Symbol Error Probability of 16-QAM

The 16-QAM signal constellation is shown in Figure 1. It consists of 16 points in the plane to form a square grid.

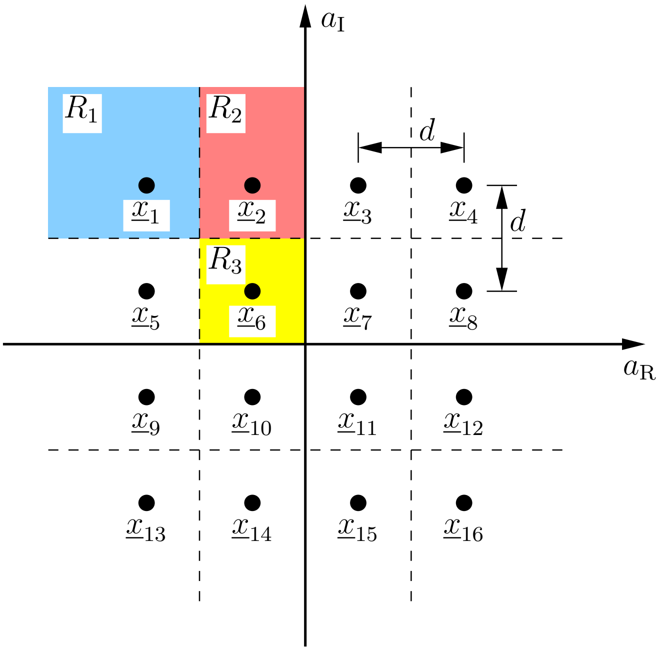

Figure 1: 16-QAM – A two-dimensional signal set with 16 points and decision regions bounded by straight lines parallel to the coordinate axes.

It has three different types of decision regions, namely, that pertaining to the four corner symbols $\underline{x}_1$, $\underline{x}_4$, $\underline{x}_{13}$ and $\underline{x}_{16}$, that pertaining to the eight symbols $\underline{x}_2$, $\underline{x}_3$, $\underline{x}_5$, $\underline{x}_8$, $\underline{x}_9$, $\underline{x}_{12}$, $\underline{x}_{14}$ and $\underline{x}_{15}$, and that pertaining to the four internal symbols $\underline{x}_6$, $\underline{x}_7$, $\underline{x}_{10}$ and $\underline{x}_{11}$.

The observed process is $\underline{Y}(t) = \underline{x}_m + \underline{N}(t)$, $m = 1, \cdots, 16$, where $\underline{N}(t) = N_\mathrm{R}(t) + \mathrm{j}N_\mathrm{I}(t)$ is a complex Gaussian distributed, white noise process with zero mean. All symbols $\underline{x}_m$ are assumed to be equally probable.

Problem

Show that if $P_{e,\mathrm{binary}} \ll 1$, then the symbol error probability of 16-QAM approximately equals

$$P_{e,\text{16-QAM}} \approx 3P_{e,\mathrm{binary}} ,$$

where

$$P_{e,\mathrm{binary}} = P\left\{N_\mathrm{R} < -\frac{d}{2}\right\} = P\left\{N_\mathrm{R} > \frac{d}{2}\right\} = P\left\{N_\mathrm{I} < -\frac{d}{2}\right\} = P\left\{N_\mathrm{I} > \frac{d}{2}\right\}$$

representing the probability that in binary modulation schemes a signal point will be mistaken as another adjacent signal point locating in distance d apart from it, and d denoting the minimum horizontal and vertical distance between pairs of signal points (see Figure 1).

Solution of Problem

The average probability of a correct symbol decision P_c is given by

$$P_c \;=\; \frac{1}{M}\sum_{m=1}^{M} P_{c|\underline{x}_m} \; . \tag{1}$$

We derive from Equation (1) that in this case

$$P_{c,\text{16-QAM}} \;=\; \frac{1}{16}\left(4\cdot P_{c|\underline{x}_1} + 8\cdot P_{c|\underline{x}_2} + 4\cdot P_{c|\underline{x}_6}\right) \; .$$

The probability $P_{c|\underline{x}_1}$ of a correct decision of the symbol $\underline{x}_1$ equals (see Figure 2)

$$\begin{aligned} P_{c|\underline{x}_1} &= P\left\{\underline{x}_1 + \underline{N} \in R_1\right\} \\ &= P\left\{N_\text{R} < \frac{d}{2},\; N_\text{I} > -\frac{d}{2}\right\} \\ &= P\left\{N_\text{R} < \frac{d}{2}\right\} P\left\{N_\text{I} > -\frac{d}{2}\right\} \\ &= \left(1 - P\left\{N_\text{R} > \frac{d}{2}\right\}\right)\left(1 - P\left\{N_\text{I} < -\frac{d}{2}\right\}\right) \\ &= \left(1 - P_{e,\text{binary}}\right)^2 \; . \end{aligned}$$

