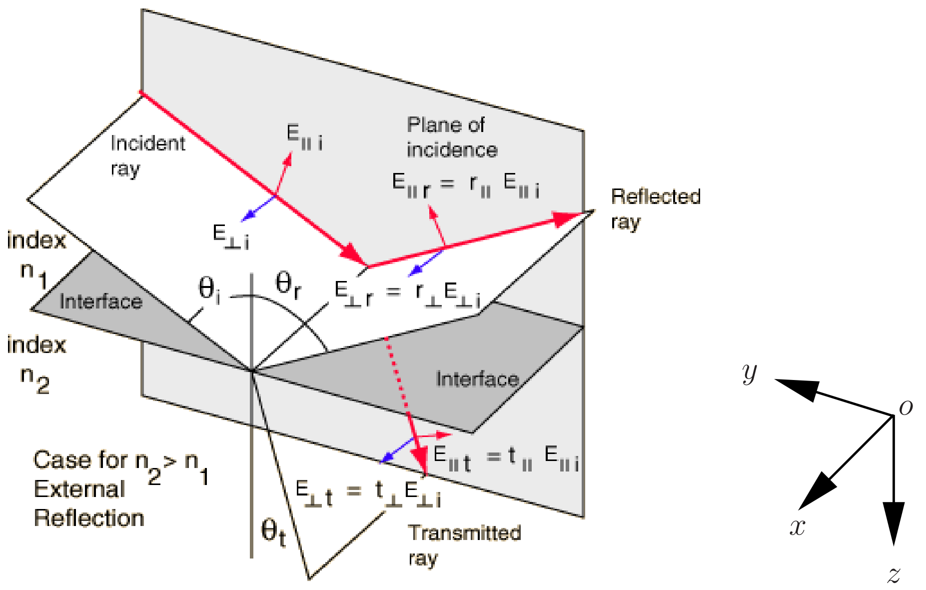


Figure 1: Illustration of reflection and refraction of the incident electric wave $\underline{\mathbf{E}}_\mathrm{i}$.

From the lecture we know that the incident electric wave can be represented as

$$\underline{\mathbf{E}}_\mathrm{i} \;=\; \left(\underline{\mathbf{E}}_{\mathrm{i}\perp} + \underline{\mathbf{E}}_{\mathrm{i}\|}\right)\mathrm{e}^{-\mathrm{j}\mathbf{k}_\mathrm{i}\cdot\mathbf{r}} \;=\; \left(\underline{E}_{\mathrm{i}\perp}\mathbf{e}_{\mathrm{i}\perp} + \underline{E}_{\mathrm{i}\|}\mathbf{e}_{\mathrm{i}\|}\right)\mathrm{e}^{-\mathrm{j}\mathbf{k}_\mathrm{i}\cdot\mathbf{r}}\,, \tag{7}$$

where $\underline{\mathbf{E}}_{\mathrm{i}\perp}$ is perpendicular to the plane of incidence (*yoz-plane*) and $\underline{\mathbf{E}}_{\mathrm{i}\|}$ is parallel to the incident plane *yoz-plane*). Therefore, $\underline{\mathbf{E}}_{\mathrm{i}\perp}$ has only x-component and $\underline{\mathbf{E}}_{\mathrm{i}\|}$ has both y-component and z-component.

1.1 Since the plane of incidence is yoz-plane, we can conclude that $\underline{\mathbf{E}}_{\mathrm{i}\perp}$ has only x-component, see Eq. (1) and Eq. (7),

$$\underline{\mathbf{E}}_{\mathrm{i}\perp} \;=\; \underline{E}_{\mathrm{i}\perp}\mathbf{e}_{\mathrm{i}\perp} \;=\; -E_0\mathbf{e}_x$$

$$\Rightarrow\quad \mathbf{e}_{\mathrm{i}\perp} \;=\; \mathbf{e}_x \quad\text{and}\quad \underline{E}_{\mathrm{i}\perp} \;=\; -E_0\,.$$

1.2 Since the plane of incidence is yoz-plane, we can conclude that $\underline{\mathbf{E}}_{\mathrm{i}\|}$ consists of both y-component and z-component, see Eq. (1) and Eq. (7),

$$\underline{\mathbf{E}}_{\mathrm{i}\|} = \underline{E}_{\mathrm{i}\|}\mathbf{e}_{\mathrm{i}\|} = E_0\left(\frac{1}{2}\mathbf{e}_y + \frac{\sqrt{3}}{2}\mathbf{e}_z\right)$$

$$\Rightarrow \quad \mathbf{e}_{\mathrm{i}\|} = \frac{1}{2}\mathbf{e}_y + \frac{\sqrt{3}}{2}\mathbf{e}_z \quad \text{and} \quad \underline{E}_{\mathrm{i}\|} = E_0 \ .$$

1.3 This problem can be solved with the following two approaches.

Assume the incident wave vector is

$$\mathbf{k}_\mathrm{i} = k_0\left(a\mathbf{e}_x + b\mathbf{e}_y + c\mathbf{e}_z\right) \ ,$$

where k_0 is the wave number of vacuum. Under the condition that $\mathbf{k}_\mathrm{i}$ has zero x-component we can obtain $a = 0$ and therefore

$$\mathbf{k}_\mathrm{i} = k_0\left(b\mathbf{e}_y + c\mathbf{e}_z\right) \ .$$

1st Approach:

