Gaussian Minimum Shift Keying (GMSK)

Using a rectangular function as the basic waveform results in a high bandwidth occupancy. For this reason we might filter the signal

$$s(t) \;=\; \sum_{k=-\infty}^{\infty} S_{\mathrm{FSK}}(k)\ \mathrm{rect}\left(\frac{t-kT_s-T_s/2}{T_s}\right) \tag{1}$$

to suppress the high frequency components. An often used approach is to filter the above signal with the so-called GAUSSIAN lowpass defined by the transfer function

$$H_{\mathrm{GAUSS}}(\omega) \;=\; \exp\left[-\frac{\ln 2}{2}\left(\frac{\omega}{\omega_{3\mathrm{dB}}}\right)^2\right] . \tag{2}$$

Problem

Find the signal

$$s_{\mathrm{LP}}(t) \;=\; s(t) * h_{\mathrm{GAUSS}}(t) , \tag{3}$$

where $h_{\mathrm{GAUSS}}(t)$ denotes the impulse response of the GAUSSIAN lowpass filter.

Solution of Problem

Let us first determine the impulse response $h_{\text{GAUSS}}(t)$ of the GAUSSIAN lowpass filter, which equals the inverse FOURIER transform of the transfer function $H_{\text{GAUSS}}(\omega)$.

To do so, we start with the known FOURIER-transform

$$\sqrt{\frac{a}{\pi}}\exp(-at^2) \quad \overset{\mathcal{F}}{\circ\!\!-\!\!\bullet} \quad \exp\left(-\frac{\omega^2}{4a}\right) . \tag{4}$$

Comparing the right side of Equation (2) with the right side of Equation (4), we conclude that

$$\frac{1}{4a} = \frac{\ln 2}{2\omega_{3\text{dB}}^2} .$$

This yields

$$a = \frac{\omega_{3\text{dB}}^2}{2\ln 2} .$$

Inserting the equation above into Equation (4), we obtain

$$\sqrt{\frac{\omega_{3\text{dB}}^2}{2\pi\ln 2}}\exp\left(-\frac{t^2\omega_{3\text{dB}}^2}{2\ln 2}\right) = \frac{\omega_{3\text{dB}}}{\sqrt{2\pi\ln 2}}\exp\left(-\frac{t^2\omega_{3\text{dB}}^2}{2\ln 2}\right) \quad \overset{\mathcal{F}}{\circ\!\!-\!\!\bullet} \quad \exp\left[-\frac{\ln 2}{2}\left(\frac{\omega}{\omega_{3\text{dB}}}\right)^2\right] .$$

Obviously, the impulse response of the GAUSSIAN lowpass filter equals

$$h_{\text{GAUSS}}(t) = \frac{\omega_{3\text{dB}}}{\sqrt{2\pi\ln 2}}\exp\left(-\frac{t^2\omega_{3\text{dB}}^2}{2\ln 2}\right) . \tag{5}$$

We will now determine the lowpass filtered signal $s_{\text{LP}}(t)$. Inserting Equation (1) into Equation (3), we obtain

$$\begin{aligned}
s_{\text{LP}}(t) &= \sum_{k=-\infty}^{\infty} S_{\text{FSK}}(k)\,\text{rect}\left(\frac{t-kT_s-T_s/2}{T_s}\right) * h_{\text{GAUSS}}(t) \\
&= \sum_{k=-\infty}^{\infty} S_{\text{FSK}}(k)\int_{-\infty}^{\infty}\text{rect}\left(\frac{\tau-kT_s-T_s/2}{T_s}\right) h_{\text{GAUSS}}(t-\tau)\,\mathrm{d}\tau \\
&\overset{!}{=} \sum_{k=-\infty}^{\infty} S_{\text{FSK}}(k)\,g(t-kT_s)
\end{aligned}$$

with

$$\begin{aligned}
g(t-kT_s) &= \int_{-\infty}^{\infty}\text{rect}\left(\frac{\tau-kT_s-T_s/2}{T_s}\right) h_{\text{GAUSS}}(t-\tau)\,\mathrm{d}\tau \\
&= \int_{kT_s}^{(k+1)T_s} h_{\text{GAUSS}}(t-\tau)\,\mathrm{d}\tau .
\end{aligned}$$

