Linear Time-Variant Systems

Problem 1

Check whether the following systems $x(t) \rightarrow y(t)$ are linear and time-variant.

$$y(t) = x(at)$$
$$y(t) = x(t) \cdot t$$

Problem 2

Given the time-variant impulse response

$$h(t,\tau) = \delta(t,\tau-\tau_0) + \delta(t,\tau-2\tau_0) + \delta(t-t_0,\tau-\tau_0) \ ,$$

where $\delta(t,\tau)$ is defined as

$$\delta(t,\tau) = 0 \ , \quad \forall\, t \neq 0 \ , \ \forall\, \tau \neq 0 \tag{1}$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \delta(t,\tau)\ \mathrm{d}t\ \mathrm{d}\tau = 1 \ , \tag{2}$$

$$\int_{-\infty}^{\infty} \delta(t,\tau)\ \mathrm{d}\tau = \delta(t) \ , \tag{3}$$

$$\int_{-\infty}^{\infty} \delta(t,\tau)\ \mathrm{d}t = \delta(\tau) \ . \tag{4}$$

2.1 Sketch the impulse response $h(t,\tau)$.

2.2 Calculate the time-variant transfer function $T(t,\omega) \bullet\!\!-\!\!\circ\ h(t,\tau)$ and sketch its magnitude.

2.3 Calculate the delay Doppler-spread function $S(\nu,\tau) \bullet\!\!-\!\!\circ\ h(t,\tau)$ and sketch its magnitude.

2.4 Calculate the output Doppler-spread function $H(\nu,\omega) \bullet\!\!-\!\!\circ\ T(t,\omega)$.

2.5 Determine the output signal $y(t)$ for the input signal $x(t) = \delta(t)$.

Solution of Problem 1

- $y(t) = x(at)$

  Linearity:

  - $x_1(t) \to y_1(t)$, $x_2(t) \to y_2(t)$:
    $y_1(t) = x_1(at)$, $y_2(t) = x_2(at)$ and $y_1(t) + y_2(t) = x_1(at) + x_2(at)$,
  - $x_1(t) + x_2(t) \to y_3(t)$:
    $y_3(t) = x_1(at) + x_2(at) = y_1(t) + y_2(t)$,

  $\Rightarrow$ linear.

  Time-variance:

  - $t \to t - t_0$:
    $x(t - t_0) \to y(t)$: $y(t) = x(at - t_0) \neq y(t - t_0) = x[a(t - t_0)]$,

  $\Rightarrow$ time-variant.

- $y(t) = x(t) \cdot t$

  Linearity:

  - $x_1(t) \to y_1(t)$, $x_2(t) \to y_2(t)$:
    $y_1(t) = x_1(t) \cdot t$, $y_2(t) = x_2(t) \cdot t$ and $y_1(t) + y_2(t) = x_1(t) \cdot t + x_2(t) \cdot t$,
  - $x_1(t) + x_2(t) \to y_3(t)$:
    $y_3(t) = x_1(t) \cdot t + x_2(t) \cdot t = y_1(t) + y_2(t)$,

  $\Rightarrow$ linear.

  Time-variance:

  - $t \to t - t_0$:
    $x(t - t_0) \to y(t)$: $y(t) = x(t - t_0) \cdot t \neq y(t - t_0) = x(t - t_0) \cdot (t - t_0)$,

  $\Rightarrow$ time-variant.

Solution of Problem 2

2.1 Using the given time-variant impulse response

$$h(t, \tau) = \delta(t, \tau - \tau_0) + \delta(t, \tau - 2\tau_0) + \delta(t - t_0, \tau - \tau_0) \ ,$$

we can sketch the magnitude of $h(t, \tau)$, for example in the discrete case with $\tau_0 = 10$ and $t_0 = 10$, as illustrated in Figure 1.

