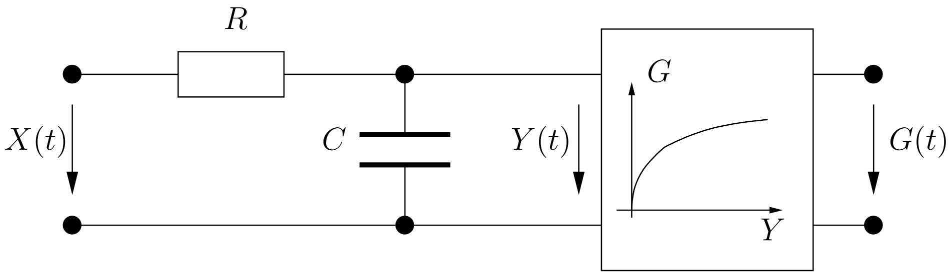
Autocorrelation, Power Spectral Density, PDF

Problem 1

A stationary Gaussian white noise $X(t)$ with power spectral density $S_{XX}(\omega) = S_0$ is input to the system in the following.



We assume that there are no backward effects. $G(t) \to Y(t)$ denotes a nonlinear memoryless system with transfer characteristic

$$G(t) = \begin{cases} 0\,, & \forall\ Y(t) \le 0\,, \\ \sqrt[4]{Y(t)}\,, & \forall\ Y(t) > 0\,. \end{cases} \tag{1}$$

1.1 Sketch the power spectral density $S_{XX}(\omega)$ and the autocorrelation function $R_{XX}(\tau)$ of the input signal.

1.2 Calculate the power spectral density $S_{YY}(\omega)$ at the output of the linear time-invariant system.

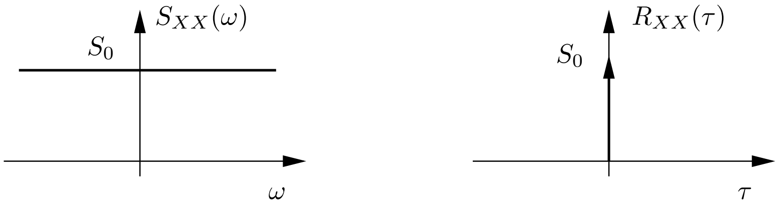
1.3 Calculate the first moment $\mathrm{E}\{Y(t)\}$ and the variance $\sigma^2_{Y(t)}$ of $Y(t)$.

1.4 Determine the probability density function $f_{Y(t)}(y)$.
Hint: Use the fact that $X(t) \to Y(t)$ is an LTI-system.

1.5 Calculate the probability density function $f_{G(t)}(g)$.

Solution of Problem 1

1.1 Since $S_{XX}(\omega) = S_0$ is a constant, we can get the following figure



$\Longrightarrow$ white noise is uncorrelated.

1.2 For a linear time-invariant (LTI) system, the power spectral density $S_{YY}(\omega)$ can be calculated by

$$S_{YY}(\omega) = S_{XX}(\omega) \cdot |H(\omega)|^2 \ , \tag{2}$$

with

$$H(\omega) = \frac{\frac{1}{\mathrm{j}\omega C}}{R + \frac{1}{\mathrm{j}\omega C}} = \frac{1}{1 + \mathrm{j}\omega RC} \ . \tag{3}$$

Utilizing $|H(\omega)|^2 = \frac{1}{1+\omega^2 R^2 C^2}$ and $S_{XX}(\omega) = S_0$ yields

$$S_{YY}(\omega) = S_0 \cdot \frac{1}{1+\omega^2 R^2 C^2} \ . \tag{4}$$

1.3 The expectation value of $Y(t)$ is

$$\begin{aligned}
\mathrm{E}\{Y(t)\} &= \mathrm{E}\{X(t) * h(t)\} = \mathrm{E}\left\{\int_{-\infty}^{+\infty} X(t-\tau)h(\tau)\mathrm{d}\tau\right\} \\
&= \int_{-\infty}^{+\infty} \mathrm{E}\{X(t-\tau)\} \cdot h(\tau)\mathrm{d}\tau \\
&= \int_{-\infty}^{+\infty} h(\tau)\mathrm{d}\tau \cdot \mathrm{E}\{X(t)\} \\
&= \underbrace{\int_{-\infty}^{+\infty} h(\tau)\mathrm{e}^{-\mathrm{j}\omega t}\mathrm{d}\tau\Big|_{\omega=0}}_{H(\omega=0)=H(0)} \cdot \mathrm{E}\{X(t)\} \\
&= H(0) \cdot \mathrm{E}\{X(t)\} = H(0) \cdot \overline{X} \\
&= H(0) \cdot 0 \\
&= 0 \ ,
\end{aligned} \tag{5}$$

