Random Generators

Autocorrelation Function and Power Spectral Density

Problem 1

Show that with

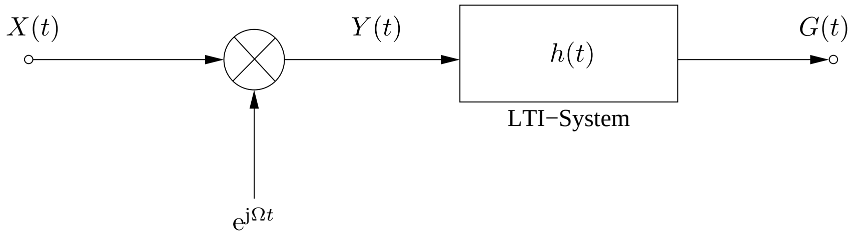
$$Y_1 = \sqrt{-2\ln X_1}\cos(2\pi X_2) \quad (1)$$

$$Y_2 = \sqrt{-2\ln X_1}\sin(2\pi X_2) \quad (2)$$

the statistically independent random variables X_1 and X_2 distributed uniformly in $[0,1]$ are transformed to "normally" distributed random variables Y_1 and Y_2.

Problem 2

Consider the sample function of a real, stationary and ergodic input signal $X(t)$ of a transmission system as given in the following.



2.1 Calculate the autocorrelation function $R_{YY}(\tau)$ and power spectral density $S_{YY}(\omega)$ of $y(t)$ in terms of $R_{XX}(\tau)$ and $S_{XX}(\omega)$.

2.2 Determine the autocorrelation function $R_{GG}(\tau)$ of the output signal $g(t)$ in terms of the autocorrelation function $R_{YY}(\tau)$ of the filter input signal $y(t)$ and the impulse response $h(t)$ as a convolution product.

2.3 Determine the power spectral density $S_{GG}(\omega)$ of the output signal $g(t)$ in terms of the power spectral density $S_{YY}(\omega)$ of the filter input signal $y(t)$ and the transfer function $H(\omega) \bullet\!\!-\!\!\circ\, h(t)$, where $h(t)$ is the real impulse response.

2.4 The system indicated as LTI-System in the plot is a short term integrator with impulse response

$$h(t) = \frac{1}{2T}\mathrm{rect}\left(\frac{t}{2T} - \frac{1}{2}\right) .$$

Write $R_{GG}(\tau)$ as an integral in terms of the integrand $R_{XX}(\tau)$ and the independent variables τ and Ω.

2.5 The displayed value of a rms-measuring device at the output of the integrator is proportional to $R_{GG}(0)$.

Which integration time T of the short term integrator is required with the aim to achieve a measured value being proportional to the power spectral density value $S_{XX}(\Omega)$ if we assume $R_{XX}(\tau) \approx 0$ for all $|\tau| > \tau_0$?

Solution of Problem 1

The probability density function (*pdf*) $f_{Y_1,Y_2}(y_1, y_2)$ can be generally calculated by

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{f_{X_1,X_2}\left[x_1\left(y_1,y_2\right),x_2(y_1,y_2)\right]}{\left|\det\dfrac{\partial(y_1,y_2)}{\partial(x_1,x_2)}\right|}\ , \tag{3}$$

where the so-called *"Jacobi-Matrix"* $\dfrac{\partial(y_1,y_2)}{\partial(x_1,x_2)}$ is defined as

$$\frac{\partial(y_1,y_2)}{\partial(x_1,x_2)} = \begin{pmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{pmatrix}\ . \tag{4}$$