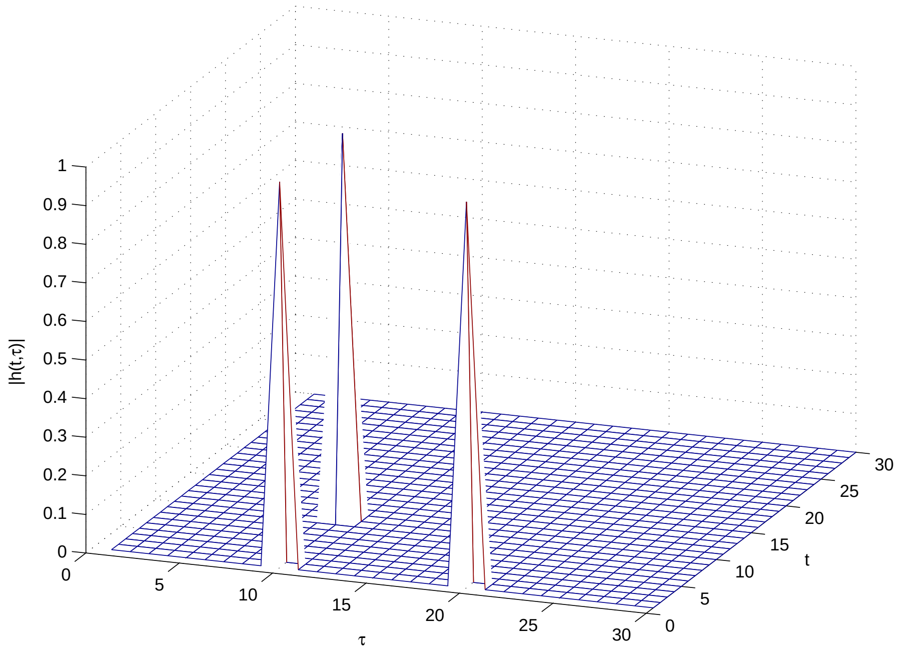

Figure 1: Magnitude of time-variant channel impulse response $|h(t,\tau)|$ (discrete case!)

2.2 Using $T(t,\omega) \bullet\!\!-\!\!\circ\ h(t,\tau)$, Equations (3) and (4) yields

$$\begin{aligned} T(t,\omega) &= \int_{-\infty}^{\infty} h(t,\tau)\mathrm{e}^{-\mathrm{j}\omega\tau}\,\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} \left[\delta(t,\tau-\tau_0)+\delta(t,\tau-2\tau_0)+\delta(t-t_0,\tau-\tau_0)\right]\mathrm{e}^{-\mathrm{j}\omega\tau}\,\mathrm{d}\tau \\ &= \delta(t)\mathrm{e}^{-\mathrm{j}\omega\tau_0}+\delta(t)\mathrm{e}^{-\mathrm{j}\omega 2\tau_0}+\delta(t-t_0)\mathrm{e}^{-\mathrm{j}\omega\tau_0} \\ &= \delta(t)\left(\mathrm{e}^{-\mathrm{j}\omega\tau_0}+\mathrm{e}^{-\mathrm{j}\omega 2\tau_0}\right)+\delta(t-t_0)\mathrm{e}^{-\mathrm{j}\omega\tau_0}\ . \end{aligned} \tag{5}$$

$$\Rightarrow \quad \left|T(t,\omega)\right| = \delta(t)\left|\mathrm{e}^{-\mathrm{j}\omega\tau_0}+\mathrm{e}^{-\mathrm{j}\omega 2\tau_0}\right| + \delta(t-t_0)\ . \tag{6}$$