Since the incident wave vector $\mathbf{k}_\mathrm{i}$ is perpendicular to the incident electric wave $\underline{\mathbf{E}}_\mathrm{i}$, i.e., $\mathbf{k}_\mathrm{i} \perp \underline{\mathbf{E}}_\mathrm{i}$, one can get

$$\mathbf{k}_\mathrm{i} \cdot \underline{\mathbf{E}}_\mathrm{i} = \begin{pmatrix} 0 \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} -1 \\ \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix} = \frac{1}{2}b + \frac{\sqrt{3}}{2}c = 0$$

$$\Rightarrow \quad b = -\sqrt{3}c \ . \tag{8}$$

Now consider the magnitude of the incident wave vector $\mathbf{k}_\mathrm{i}$

$$|\mathbf{k}_\mathrm{i}| = \omega\sqrt{\varepsilon\mu} = \underbrace{\omega\sqrt{\varepsilon_0\mu_0}}_{=k_0}\underbrace{\sqrt{\varepsilon_{\mathrm{r}1}\mu_{\mathrm{r}1}}}_{=1} = k_0$$

and hence

$$\frac{|\mathbf{k}_\mathrm{i}|}{k_0} = \sqrt{b^2 + c^2} \overset{(8)}{=} \sqrt{3c^2 + c^2} = 1$$

$$\Rightarrow \quad c = \pm\frac{1}{2} \quad \text{and} \quad b = -\sqrt{3}c = \mp\frac{\sqrt{3}}{2} \ . \tag{9}$$

Inserting Eq. (9) into $\mathbf{k}_\mathrm{i} = k_0\left(b\mathbf{e}_y + c\mathbf{e}_z\right)$ we can get

$$\mathbf{k}_\mathrm{i} = \pm k_0\left(-\frac{\sqrt{3}}{2}\mathbf{e}_y + \frac{1}{2}\mathbf{e}_z\right) \ .$$

However, from Figure 2, note that the y-component of $\mathbf{k}_\mathrm{i}$ propagates in the direction of negative y-axis and z-component of $\mathbf{k}_\mathrm{i}$ propagates in the direction of positive z-axis, then

$$c = \frac{1}{2} \quad \text{and} \quad b = -\sqrt{3}c = -\frac{\sqrt{3}}{2}$$

and finally the incident wave vector is

$$\mathbf{k}_\mathrm{i} = k_0 \left(-\frac{\sqrt{3}}{2}\mathbf{e}_y + \frac{1}{2}\mathbf{e}_z \right) .$$

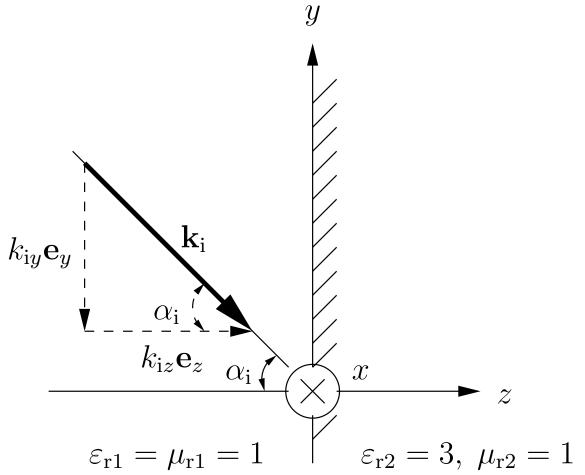


Figure 2: Illustration of the incident wave vector $\mathbf{k}_\mathrm{i} = k_{\mathrm{i}y}\mathbf{e}_y + k_{\mathrm{i}z}\mathbf{e}_z$.

2nd Approach:

Start from $\mathbf{k}_\mathrm{i} \perp \underline{\mathbf{E}}_{\mathrm{i}\perp}$ and $\mathbf{k}_\mathrm{i} \perp \underline{\mathbf{E}}_{\mathrm{i}\|}$, see Figure 1, and note that E_0 is a constant we can conclude that

$$\mathbf{k}_\mathrm{i} \parallel \left(\underline{\mathbf{E}}_{\mathrm{i}\perp} \times \underline{\mathbf{E}}_{\mathrm{i}\|}\right) \parallel (-\mathbf{e}_x) \times \left(\frac{1}{2}\mathbf{e}_y + \frac{\sqrt{3}}{2}\mathbf{e}_z \right)$$

$$= \begin{vmatrix} \mathbf{e}_x & \mathbf{e}_y & \mathbf{e}_z \\ -1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \end{vmatrix} = \frac{\sqrt{3}}{2}\mathbf{e}_y - \frac{1}{2}\mathbf{e}_z . \tag{10}$$

Consider the magnitude of the incident wave vector $\mathbf{k}_\mathrm{i}$

$$|\mathbf{k}_\mathrm{i}| = \omega\sqrt{\varepsilon\mu} = \underbrace{\omega\sqrt{\varepsilon_0\mu_0}}_{=k_0}\underbrace{\sqrt{\varepsilon_\mathrm{r1}\mu_\mathrm{r1}}}_{=1} = k_0 \quad \text{and} \quad \left| \frac{\sqrt{3}}{2}\mathbf{e}_y - \frac{1}{2}\mathbf{e}_z \right| = 1$$

we can conclude that

$$\mathbf{k}_\mathrm{i} \;=\; \pm k_0\left(\frac{\sqrt{3}}{2}\mathbf{e}_y - \frac{1}{2}\mathbf{e}_z\right) \;=\; \pm k_0\left(-\frac{\sqrt{3}}{2}\mathbf{e}_y + \frac{1}{2}\mathbf{e}_z\right) \;.$$

Figure 2 illustrates that the y-component of $\mathbf{k}_\mathrm{i}$ propagates in the direction of negative y-axis and z-component of $\mathbf{k}_\mathrm{i}$ propagates in the direction of positive z-axis, then finally

$$\mathbf{k}_\mathrm{i} \;=\; k_0\left(-\frac{\sqrt{3}}{2}\mathbf{e}_y + \frac{1}{2}\mathbf{e}_z\right) \;.$$

1.4 From the results of subproblem 1.3 and Figure 3 we can get

$$\tan\alpha_\mathrm{i} \;=\; \left|\frac{k_{\mathrm{i}y}}{k_{\mathrm{i}z}}\right| \;=\; \left|\frac{-\frac{\sqrt{3}}{2}k_0}{\frac{1}{2}k_0}\right| \;=\; \sqrt{3}$$

$$\Rightarrow \quad \alpha_\mathrm{r} \;=\; \alpha_\mathrm{i} \;=\; \arctan\sqrt{3} \;=\; 60^\circ \;.$$