since for DC-power

$$\begin{aligned}\overline{X^2} &= \lim_{\epsilon\to 0}\int_{-\epsilon}^{+\epsilon} S_{XX}(\omega)\mathrm{d}\omega \\ &= \lim_{\epsilon\to 0} 2S_0\epsilon = 0 \\ \Rightarrow \quad \overline{X} &= 0\,.\end{aligned}$$

The variance is

$$\begin{aligned}\sigma^2_{Y(t)} &= \mathrm{E}\{Y(t)Y(t)\} - \mathrm{E}^2\{Y(t)\} = R_{YY}(0) - 0 \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{YY}(\omega)\mathrm{d}\omega = \frac{S_0}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{1+\omega^2R^2C^2}\mathrm{d}\omega\,. \qquad (6)\end{aligned}$$

If let $x = \omega RC$, we can get $\mathrm{d}x = RC\cdot\mathrm{d}\omega \quad\Rightarrow\quad \mathrm{d}\omega = \frac{1}{RC}\mathrm{d}x$. Then,

$$\begin{aligned}\sigma^2_{Y(t)} &= \frac{S_0}{2\pi RC}\int_{-\infty}^{+\infty}\frac{1}{1+x^2}\mathrm{d}x \\ &= \frac{S_0}{2\pi RC}\underbrace{[\arctan x]_{-\infty}^{+\infty}}_{\frac{\pi}{2}-\left(-\frac{\pi}{2}\right)=\pi} = \frac{S_0}{2RC}\,. \qquad (7)\end{aligned}$$

1.4 Because $X(t)$ is Gaussian and $X(t)\to Y(t)$ is an LTI-system, also $Y(t)$ is Gaussian distributed, which is completely determined by its first and second moment. From 1.3, we have already obtained $\overline{Y} = \mathrm{E}\{Y(t)\} = 0$ and $\sigma_Y^2 = \frac{S_0}{2RC}$, then we can determine the probability density function

$$\begin{aligned}f_{Y(t)}(y) &= \frac{1}{\sqrt{2\pi}\sigma_Y}\,\mathrm{e}^{-\frac{(y-\overline{Y})^2}{2\sigma_Y^2}} \\ &= \sqrt{\frac{RC}{S_0\pi}}\,\mathrm{e}^{-\frac{RC}{S_0}y^2}\,. \qquad (8)\end{aligned}$$

1.5 **Range 1:** $g < 0$
There is no outcome of Y which is mapped to values $G < 0$.
Hence, we obtain $\Pr\{G<0\} = 0$ and consequently $f_{G(t)}(g) = 0\,,\ \forall\, g<0$.

Range 2: $g = 0$
All values $Y \le 0$ are mapped to the same value $G = 0$!

$$\begin{aligned}\Pr\{G=0\} &= \Pr\{-\infty < Y \le 0\} = \int_{-\infty}^{0} f_{Y(t)}(y)\mathrm{d}y \\ &= \frac{1}{2} \quad \text{(because of symmetry and normalization condition)}\end{aligned}$$

and consequently

$$f_{G(t)}(g) \;=\; \frac{1}{2}\delta(g)\;.$$

Range 3: $g > 0$
Within this range, $g = v(y)$ is monotonously increasing.
With $g = v(y) = \sqrt[4]{y} \quad\Rightarrow\quad y = g^4$ and $v'(y) = \frac{1}{4}y^{-\frac{3}{4}}$, the probability density function is

$$f_{G(t)}(g) \;=\; \frac{f_{Y(t)}[y(g)]}{|v'[y(g)]|} \;=\; 4g^3\sqrt{\frac{RC}{S_0\pi}}\mathrm{e}^{-\frac{RC}{S_0}g^8}\;, \quad \forall\, g > 0\;. \tag{9}$$

Finally, the probability density function $f_{G(t)}(g)$ is then

$$f_{G(t)}(g) \;=\; \frac{1}{2}\delta(g) + \varepsilon(g)\cdot 4g^3\sqrt{\frac{RC}{S_0\pi}}\mathrm{e}^{-\frac{RC}{S_0}g^8}\;. \tag{10}$$