Note that

$$\begin{aligned} \left|\mathrm{e}^{-\mathrm{j}\omega\tau_0}+\mathrm{e}^{-\mathrm{j}\omega 2\tau_0}\right| &= \Big|\underbrace{\cos(-\omega\tau_0)+\mathrm{j}\sin(-\omega\tau_0)}_{\mathrm{e}^{-\mathrm{j}\omega\tau_0}}+\underbrace{\cos(-2\omega\tau_0)+\mathrm{j}\sin(-2\omega\tau_0)}_{\mathrm{e}^{-\mathrm{j}\omega 2\tau_0}}\Big| \\ &= \Big|\big[\cos(\omega\tau_0)+\cos(2\omega\tau_0)\big]-\mathrm{j}\big[\sin(\omega\tau_0)+\sin(2\omega\tau_0)\big]\Big| \\ &= \sqrt{\big[\cos(\omega\tau_0)+\cos(2\omega\tau_0)\big]^2+\big[\sin(\omega\tau_0)+\sin(2\omega\tau_0)\big]^2} \\ &= \sqrt{2+2\big[\cos(2\omega\tau_0)\cos(\omega\tau_0)+\sin(2\omega\tau_0)\sin(\omega\tau_0)\big]} \\ &= \sqrt{2+2\cos(\omega\tau_0)} \end{aligned}$$

and hence,

$$\left|T(t,\omega)\right| \;=\; \delta(t)\sqrt{2+2\cos(\omega\tau_0)} + \delta(t-t_0)\ .$$

The magnitude of $T(t,\omega)$, for example in the discrete case with $t_0 = 10$ and $\omega = \dfrac{2\pi}{\tau_0}$, as illustrated in Figure 2.

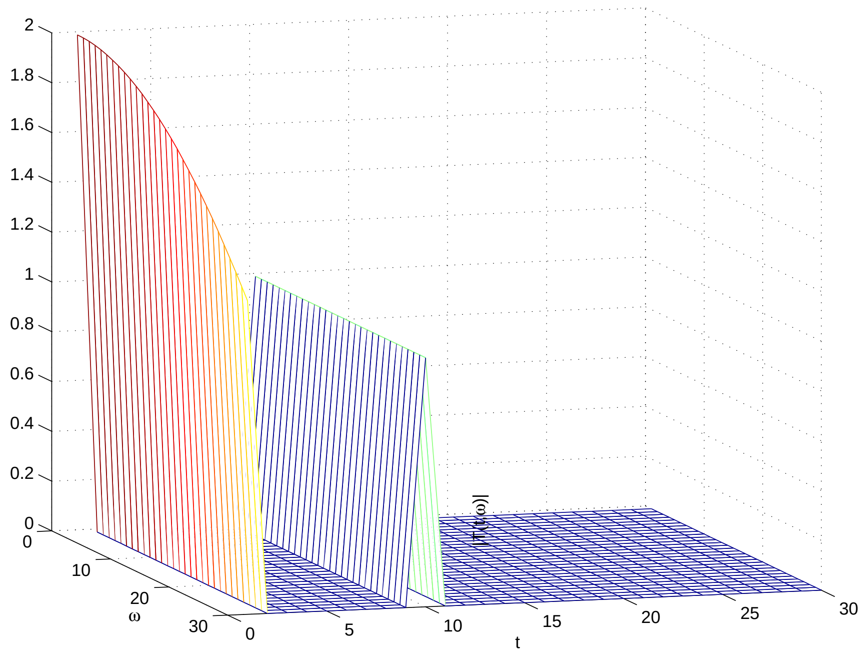

Figure 2: Magnitude of time-variant transfer function $|T(t,\omega)|$ (discrete case!)

2.3 Using $S(\nu,\tau) \bullet\!\!-\!\!\circ\ h(t,\tau)$, Equations (3) and (4) yields

$$\begin{aligned} S(\nu,\tau) &= \int_{-\infty}^{\infty} h(t,\tau)\mathrm{e}^{-\mathrm{j}\nu t}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \big[\delta(t,\tau-\tau_0)+\delta(t,\tau-2\tau_0)+\delta(t-t_0,\tau-\tau_0)\big]\mathrm{e}^{-\mathrm{j}\nu t}\,\mathrm{d}t \\ &= \delta(\tau-\tau_0)+\delta(\tau-2\tau_0)+\mathrm{e}^{-\mathrm{j}\nu t_0}\delta(\tau-\tau_0) \\ &= \delta(\tau-\tau_0)\left(1+\mathrm{e}^{-\mathrm{j}\nu t_0}\right)+\delta(\tau-2\tau_0)\ . \end{aligned} \tag{7}$$

$$\Rightarrow \quad \left|S(\nu,\tau)\right| \;=\; \delta(\tau-\tau_0)\left|1+\mathrm{e}^{-\mathrm{j}\nu t_0}\right| + \delta(\tau-2\tau_0)\ . \tag{8}$$