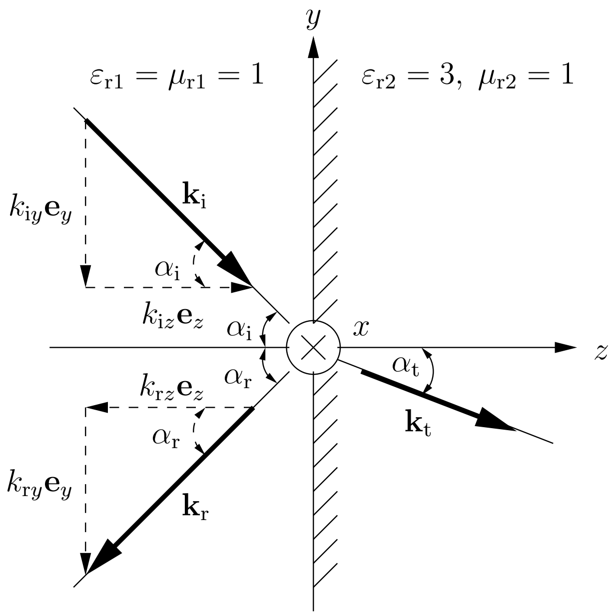


Figure 3: Illustration of the incident wave vector $\mathbf{k}_\mathrm{i} = k_{\mathrm{i}y}\mathbf{e}_y + k_{\mathrm{i}z}\mathbf{e}_z$, reflected wave vector $\mathbf{k}_\mathrm{r} = k_{\mathrm{r}y}\mathbf{e}_y + k_{\mathrm{r}z}\mathbf{e}_z$ and the transmitted wave vector $\mathbf{k}_\mathrm{t}$.

1.5 For the reflected wave, see Figure 3, it is valid with the results of subproblem 1.3

$$k_{\mathrm{r}y} \;=\; k_{\mathrm{i}y} \;=\; -\frac{\sqrt{3}}{2}k_0$$

$$k_{\mathrm{r}z} \;=\; -k_{\mathrm{i}z} \;=\; -\frac{1}{2}k_0$$

$$\Rightarrow \quad \mathbf{k}_\mathrm{r} \;=\; k_0\left(-\frac{1}{2}\sqrt{3}\mathbf{e}_y - \frac{1}{2}\mathbf{e}_z\right) \;.$$

1.6 From the Fresnel's equation we can calculate the reflection coefficients for the parallel (to the plane of incidence) and perpendicular (to the plane of incidence) part of the electric field and note that the refraction index $n = \sqrt{\varepsilon_\mathrm{r}}$ and $\alpha_\mathrm{i} = 60°$

$$\underline{r}_\perp = \frac{\underline{E}_{\mathrm{r}\perp}}{\underline{E}_{\mathrm{i}\perp}} = -\frac{n_2^2\cos\alpha_\mathrm{i} - n_1\sqrt{n_2^2 - n_1^2\sin^2\alpha_\mathrm{i}}}{n_2^2\cos\alpha_\mathrm{i} + n_1\sqrt{n_2^2 - n_1^2\sin^2\alpha_\mathrm{i}}} = -\frac{\frac{3}{2}-\frac{3}{2}}{\frac{3}{2}+\frac{3}{2}} = 0$$

$$\Rightarrow \quad \alpha_\mathrm{i} \text{ is Brewster's angle}$$

$$\underline{r}_\parallel = \frac{\underline{E}_{\mathrm{r}\parallel}}{\underline{E}_{\mathrm{i}\parallel}} = \frac{n_1\cos\alpha_\mathrm{i} - \sqrt{n_2^2 - n_1^2\sin^2\alpha_\mathrm{i}}}{n_1\cos\alpha_\mathrm{i} + \sqrt{n_2^2 - n_1^2\sin^2\alpha_\mathrm{i}}} = \frac{\frac{1}{2}-\frac{3}{2}}{\frac{1}{2}+\frac{3}{2}} = -\frac{1}{2}\,.$$

$\underline{E}_{\mathrm{r}\perp}$ and $\underline{E}_{\mathrm{r}\parallel}$ are then (coefficients do not contain information about directions) with the results of subproblems 1.1 and 1.2

$$\underline{E}_{\mathrm{r}\perp} = \underline{r}_\perp\underline{E}_{\mathrm{i}\perp} = 0$$

$$\underline{E}_{\mathrm{r}\parallel} = \underline{r}_\parallel\underline{E}_{\mathrm{i}\parallel} = -\frac{1}{2}\underline{E}_{\mathrm{i}\parallel} = -\frac{1}{2}E_0\,.$$

Solution of Problem 2

2.1 From the representation of the incident electric field

$$\underline{\mathbf{E}}_\mathrm{i} = E_0\mathbf{e}_{E_\mathrm{i}}\mathrm{e}^{-\mathrm{j}\,k_0\left(\frac{\sqrt{3}}{2}\mathbf{e}_y+\frac{3}{2}\mathbf{e}_z\right)\cdot\mathbf{r}}$$

we can know that the incident wave vector

$$\mathbf{k}_\mathrm{i} = k_0\left(\frac{\sqrt{3}}{2}\mathbf{e}_y + \frac{3}{2}\mathbf{e}_z\right) \tag{11}$$

has zero x-component, i.e., the plane of incidence is *yoz-plane.*

Note that the magnitude of the incident wave vector holds

$$|\mathbf{k}_\mathrm{i}| = \omega\sqrt{\varepsilon\mu} = \underbrace{\omega\sqrt{\varepsilon_0\mu_0}}_{=k_0}\sqrt{\varepsilon_{\mathrm{r}1}\mu_{\mathrm{r}1}} = k_0\sqrt{\varepsilon_{\mathrm{r}1}\mu_{\mathrm{r}1}}\,. \tag{12}$$