With the four derivatives

$$\frac{\partial y_1}{\partial x_1} = -\frac{2}{x_1}\frac{1}{2\sqrt{-2\ln x_1}}\cos(2\pi x_2) \tag{5}$$

$$\frac{\partial y_1}{\partial x_2} = \sqrt{-2\ln x_1}\left(-\sin(2\pi x_2)\cdot 2\pi\right) \tag{6}$$

$$\frac{\partial y_2}{\partial x_1} = -\frac{2}{x_1}\frac{1}{2\sqrt{-2\ln x_1}}\sin(2\pi x_2) \tag{7}$$

$$\frac{\partial y_2}{\partial x_2} = \sqrt{-2\ln x_1}\left(\cos(2\pi x_2)\cdot 2\pi\right) \tag{8}$$

inserted into the determinant yields

$$\begin{aligned}\left|\det\frac{\partial(y_1,y_2)}{\partial(x_1,x_2)}\right| &= \left|\det\begin{pmatrix} \dfrac{\partial y_1}{\partial x_1} & \dfrac{\partial y_1}{\partial x_2} \\ \dfrac{\partial y_2}{\partial x_1} & \dfrac{\partial y_2}{\partial x_2} \end{pmatrix}\right| \\ &= \left|\frac{\partial y_1}{\partial x_1}\cdot\frac{\partial y_2}{\partial x_2} - \frac{\partial y_1}{\partial x_2}\cdot\frac{\partial y_2}{\partial x_1}\right| \\ &= \left|-\frac{1}{x_1}2\pi\left(\cos^2(2\pi x_2)+\sin^2(2\pi x_2)\right)\right| \\ &= \frac{2\pi}{x_1}\ .\end{aligned} \tag{9}$$

Since the probability density function $f_{Y_1,Y_2}(y_1, y_2)$ is depended on y_1 and y_2, we have to substitute x_1 with the new variables y_1 and y_2.

With the relation

$$\begin{aligned} y_1^2 + y_2^2 &= -2\ln x_1\cos^2\left(2\pi x_2\right) - 2\ln x_1\sin^2\left(2\pi x_2\right) \\ &= -2\ln x_1\ , \end{aligned} \tag{10}$$

x_1 can be represented in terms of y_1 and y_2

$$\begin{aligned} y_1^2 + y_2^2 &= -2\ln x_1 \\ \Rightarrow \quad x_1 &= \mathrm{e}^{-\frac{1}{2}(y_1^2+y_2^2)} \,. \end{aligned} \tag{11}$$

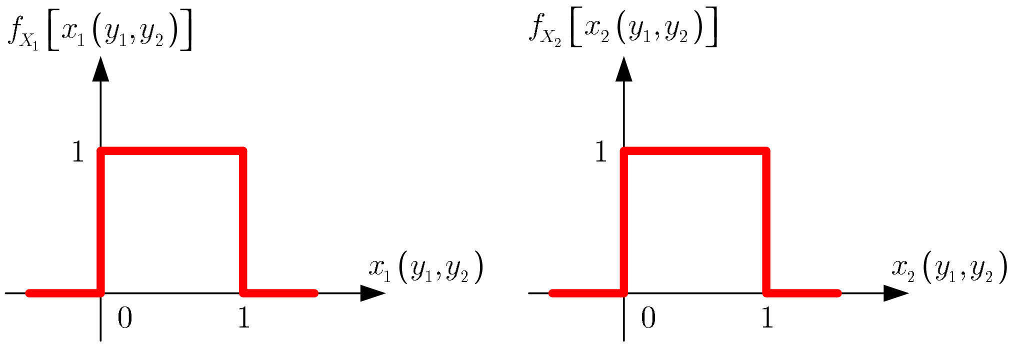
Inserting (11) and (9) into Equation (3) yields

$$f_{Y_1,Y_2}(y_1,y_2) = \frac{1}{2\pi}\mathrm{e}^{-\frac{1}{2}(y_1^2+y_2^2)} f_{X_1,X_2}\left[x_1(y_1,y_2), x_2(y_1,y_2)\right] \,. \tag{12}$$

Since X_1 and X_2 are statistically independent, uniformly distributed in $[0,1]$, we can get

$$\begin{aligned} f_{X_1,X_2}\left[x_1(y_1,y_2), x_2(y_1,y_2)\right] &= f_{X_1}\left[x_1(y_1,y_2)\right] \cdot f_{X_2}\left[x_2(y_1,y_2)\right] \\ &= \begin{cases} 1 & \text{if} \quad 0 \le x_1(y_1,y_2), x_2(y_1,y_2) \le 1, \\ 0 & \text{else}, \end{cases} \end{aligned} \tag{13}$$

for arbitrary y_1 and y_2. The following figure illustrates the probability density functions $f_{X_1}\left[x_1(y_1,y_2)\right]$ and $f_{X_2}\left[x_2(y_1,y_2)\right]$.