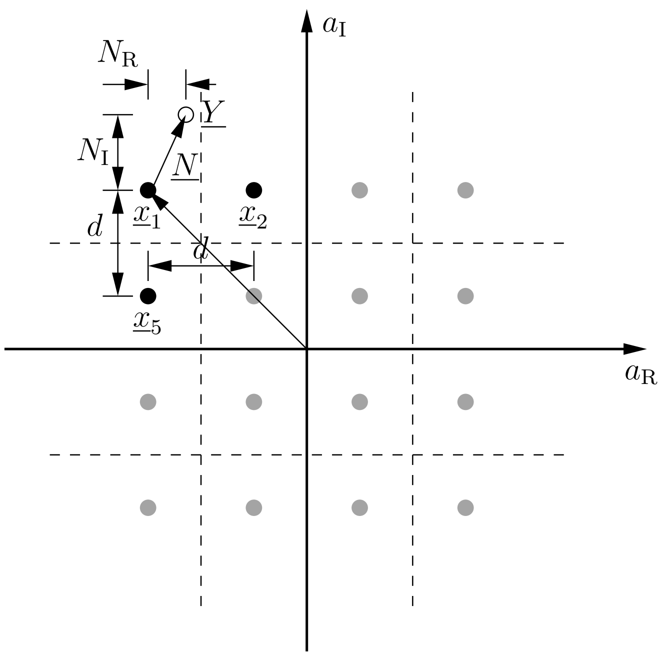

Figure 2: Geometric representation of $\underline{Y} \;=\; \underline{x}_1 + \underline{N}$.

Similarly, the probability $P_{c|\underline{x}_2}$ of a correct decision of the symbol $\underline{x}_2$ equals

$$\begin{aligned}
P_{c|\underline{x}_2} &= P\left\{\underline{x}_2 + \underline{N} \in R_2\right\} \\
&= P\left\{-\frac{d}{2} < N_\mathrm{R} < \frac{d}{2},\ N_\mathrm{I} > -\frac{d}{2}\right\} \\
&= \left(P\left\{N_\mathrm{R} > -\frac{d}{2}\right\} - P\left\{N_\mathrm{R} > \frac{d}{2}\right\}\right) P\left\{N_\mathrm{I} > -\frac{d}{2}\right\} \\
&= \left(1 - P\left\{N_\mathrm{R} < -\frac{d}{2}\right\} - P\left\{N_\mathrm{R} > \frac{d}{2}\right\}\right)\left(1 - P\left\{N_\mathrm{I} < -\frac{d}{2}\right\}\right) \\
&= \left(1 - 2P_{e,\mathrm{binary}}\right)\left(1 - P_{e,\mathrm{binary}}\right) .
\end{aligned}$$

Finally, the probability $P_{c|\underline{x}_6}$ of a correct decision of the symbol $\underline{x}_6$ equals

$$\begin{aligned}
P_{c|\underline{x}_6} &= P\left\{\underline{x}_6 + \underline{N} \in R_3\right\} \\
&= P\left\{-\frac{d}{2} < N_\mathrm{R} < \frac{d}{2},\ -\frac{d}{2} < N_\mathrm{I} < \frac{d}{2}\right\} \\
&= \left(P\left\{N_\mathrm{R} > -\frac{d}{2}\right\} - P\left\{N_\mathrm{R} > \frac{d}{2}\right\}\right)\left(P\left\{N_\mathrm{I} > -\frac{d}{2}\right\} - P\left\{N_\mathrm{I} > \frac{d}{2}\right\}\right) \\
&= \left(1 - P\left\{N_\mathrm{R} < -\frac{d}{2}\right\} - P\left\{N_\mathrm{R} > \frac{d}{2}\right\}\right)\left(1 - P\left\{N_\mathrm{I} < -\frac{d}{2}\right\} - P\left\{N_\mathrm{I} > \frac{d}{2}\right\}\right) \\
&= \left(1 - 2P_{e,\mathrm{binary}}\right)^2 .
\end{aligned}$$

In conclusion, from Equation (1) and the latter Equations we obtain

$$\begin{aligned}
P_{c,16\text{-QAM}} &= \frac{1}{4}\left[\left(1 - P_{e,\mathrm{binary}}\right)^2 + 2\left(1 - 2P_{e,\mathrm{binary}}\right)\left(1 - P_{e,\mathrm{binary}}\right) + \left(1 - 2P_{e,\mathrm{binary}}\right)^2\right] \\
&= 1 - 3P_{e,\mathrm{binary}} + \frac{9}{4}P_{e,\mathrm{binary}}^2
\end{aligned}$$

and hence

$$P_{e,16\text{-QAM}} = 1 - P_{c,16\text{-QAM}} = 3P_{e,\mathrm{binary}} - \frac{9}{4}P_{e,\mathrm{binary}}^2 .$$

We observe that for $P_{e,\mathrm{binary}} \ll 1$, i.e., $P_{e,\mathrm{binary}}^2 \ll P_{e,\mathrm{binary}}$ (a situation that should always occur for reliable transmission) we have

$$P_{e,16\text{-QAM}} \approx 3P_{e,\mathrm{binary}} . \tag{2}$$

It is interesting to interpret Equation (2). It can be seen from Figure 1 that the four corner signal points similar to $\underline{x}_1$ have 2 nearest neighbours (i.e., point in distance d away), the eight signal points similar to $\underline{x}_2$ have 3 nearest neighbours, and the four internal signal points similar to $\underline{x}_6$ have 4 nearest neighbours. The average number of nearest neighbours in this constellation is then

$$\bar{n} = \frac{1}{16}(4 \cdot 2 + 8 \cdot 3 + 4 \cdot 4) = 3 . \tag{3}$$

Comparing Equations (2) and (3), we conclude that the symbol error probability is approximately equal to, for low enough noise, the product of the binary error probability $P_{e,\text{binary}}$ multiplied by a factor equal to the average number of signal points in the minimum distance.