Combining Eq. (11) and Eq. (12) we can obtain with $\mu_{\mathrm{r}1} = 1$

$$|\mathbf{k}_\mathrm{i}| = k_0\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\right)^2} = k_0\sqrt{3} \stackrel{!}{=} k_0\sqrt{\varepsilon_{\mathrm{r}1}\mu_{\mathrm{r}1}} = k_0\sqrt{\varepsilon_{\mathrm{r}1}}$$

$$\Rightarrow \quad \varepsilon_{\mathrm{r}1} = 3\,.$$

2.2 $E_0\mathbf{e}_{E_1}$ can be written as

$$E_0\mathbf{e}_{E_1} \;=\; E_{\mathrm{i}x}\mathbf{e}_x + E_{\mathrm{i}y}\mathbf{e}_y + E_{\mathrm{i}z}\mathbf{e}_z \;. \tag{13}$$

Since $\mathbf{k}_\mathrm{i} \perp E_0\mathbf{e}_{E_1}$, it equivalently means $\mathbf{k}_\mathrm{i} \cdot E_0\mathbf{e}_{E_1} = 0$, we can get

$$\begin{pmatrix} 0 \\ \frac{\sqrt{3}}{2} \\ \frac{3}{2} \end{pmatrix} \cdot \begin{pmatrix} E_{\mathrm{i}x} \\ E_{\mathrm{i}y} \\ E_{\mathrm{i}z} \end{pmatrix} \;=\; \frac{\sqrt{3}}{2}E_{\mathrm{i}y} + \frac{3}{2}E_{\mathrm{i}z} \;=\; 0 \quad\Rightarrow\quad E_{\mathrm{i}y} \;=\; -\sqrt{3}E_{\mathrm{i}z} \;. \tag{14}$$

With the assumption $|E_{\mathrm{i}x}| \;=\; 2\,|E_{\mathrm{i}z}|$ one obtains

$$E_{\mathrm{i}x} \;=\; \pm 2E_{\mathrm{i}z} \;. \tag{15}$$

Inserting Eq. (14) and Eq. (15) into Eq. (13) yields

$$E_0\mathbf{e}_{E_1} \;=\; \pm 2E_{\mathrm{i}z}\mathbf{e}_x - \sqrt{3}E_{\mathrm{i}z}\mathbf{e}_y + E_{\mathrm{i}z}\mathbf{e}_z \;.$$

The absolute value of $E_0\mathbf{e}_{E_1}$ is given by $|E_0\mathbf{e}_{E_i}| = \sqrt{2}\,\frac{\mathrm{V}}{\mathrm{m}}$, therefore

$$\begin{aligned} |E_0\mathbf{e}_{E_i}| \;&=\; \sqrt{4E_{\mathrm{i}z}^2 + 3E_{\mathrm{i}z}^2 + E_{\mathrm{i}z}^2} \;=\; \sqrt{8}\,|E_{\mathrm{i}z}| \;\stackrel{!}{=}\; \sqrt{2}\,\frac{\mathrm{V}}{\mathrm{m}} \\ \Rightarrow\quad E_{\mathrm{i}z} \;&=\; \pm\frac{1}{2}\,\frac{\mathrm{V}}{\mathrm{m}} \\ \Rightarrow\quad E_{\mathrm{i}x} \;&\stackrel{(15)}{=}\; \pm 1\,\frac{\mathrm{V}}{\mathrm{m}} \\ \Rightarrow\quad E_{\mathrm{i}y} \;&\stackrel{(14)}{=}\; \mp\frac{\sqrt{3}}{2}\,\frac{\mathrm{V}}{\mathrm{m}} \;. \end{aligned}$$

Then, the amplitude vector $E_0\mathbf{e}_{E_i}$ of the incident electric field $\underline{\mathbf{E}}_\mathrm{i}$ with the assumptions $E_{\mathrm{i}x} > 0$ and $E_{\mathrm{i}y} > 0$ can be obtained as

$$E_0\mathbf{e}_{E_i} \;=\; \left(1\,\frac{\mathrm{V}}{\mathrm{m}}\,\mathbf{e}_x + \frac{\sqrt{3}}{2}\,\frac{\mathrm{V}}{\mathrm{m}}\,\mathbf{e}_y - \frac{1}{2}\,\frac{\mathrm{V}}{\mathrm{m}}\,\mathbf{e}_z\right) \;.$$

2.3 Here, $\mathbf{e}_\mathrm{n}$ is defined as the unit vector that is normal to the boundary surface $z = 0$. Hence, from

$$\mathbf{k}_\mathrm{i} \;=\; k_0\left(\frac{\sqrt{3}}{2}\mathbf{e}_y + \frac{3}{2}\mathbf{e}_z\right)$$

and

$$\mathbf{e}_\mathrm{n}\cdot\mathbf{k}_\mathrm{i} \;=\; -\mathbf{e}_\mathrm{n}\cdot\mathbf{k}_\mathrm{r}$$

one can get as illustrated in Figure 4

$$\begin{aligned} k_{\mathrm{r}y} \;&=\; k_{\mathrm{i}y} \;=\; \frac{\sqrt{3}}{2}k_0 \\ k_{\mathrm{r}z} \;&=\; -k_{\mathrm{i}z} \;=\; -\frac{3}{2}k_0 \;. \end{aligned}$$

Finally we can obtain

$$\mathbf{k}_\mathrm{r} \;=\; k_0\left(\frac{\sqrt{3}}{2}\mathbf{e}_y - \frac{3}{2}\mathbf{e}_z\right) \;.$$