Therefore, we can get

$$\begin{aligned} f_{Y_1,Y_2}(y_1,y_2) &= \frac{1}{2\pi}\mathrm{e}^{-\frac{1}{2}(y_1^2+y_2^2)} \cdot 1 \\ &= \underbrace{\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}y_1^2}}_{f_{Y_1}(y_1)} \cdot \underbrace{\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{1}{2}y_2^2}}_{f_{Y_2}(y_2)} \qquad (14) \\ &= f_{Y_1}(y_1) \cdot f_{Y_2}(y_2) \,, \qquad (15) \end{aligned}$$

which is the product of the probability density functions of two statistically independent, normally distributed random variables Y_1 and Y_2.

Solution of Problem 2

2.1

$$\begin{aligned} R_{YY}(\tau) &= \mathrm{E}\{Y(t)Y^*(t+\tau)\} \\ &= \mathrm{E}\left\{X(t)\mathrm{e}^{\mathrm{j}\Omega t}X^*(t+\tau)\mathrm{e}^{-\mathrm{j}\Omega(t+\tau)}\right\} \\ &= \mathrm{E}\{X(t)X^*(t+\tau)\}\,\mathrm{e}^{-\mathrm{j}\Omega\tau} \\ &= R_{XX}(\tau)\mathrm{e}^{-\mathrm{j}\Omega\tau}\ . \end{aligned} \tag{16}$$

$$\begin{aligned} S_{YY}(\omega) &= \mathcal{F}\{R_{YY}(\tau)\} = \int_{-\infty}^{+\infty} R_{YY}(\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau \\ &= \int_{-\infty}^{+\infty} R_{XX}(\tau)\mathrm{e}^{-\mathrm{j}\Omega\tau}\mathrm{e}^{-\mathrm{j}\omega\tau}\mathrm{d}\tau \\ &= \int_{-\infty}^{+\infty} R_{XX}(\tau)\mathrm{e}^{-\mathrm{j}(\omega+\Omega)\tau}\mathrm{d}\tau \\ &= S_{XX}(\omega+\Omega)\ . \end{aligned} \tag{17}$$

2.2 With $R_{YG}(\tau) = R_{YY}(\tau) * h(\tau)$, it can be shown that for a linear time invariant system with the real impulse response $h(t)$ is valid

$$R_{GG}(\tau) = R_{YY}(\tau) * h(\tau) * h(-\tau)\ . \tag{18}$$

2.3 We can obtain straightforward

$$S_{GG}(\tau) = S_{YY}(\omega)\cdot H(\omega)\cdot H^*(\omega) = S_{YY}(\omega)\,|H(\omega)|^2\ . \tag{19}$$

2.4 The rect-function is defined as

$$\mathrm{rect}(t) = \begin{cases} 1 & \text{for } |t| \le \frac{1}{2}, \\ 0 & \text{else.} \end{cases}$$

Figure 1 illustrates the impulse response $h(t) = \frac{1}{2T}\mathrm{rect}\left(\frac{t}{2T} - \frac{1}{2}\right)$.

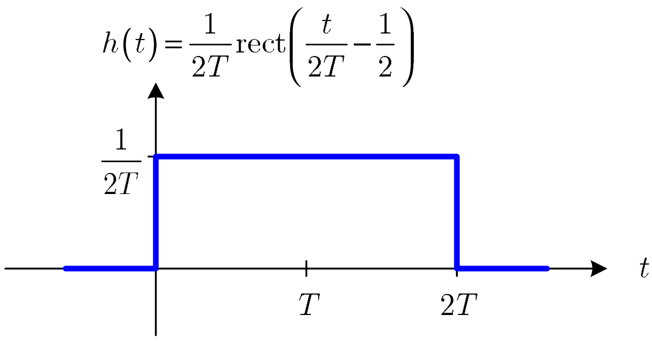


Figure 1: Illustration of the real-valued impulse response $h(t)$.