Utilizing the so-called *Euler Formula*

$$\mathrm{e}^{\mathrm{j}\alpha} \;=\; \cos(\alpha)+\mathrm{j}\sin(\alpha)$$

yields

$$\begin{aligned}
\left|1+\mathrm{e}^{-\mathrm{j}\nu t_0}\right| &= \Big|1+\underbrace{\cos(-\nu t_0)+\mathrm{j}\sin(-\nu t_0)}_{\mathrm{e}^{-\mathrm{j}\nu t_0}}\Big| \\
&= \Big|\big[1+\cos(\nu t_0)\big]-\mathrm{j}\sin(\nu t_0)\Big| \\
&= \sqrt{\big[1+\cos(\nu t_0)\big]^2+\sin^2(\nu t_0)} \\
&= \sqrt{1+\cos^2(\nu t_0)+\sin^2(\nu t_0)+2\cos(\nu t_0)} \\
&= \sqrt{2+2\cos(\nu t_0)}
\end{aligned}$$

and therefore,

$$\Big|S(\nu,\tau)\Big| = \delta(\tau-\tau_0)\sqrt{2+2\cos(\nu t_0)}+\delta(\tau-2\tau_0)\ .$$

The magnitude of $S(\nu,\tau)$, for example in the discrete case with $\tau_0 = 10$ and $\nu = \dfrac{2\pi}{t_0}$, as illustrated in Figure 3.

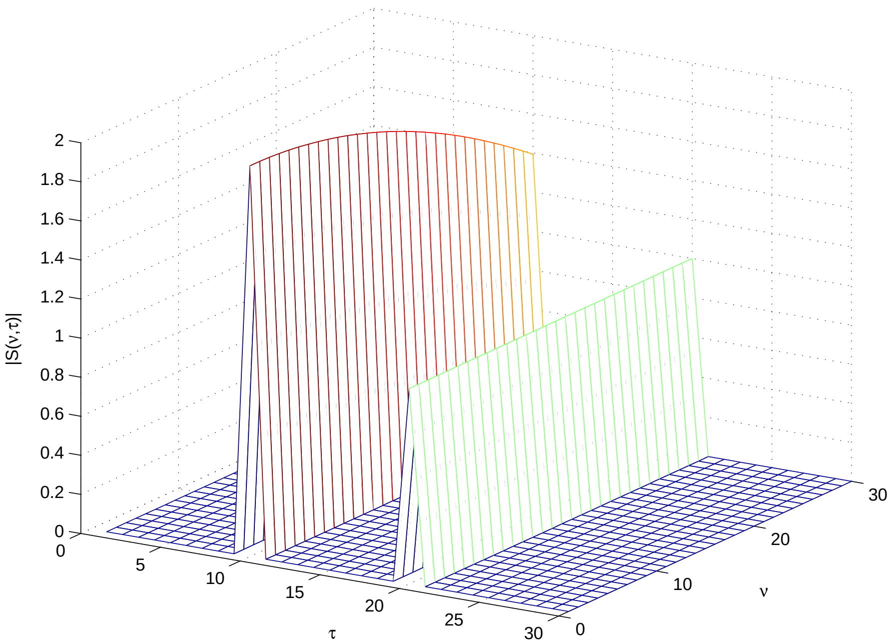

Figure 3: Magnitude of delay Doppler-spread function $|S(\nu,\tau)|$ (discrete case!)