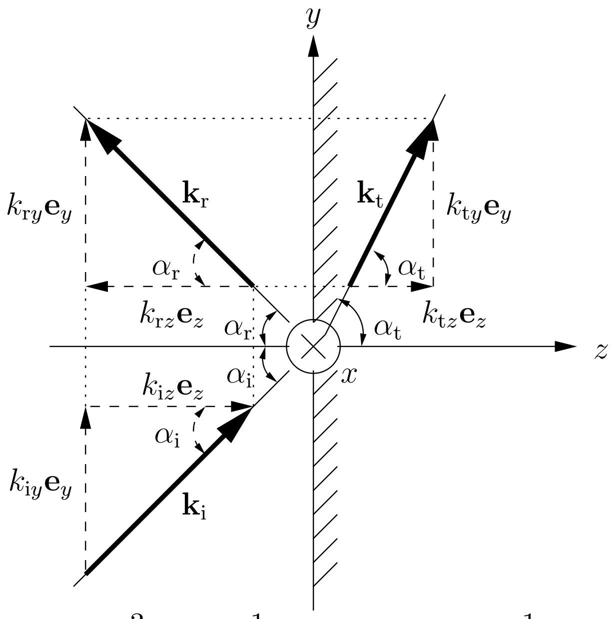


Figure 4: Illustration of the incident wave vector $\mathbf{k}_\mathrm{i} = k_{\mathrm{i}y}\mathbf{e}_y + k_{\mathrm{i}z}\mathbf{e}_z$, reflected wave vector $\mathbf{k}_\mathrm{r} = k_{\mathrm{r}y}\mathbf{e}_y + k_{\mathrm{r}z}\mathbf{e}_z$ and the transmitted wave vector $\mathbf{k}_\mathrm{t} = k_{\mathrm{t}y}\mathbf{e}_y + k_{\mathrm{t}z}\mathbf{e}_z$.

2.4 Because $\mathbf{k}_\mathrm{tan}$ is continuous and only the velocity in normal direction can change, we have

$$k_{\mathrm{t}y} \;=\; k_{\mathrm{r}y} \;=\; k_{\mathrm{i}y} \;=\; \frac{\sqrt{3}}{2}k_0 \;, \tag{16}$$

as illustrated in Figure 4.

Moreover, we know from Snell's law, $\varepsilon_{\mathrm{r}1} = 3$ from subproblem 2.1 and $\varepsilon_{\mathrm{r}2} = 1$ from problem condition

$$\begin{aligned} n_1 \sin\alpha_\mathrm{i} &= n_2 \sin\alpha_\mathrm{t} \\ \Leftrightarrow \quad \sqrt{\varepsilon_{\mathrm{r}1}}\sin\alpha_\mathrm{i} &= \sqrt{\varepsilon_{\mathrm{r}2}}\sin\alpha_\mathrm{t} \\ \Rightarrow \quad \sin\alpha_\mathrm{t} &= \sqrt{\frac{\varepsilon_{\mathrm{r}1}}{\varepsilon_{\mathrm{r}2}}}\sin\alpha_\mathrm{i} \;=\; \sqrt{3}\sin\alpha_\mathrm{i} \;. \end{aligned} \tag{17}$$

From Figure 4 we know that

$$\sin\alpha_\mathrm{i} \;=\; \frac{|k_{\mathrm{i}y}|}{|\mathbf{k}_\mathrm{i}|} \;=\; \frac{k_0\frac{\sqrt{3}}{2}}{k_0\sqrt{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(\frac{3}{2}\right)^2}} \;=\; \frac{1}{2}$$

$$\Rightarrow \quad \alpha_\mathrm{i} \;=\; 30^\circ \;. \tag{18}$$

Inserting Eq. (18) into Eq. (17) one can obtain

$$\sin\alpha_{\mathrm{t}} = \frac{\sqrt{3}}{2} \quad \Rightarrow \quad \alpha_{\mathrm{t}} = 60^\circ \,. \tag{19}$$

From Figure 4 we can conclude that the z-component $k_{\mathrm{t}z}\mathbf{e}_z$ of the transmitted wave vector $\mathbf{k}_{\mathrm{t}}$ in medium 2 has the direction of positive z-axis, i.e., $k_{\mathrm{t}z} > 0$, and note that

$$k_{\mathrm{t}z} = \frac{|k_{\mathrm{t}y}|}{\tan\alpha_{\mathrm{t}}} \overset{(16),\,(19)}{=} \frac{\frac{\sqrt{3}}{2}k_0}{\tan 60^\circ} = \frac{\frac{\sqrt{3}}{2}k_0}{\sqrt{3}} = \frac{1}{2}k_0 \,. \tag{20}$$

Therefore, the wave vector of the transmitted wave is

$$\begin{aligned} \mathbf{k}_{\mathrm{t}} &= k_{\mathrm{t}y}\mathbf{e}_y + k_{\mathrm{t}z}\mathbf{e}_z \\ &\overset{(16),\,(20)}{=} k_0\left(\frac{\sqrt{3}}{2}\mathbf{e}_y + \frac{1}{2}\mathbf{e}_z\right) \,. \end{aligned}$$

2.5 From the Fresnel's equation we can calculate the transmission coefficients for the perpendicular (to the incident plane) and parallel (to the incident plane) part of the electric field and note that $n_1 = \sqrt{\varepsilon_{\mathrm{r}1}} = \sqrt{3}$, $n_2 = \sqrt{\varepsilon_{\mathrm{r}2}} = 1$ and $\alpha_{\mathrm{i}} = 30^\circ$ from Eq. (18)

$$\begin{aligned} \underline{t}_\perp &= \frac{\underline{E}_{\mathrm{t}\perp}}{\underline{E}_{\mathrm{i}\perp}} = -\frac{2n_1 n_2\cos\alpha_{\mathrm{i}}}{n_2^2\cos\alpha_{\mathrm{i}} + n_1\sqrt{n_2^2 - n_1^2\sin^2\alpha_{\mathrm{i}}}} \\ &= -\frac{2\cdot\sqrt{3}\cdot 1\cdot\frac{\sqrt{3}}{2}}{1\cdot\frac{\sqrt{3}}{2} + \sqrt{3}\cdot\sqrt{1 - 3\cdot\frac{1}{4}}} = -\sqrt{3} \end{aligned}$$

$$\begin{aligned} \underline{t}_\parallel &= \frac{\underline{E}_{\mathrm{t}\parallel}}{\underline{E}_{\mathrm{i}\parallel}} = \frac{2n_1\cos\alpha_{\mathrm{i}}}{n_1\cos\alpha_{\mathrm{i}} + \sqrt{n_2^2 - n_1^2\sin^2\alpha_{\mathrm{i}}}} \\ &= \frac{2\cdot\sqrt{3}\cdot\frac{\sqrt{3}}{2}}{\sqrt{3}\cdot\frac{\sqrt{3}}{2} + \sqrt{1 - 3\cdot\frac{1}{4}}} = \frac{3}{2} \,. \end{aligned}$$