From the above and Equation (5) it follows that

$$g(t-kT_s) \;=\; \frac{\omega_{3\mathrm{dB}}}{\sqrt{2\pi\ln 2}}\int_{kT_s}^{(k+1)T_s}\exp\left[-\frac{(t-\tau)^2\omega_{3\mathrm{dB}}^2}{2\ln 2}\right]\,\mathrm{d}\tau\;. \tag{6}$$

To solve this integral, we change the variables according to

$$\xi(\tau) \;=\; \frac{(t-\tau)\omega_{3\mathrm{dB}}}{\sqrt{2\ln 2}}\;.$$

We obtain

$$\begin{aligned}\frac{\mathrm{d}\xi}{\mathrm{d}\tau} &= -\frac{\omega_{3\mathrm{dB}}}{\sqrt{2\ln 2}}\;,\\ \xi\left[(k+1)T_s\right] &= \frac{\omega_{3\mathrm{dB}}}{\sqrt{2\ln 2}}\left[t-(k+1)T_s\right] \;=\; \frac{\alpha}{T_s}\left[t-(k+1)T_s\right]\;,\\ \xi(kT_s) &= \frac{\omega_{3\mathrm{dB}}}{\sqrt{2\ln 2}}(t-kT_s) \;=\; \frac{\alpha}{T_s}(t-kT_s)\;,\end{aligned}$$

where

$$\alpha \;=\; \frac{\omega_{3\mathrm{dB}}T_s}{\sqrt{2\ln 2}}\;.$$

Applying the above, Equation (6) yields

$$g(t-kT_s) \;=\; -\frac{1}{\sqrt{\pi}}\int_{\frac{\alpha}{T_s}(t-kT_s)}^{\frac{\alpha}{T_s}[t-(k+1)T_s]}\exp\left(-\xi^2\right)\,\mathrm{d}\xi$$

or equivalently

$$\begin{aligned}g(t) &= -\frac{1}{\sqrt{\pi}}\int_{\frac{\alpha}{T_s}t}^{\frac{\alpha}{T_s}(t-T_s)}\exp\left(-\xi^2\right)\,\mathrm{d}\xi\\ &= \frac{1}{\sqrt{\pi}}\int_{\frac{\alpha}{T_s}(t-T_s)}^{\frac{\alpha}{T_s}t}\exp\left(-\xi^2\right)\,\mathrm{d}\xi\\ &= \frac{1}{\sqrt{\pi}}\left[\int_0^{\frac{\alpha}{T_s}t}\exp\left(-\xi^2\right)\,\mathrm{d}\xi-\int_0^{\frac{\alpha}{T_s}(t-T_s)}\exp\left(-\xi^2\right)\,\mathrm{d}\xi\right]\;.\end{aligned} \tag{7}$$

The error function $\mathrm{erf}(x)$ equals by its definition

$$\mathrm{erf}(x) \;=\; \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,\mathrm{d}t\;.$$

Comparing this definition with Equation (7) we conclude that

$$g(t) \;=\; \frac{1}{2}\left[\mathrm{erf}\left(\alpha\frac{t}{T_s}\right)-\mathrm{erf}\left(\alpha\frac{t-T_s}{T_s}\right)\right]\;.$$

Finally, replacing $g(t)$ by $g_{\text{GMSK}}(t) \; = \; 2 \cdot g(t)$, we obtain

$$s_{\text{LP}}(t) \; = \; \frac{1}{2} \sum_{k=-\infty}^{\infty} S_{\text{FSK}}(k) \, g_{\text{GMSK}}(t - kT_s) \; ,$$

where

$$g_{\text{GMSK}}(t) \; = \; \text{erf}\left(\alpha \frac{t}{T_s}\right) - \text{erf}\left(\alpha \frac{t - T_s}{T_s}\right) \; .$$