2.4 Using $H(\nu,\omega) \bullet\!\!-\!\!\circ\, T(t,\omega)$, Equations (3) and (4) yields

$$\begin{aligned} H(\nu,\omega) &= \int_{-\infty}^{\infty} T(t,\omega)\mathrm{e}^{-\mathrm{j}\nu t}\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \left[\delta(t)\left(\mathrm{e}^{-\mathrm{j}\omega\tau_0} + \mathrm{e}^{-\mathrm{j}\omega 2\tau_0}\right) + \delta(t-t_0)\mathrm{e}^{-\mathrm{j}\omega\tau_0}\right]\mathrm{e}^{-\mathrm{j}\nu t}\,\mathrm{d}t \\ &= \mathrm{e}^{-\mathrm{j}\omega\tau_0} + \mathrm{e}^{-\mathrm{j}\omega 2\tau_0} + \mathrm{e}^{-\mathrm{j}\omega\tau_0}\mathrm{e}^{-\mathrm{j}\nu t_0}\ . \end{aligned} \tag{9}$$

Figure 4 illustrates the magnitude of output Doppler-spread function $|H(\nu,\omega)|$ in the discrete case.

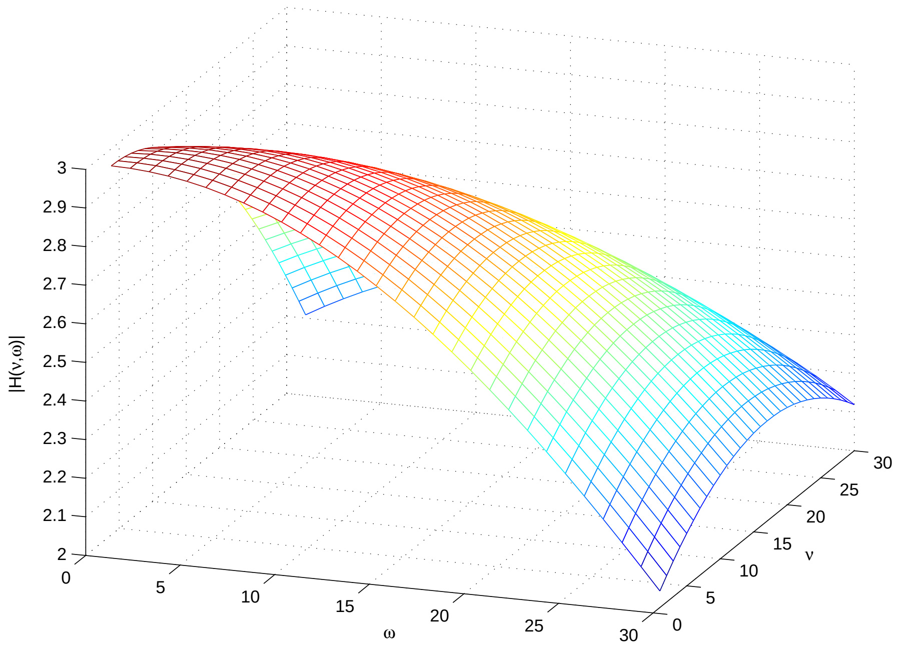

Figure 4: Magnitude of output Doppler-spread function $|H(\nu,\omega)|$ (discrete case!)

2.5 The output signal $y(t)$ for the input signal for the input signal $x(t) = \delta(t)$ can be calculated as follows.

$$\begin{aligned} y(t) &= x(t) * h(t,\tau) \\ &= \int_{-\infty}^{\infty} x(t-\tau)h(t,\tau)\,\mathrm{d}\tau \\ &= \int_{-\infty}^{\infty} \delta(t-\tau)h(t,\tau)\,\mathrm{d}\tau \\ &= h(t,t) = \begin{cases} \delta(t-t_0) & , \quad \tau_0 = t_0\ , \\ 0 & , \quad \tau_0 \neq t_0\ . \end{cases} \end{aligned} \tag{10}$$