2.6

$$\sin\alpha_{\mathrm{i,tot}} > \frac{n_2}{n_1} = \frac{1}{\sqrt{3}} \quad \Rightarrow \quad \alpha_{\mathrm{i,tot}} > 35.26^\circ \,.$$

Solution of Problem 3

3.1 Since $\mathbf{k}_2 = |\mathbf{k}_2|\,\mathbf{n}_2 = k_2\mathbf{n}_2$, we can assume that

$$\mathbf{n}_2 = a\mathbf{e}_x + b\mathbf{e}_y + c\mathbf{e}_z$$

and therefore

$$\mathbf{k}_2 \;=\; k_{2x}\mathbf{e}_x + k_{2y}\mathbf{e}_y + k_{2z}\mathbf{e}_z \;=\; k_2\left(a\mathbf{e}_x + b\mathbf{e}_y + c\mathbf{e}_z\right) \; .$$

From the assumption of the problem $k_{2z} = 0$, we have $c = \dfrac{k_{2z}}{k_2} = 0$. Thus,

$$\mathbf{n}_2 \;=\; a\mathbf{e}_x + b\mathbf{e}_y \; .$$

We know that $\mathbf{k}_2 \perp \underline{\mathbf{E}}_2$, equivalently we can get

$$\begin{aligned} \mathbf{n}_2 \perp \underline{\mathbf{E}}_2 \quad &\Leftrightarrow \quad \mathbf{n}_2 \cdot \underline{\mathbf{E}}_2 \;=\; 0 \\ &\Rightarrow \quad \begin{pmatrix} a \\ b \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -3 \\ 0 \end{pmatrix} \;=\; a - 3b \;\stackrel{!}{=}\; 0 \\ &\Rightarrow \quad b \;=\; \frac{a}{3} \; . \end{aligned} \tag{21}$$

Using the definition of *normalized direction vector*, one can obtain

$$\begin{aligned} |\mathbf{n}_2| \;=\; 1 \quad &\Leftrightarrow \quad |\mathbf{n}_2| \;=\; |a\mathbf{e}_x + b\mathbf{e}_y| \;\stackrel{(21)}{=}\; \sqrt{a^2 + \frac{a^2}{9}} \;=\; 1 \\ &\Leftrightarrow \quad a \;=\; \pm\frac{3}{\sqrt{10}} \; . \end{aligned} \tag{22}$$

From the assumption $k_{2x} = k_2 a > 0$ we have

$$\begin{aligned} a \;&=\; \frac{3}{\sqrt{10}} \\ \Rightarrow \quad b \;&\stackrel{(21)}{=}\; \frac{a}{3} \;=\; \frac{1}{\sqrt{10}} \; . \end{aligned} \tag{23}$$

Finally, man can get the normalized direction vector in medium 2

$$\mathbf{n}_2 \;=\; \frac{1}{\sqrt{10}}\left(3\mathbf{e}_x + \mathbf{e}_y\right) \; .$$

3.2 From the assumption of the problem that there is no reflected wave we can draw that $\underline{r}_{\parallel} \;=\; 0$ or $\underline{r}_{\perp} \;=\; 0$. Since the refraction index in medium 2 $n_2 \;=\; \sqrt{\varepsilon_{\mathrm{r}2}} \;>\; 1$ is larger than the refraction index in medium 1 $n_1 \;=\; \sqrt{\varepsilon_{\mathrm{r}1}} \;=\; 1$, $\underline{r}_{\parallel}$ is nonzero and $\underline{r}_{\perp} \;=\; 0$. Thus, the angle of incidence α_{i} is *Brewster angle*. In the following let's prove that *if the angle of incidence α_{i} is Brewster angle, then the reflected wave is perpendicular to the transmitted (refracted) wave, i.e., $\alpha_{\mathrm{r}} + \alpha_{\mathrm{t}} = 90°$.*

Proof: When the angle of incidence α_{i} is Brewster angle, we have

$$\sin^2\alpha_{\mathrm{i}} \;=\; \frac{n_2^2}{n_1^2 + n_2^2} \; ,$$

where $n_1 = \sqrt{\varepsilon_{\text{r1}}}$ and $n_2 = \sqrt{\varepsilon_{\text{r2}}}$ are the refraction indices in the medium 1 and medium 2, respectively. Since $\alpha_\text{r} = \alpha_\text{i}$, man can get

$$\begin{aligned} \sin^2\alpha_\text{r} &= \sin^2\alpha_\text{i} = \frac{n_2^2}{n_1^2+n_2^2} \\ \Rightarrow \quad \cos^2\alpha_\text{r} &= 1-\sin^2\alpha_\text{r} = \frac{n_1^2}{n_1^2+n_2^2} \ . \end{aligned} \tag{24}$$

From $n_1 \cdot \sin\alpha_\text{i} = n_2 \cdot \sin\alpha_\text{t}$, law of refraction – Snell's law, we can get

$$\sin^2\alpha_\text{t} = \frac{n_1^2}{n_2^2}\cdot\sin^2\alpha_\text{i} = \frac{n_1^2}{n_2^2}\cdot\frac{n_2^2}{n_1^2+n_2^2} = \frac{n_1^2}{n_1^2+n_2^2} \ . \tag{25}$$

Compare (24) with (25) and note that $0^\circ \le \alpha_\text{r}, \alpha_\text{t} < 90^\circ$, we have

$$\cos\alpha_\text{r} = \sin\alpha_\text{t} \quad \Rightarrow \quad \alpha_\text{r}+\alpha_\text{t} = 90^\circ \ .$$

So, it follows that the reflected wave is perpendicular to the transmitted (refracted) wave when the angle of incidence α_i is Brewster angle.