The convolution of the two rect-functions is a triangular function

$$h(\tau) * h(-\tau) \;=\; \frac{1}{2T}\left(1-\frac{|\tau|}{2T}\right)\mathrm{rect}\left(\frac{\tau}{4T}\right) \;.$$

Figure 2 illustrates the convolution $h(\tau) * h(-\tau)$.

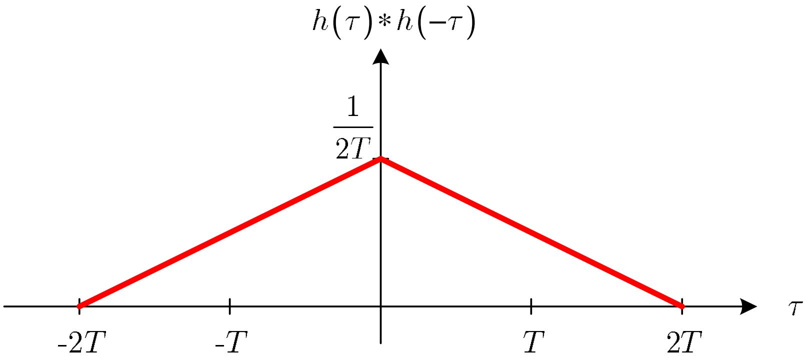


Figure 2: Convolution of $h(\tau)$ and $h(-\tau)$.

Hence, from (16) we have

$$\begin{aligned} R_{GG}(\tau) &= R_{YY}(\tau) * h(\tau) * h(-\tau) \\ &= \frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|u|}{2T}\right)R_{YY}(\tau-u)\mathrm{d}u \\ &= \frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|u|}{2T}\right)R_{XX}(\tau-u)\mathrm{e}^{-\mathrm{j}\Omega(\tau-u)}\mathrm{d}u \;. \end{aligned} \tag{20}$$

2.5 From 2.4 follows for $\tau = 0$

$$\begin{aligned} R_{GG}(0) &= \frac{1}{2T}\int_{-2T}^{2T}\left(1-\frac{|u|}{2T}\right)R_{XX}(-u)\mathrm{e}^{-\mathrm{j}\Omega(-u)}\mathrm{d}u \\ &= \frac{1}{2T}\int_{-2T}^{2T}R_{XX}(u)\mathrm{e}^{-\mathrm{j}\Omega u}\mathrm{d}u - \frac{1}{2T}\int_{-2T}^{2T}\frac{|u|}{2T}R_{XX}(u)\mathrm{e}^{-\mathrm{j}\Omega u}\mathrm{d}u \;. \end{aligned} \tag{21}$$

The first term of Equation (21) is similar to the Fourier transform of the autocorrelation function $R_{XX}(\tau)$

$$\begin{aligned} S_{XX}(\Omega) &= \int_{-\infty}^{+\infty} R_{XX}(\tau)\mathrm{e}^{-\mathrm{j}\Omega\tau}\mathrm{d}\tau \\ &= \int_{-\tau_0}^{\tau_0} R_{XX}(\tau)\mathrm{e}^{-\mathrm{j}\Omega\tau}\mathrm{d}\tau \;, \end{aligned} \tag{22}$$

because we assume that $R_{XX}(\tau) \approx 0, \forall\ |\tau| > \tau_0$.

For $\frac{\tau_0}{2T} \ll 1$, (i.e. if the integration time of the short term integrator is much larger than the *"correlation time, i.e. the width of the correlation function"*, the second term in Equation (21) can be neglected because the integration extends only up to $u = \tau_0$) and thus we approximately obtain

$$R_{GG}(0) \approx \frac{1}{2T} S_{XX}(\Omega) . \tag{23}$$

Therefore, for a certain frequency Ω we can approximately determine with this circuit apart from the constant factor $\frac{1}{2T}$ the value of the power spectral density at this frequency.