From the conclusion above we can get $\mathbf{n}_{1\text{r}} \perp \mathbf{n}_2$, i.e., $\mathbf{n}_{1\text{r}} \cdot \mathbf{n}_2 = 0$.

Assume that

$$\mathbf{n}_{1\text{r}} = a\mathbf{e}_x + b\mathbf{e}_y + c\mathbf{e}_z \ ,$$

one can obtain from $\mathbf{n}_{1\text{r}} \cdot \mathbf{n}_2 = 0$ and (23) as follows

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \cdot \begin{pmatrix} \frac{3}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{10}}(3a+b) \stackrel{!}{=} 0 \quad \Rightarrow \quad b = -3a \ . \tag{26}$$

Since the incident plane is *xoy-plane*, i.e., the plane of $z = 0$, we can get that $c = 0$. Consequently,

$$\mathbf{n}_{1\text{r}} = a\mathbf{e}_x + b\mathbf{e}_y \stackrel{(26)}{=} a\mathbf{e}_x - 3a\mathbf{e}_y \tag{27}$$

can be acquired. Thus, according to

$$\begin{aligned} |\mathbf{n}_{1\text{r}}| &\stackrel{(27)}{=} \sqrt{a^2+(-3a)^2} = \sqrt{10a^2} = 1 \\ \Rightarrow \quad a &= \pm\frac{1}{\sqrt{10}} \\ \Rightarrow \quad b &\stackrel{(26)}{=} \mp\frac{3}{\sqrt{10}} \end{aligned} \tag{28}$$

the normalized reflected direction vector $\mathbf{n}_{1\text{r}}$ can be written as

$$\mathbf{n}_{1\text{r}} = \pm\frac{1}{\sqrt{10}}(\mathbf{e}_x - 3\mathbf{e}_y) \ .$$

From the sketch of Figure 5, we know that

$$\begin{aligned} \mathbf{n}_{1x} &= -\mathbf{n}_{1\mathrm{r}x} \\ \mathbf{n}_{1y} &= \mathbf{n}_{1\mathrm{r}y} \\ \Rightarrow \quad \mathbf{n}_1 &\overset{(28)}{=} \pm\frac{1}{\sqrt{10}}\left(-\mathbf{e}_x - 3\mathbf{e}_y\right) = \pm\frac{1}{\sqrt{10}}\left(\mathbf{e}_x + 3\mathbf{e}_y\right) . \end{aligned}$$

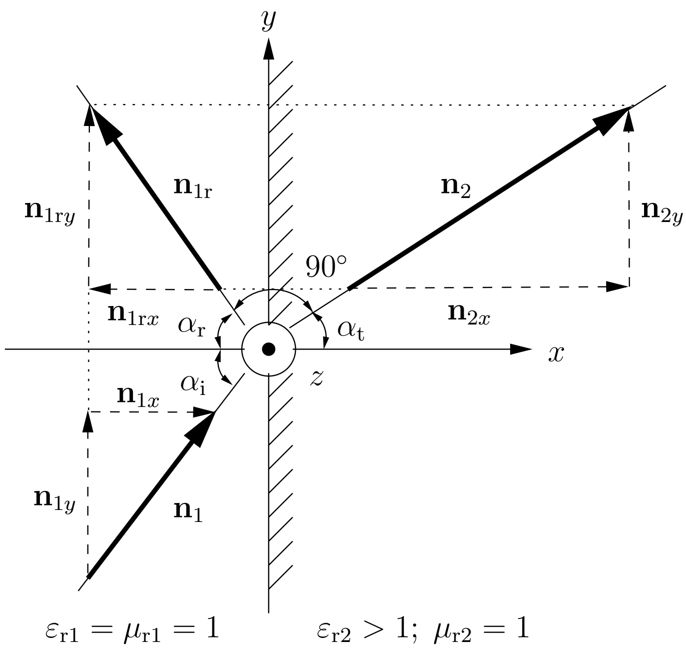


Figure 5: Illustration of the normalized incident direction vector $\mathbf{n}_1 = \mathbf{n}_{1x} + \mathbf{n}_{1y}$, normalized reflected direction vector $\mathbf{n}_{1\mathrm{r}} = \mathbf{n}_{1\mathrm{r}x} + \mathbf{n}_{1\mathrm{r}y}$ and the normalized transmitted direction vector $\mathbf{n}_2 = \mathbf{n}_{2x} + \mathbf{n}_{2y}$, when the angle of incidence α_i is Brewster angle.

From the assumption of the problem that $k_{1x} > 0$, i.e., $n_{1x} = |\mathbf{n}_{1x}| > 0$, we can finally have

$$\mathbf{n}_1 = \frac{1}{\sqrt{10}}\left(\mathbf{e}_x + 3\mathbf{e}_y\right) \quad \text{and} \quad \mathbf{n}_{1\mathrm{r}} = \frac{1}{\sqrt{10}}\left(-\mathbf{e}_x + 3\mathbf{e}_y\right) .$$

3.3 From

$$\mathbf{k}_1 = k_1\,\mathbf{n}_1 = \omega\sqrt{\varepsilon_1\mu_1}\,\mathbf{n}_1 = \underbrace{\omega\sqrt{\varepsilon_0\mu_0}}_{=k_0}\underbrace{\sqrt{\varepsilon_{\mathrm{r}1}\mu_{\mathrm{r}1}}}_{=1}\,\mathbf{n}_1$$

we can get

$$\mathbf{k}_1 = k_0\,\mathbf{n}_1 = k_0\,\frac{1}{\sqrt{10}}\left(\mathbf{e}_x + 3\mathbf{e}_y\right) = \underbrace{\frac{k_0}{\sqrt{10}}}_{=k_{1x}}\mathbf{e}_x + \underbrace{\frac{3k_0}{\sqrt{10}}}_{=k_{1y}}\mathbf{e}_y .$$

Similarly,

$$\mathbf{k}_2 = k_2\,\mathbf{n}_2 = \underbrace{\omega\sqrt{\varepsilon_0\mu_0}}_{=k_0}\,\underbrace{\sqrt{\varepsilon_{r2}\mu_{r2}}}_{=\sqrt{\varepsilon_{r2}}}\,\mathbf{n}_2 = \sqrt{\varepsilon_{r2}}\,k_0\left(\frac{3}{\sqrt{10}}\mathbf{e}_x + \frac{1}{\sqrt{10}}\mathbf{e}_y\right)$$

$$\Rightarrow \quad \mathbf{k}_2 = \underbrace{\frac{3k_0\sqrt{\varepsilon_{r2}}}{\sqrt{10}}}_{=k_{2x}}\mathbf{e}_x + \underbrace{\frac{k_0\sqrt{\varepsilon_{r2}}}{\sqrt{10}}}_{=k_{2y}}\mathbf{e}_y\ .$$

Since $\mathbf{k}_{\mathrm{tan}}$ is continuous and only the velocity in normal direction can change, we have $k_{1y} = k_{2y}$. Therefore,

$$\underbrace{\frac{3k_0}{\sqrt{10}}}_{=k_{1y}} = \underbrace{\frac{k_0\sqrt{\varepsilon_{r2}}}{\sqrt{10}}}_{=k_{2y}}$$

$$\Rightarrow \quad \sqrt{\varepsilon_{r2}} = 3 \quad \Rightarrow \quad \varepsilon_{r2} = 9\ .$$

3.4 Since

$$\underline{\mathbf{E}}_2 = E_0\,(\mathbf{e}_x - 3\mathbf{e}_y)\,\mathrm{e}^{-\mathrm{j}\mathbf{k}_2\cdot\mathbf{r}} = (E_0\mathbf{e}_x - 3E_0\mathbf{e}_y)\,\mathrm{e}^{-\mathrm{j}\mathbf{k}_2\cdot\mathbf{r}}\ .$$

we know that

$$\underline{E}_{2x} = E_0 \quad \text{and} \quad \underline{E}_{2y} = -3E_0\ . \tag{29}$$

Keep in mind that $\underline{\mathbf{E}}_{\mathrm{tan}}$ and $\varepsilon\underline{\mathbf{E}}_{\mathrm{norm}}$ is continuous at the dielectric boundary surface. Therefore, for $\underline{\mathbf{E}}_{\mathrm{tan}}$ component we have

$$\underline{E}_{1y} = \underline{E}_{2y} \overset{(29)}{=} -3E_0$$

and for $\varepsilon\underline{\mathbf{E}}_{\mathrm{norm}}$ component we have

$$\varepsilon_{r1}\underline{E}_{1x} = \varepsilon_{r2}\underline{E}_{2x} \quad \Leftrightarrow \quad \underline{E}_{1x} = \frac{\varepsilon_{r2}}{\varepsilon_{r1}}\underline{E}_{2x} \overset{(29)}{=} 9E_0\ .$$

Finally we can know that

$$\underline{\mathbf{E}}_1 = (9E_0\mathbf{e}_x - 3E_0\mathbf{e}_y)\,\mathrm{e}^{-\mathrm{j}\mathbf{k}_1\cdot\mathbf{r}}\ . \tag{30}$$

3.5 Firstly, consider the relationship between the wave vector $\mathbf{k}_1$ and the normalized direction vector $\mathbf{n}_1$.

$$\mathbf{k}_1 = k_1\mathbf{n}_1 \quad \Rightarrow \quad \mathbf{n}_1 = \frac{\mathbf{k}_1}{k_1}\ , \tag{31}$$

where

$$k_1 = \omega\sqrt{\varepsilon_1\mu_1} = \underbrace{\omega\sqrt{\varepsilon_0\mu_0}}_{=k_0}\,\underbrace{\sqrt{\varepsilon_{r1}\mu_{r1}}}_{=1} = k_0 \tag{32}$$

is the scalar wave number in medium 1. Combining (31) and (32) we have

$$\mathbf{k}_1 = k_0\,\mathbf{n}_1\ . \tag{33}$$

Then we can calculate the magnetic field $\underline{\mathbf{H}}_1$ in medium 1 with Maxwell's equation.

$$\begin{aligned}
\underline{\mathbf{H}}_1 &\stackrel{!}{=} -\frac{1}{\mathrm{j}\omega\mu_1}\mathrm{rot}\,\underline{\mathbf{E}}_1 \\
&\stackrel{!}{=} \frac{\mathbf{k}_1}{\omega\mu_1}\times\underline{\mathbf{E}}_1 \stackrel{(33)}{=} \frac{k_0}{\omega\mu_1}\mathbf{n}_1\times\underline{\mathbf{E}}_1 \\
&= \frac{\omega\sqrt{\varepsilon_0\mu_0}}{\omega\mu_0\mu_{\mathrm{r1}}}\left(\frac{1}{\sqrt{10}}\mathbf{e}_x+\frac{3}{\sqrt{10}}\mathbf{e}_y\right)\times\left(9E_0\mathbf{e}_x-3E_0\mathbf{e}_y\right)\mathrm{e}^{-\mathrm{j}\mathbf{k}_1\cdot\mathbf{r}} \\
&= -\sqrt{\frac{\varepsilon_0}{\mu_0}}\frac{30}{\sqrt{10}}E_0\mathbf{e}_z\mathrm{e}^{-\mathrm{j}\mathbf{k}_1\cdot\mathbf{r}}\,.
\end{aligned}$$