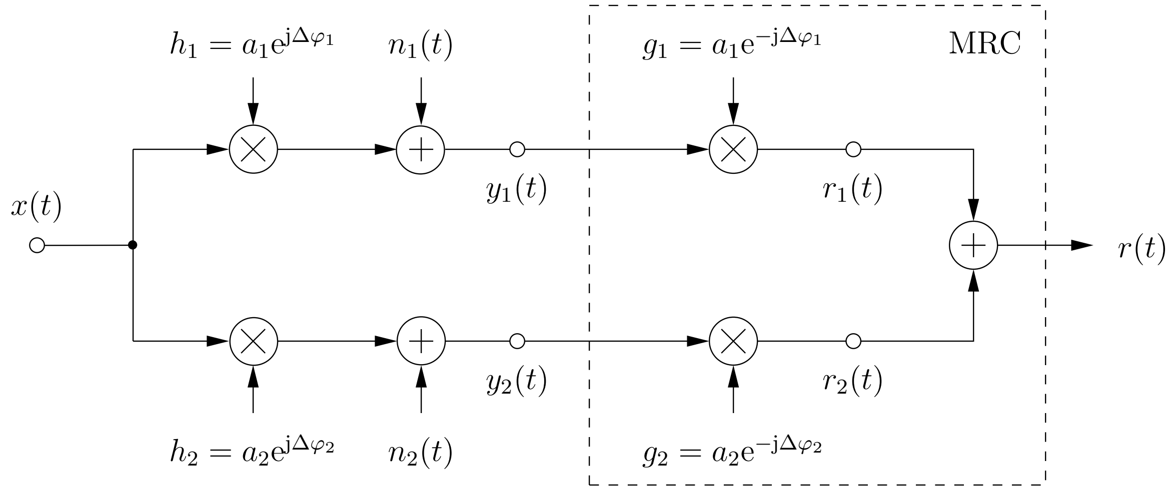
Diversity and Maximum Ratio Combining (MRC)

Problem 1

A simplified model for a baseband representation of a signal transmission and reception is shown in the following figure.



A random signal $x(t)$ with zero mean and the variance $\mathrm{E}\{x(t)x^*(t)\} = \sigma_X^2$ is transmitted through two independent RAYLEIGH fading channels, where $h_1 = a_1 e^{j\Delta\varphi_1}$ and $h_2 = a_2 e^{j\Delta\varphi_2}$ denote the fading coefficients. Hence, for $i = 1$ and 2, $|h_i| = |a_i|$ is RAYLEIGH distributed and $\Delta\varphi_i$ is assumed to be uniformly distributed in $[0, 2\pi]$. The additive noise signals $n_1(t)$ and $n_2(t)$ are independent, zero mean GAUSSIAN distributed with the same variance $\mathrm{E}\{n_1(t)n_1^*(t)\} = \mathrm{E}\{n_2(t)n_2^*(t)\} = \sigma_N^2$. The receiver uses maximum ratio combining (MRC) scheme with the respective coefficients $g_1 = a_1 e^{-j\Delta\varphi_1}$ and $g_2 = a_2 e^{-j\Delta\varphi_2}$.

1.1 Give the expression for the output signal of the combiner $r(t)$ as a function of the transmitted signal $x(t)$ and additive GAUSSIAN noise signals $n_1(t)$ and $n_2(t)$.

1.2 Calculate the variance of the noise contribution in the output combining signal $r(t)$ as a function of σ_N^2.

1.3 Calculate the instantaneous signal-to-noise ratio (SNR) γ_1 and γ_2 in terms of σ_X^2 and σ_N^2 for each branch, respectively.

1.4 What kind of relation is between γ_{MRC} and γ_i, where γ_{MRC} is the SNR after the optimum linear combination and γ_i with $i = 1$ and 2 is the SNR of each branch? Justify your answer with specifically mathematical derivations.

1.5 Now assume that the both branches have the same average SNR with $\mathrm{E}\{\gamma_1\} = \mathrm{E}\{\gamma_2\} = \Gamma = 10$. The maximum ratio combiner at the receiver side requires a minimum SNR threshold $\gamma_{\mathrm{s}} = 1$.
Calculate the outage probability $P_{\mathrm{out}} = \Pr\{\gamma_{\mathrm{MRC}} \leq \gamma_{\mathrm{s}}\}$.

Solution of Problem 1

1.1 The output signal of the combiner can be represented as

$$\begin{aligned} r(t) &= r_1(t) + r_2(t) = g_1 y_1(t) + g_2 y_2(t) \\ &= a_1 e^{-j\Delta\varphi_1} y_1(t) + a_2 e^{-j\Delta\varphi_2} y_2(t) \end{aligned} \quad (1)$$

with

$$y_1(t) = h_1 x(t) + n_1(t) = a_1 e^{j\Delta\varphi_1} x(t) + n_1(t) \quad (2)$$

and

$$y_2(t) = h_2 x(t) + n_2(t) = a_2 e^{j\Delta\varphi_2} x(t) + n_2(t) \,. \quad (3)$$

Inserting (2) and (3) into (1) yields

$$\begin{aligned} r(t) &= a_1 e^{-j\Delta\varphi_1} \left[a_1 e^{j\Delta\varphi_1} x(t) + n_1(t)\right] + a_2 e^{-j\Delta\varphi_2} \left[a_2 e^{j\Delta\varphi_2} x(t) + n_2(t)\right] \\ &= |a_1|^2 x(t) + a_1 e^{-j\Delta\varphi_1} n_1(t) + |a_2|^2 x(t) + a_2 e^{-j\Delta\varphi_2} n_2(t) \\ &= \left(|a_1|^2 + |a_2|^2\right) x(t) + \left[a_1 e^{-j\Delta\varphi_1} n_1(t) + a_2 e^{-j\Delta\varphi_2} n_2(t)\right] \,. \end{aligned} \quad (4)$$

1.2 From (4) we can know that the contribution of noise signals in the output of the combiner $r(t)$ is

$$g_1 n_1(t) + g_2 n_2(t) = a_1 e^{-j\Delta\varphi_1} n_1(t) + a_2 e^{-j\Delta\varphi_2} n_2(t) \,.$$

Since $n_1(t)$ and $n_2(t)$ are independent, zero mean, additive Gaussian noise signals with the same variance $\mathrm{E}\{n_1(t)n_1^*(t)\} = \mathrm{E}\{n_2(t)n_2^*(t)\} = \sigma_N^2$, the variance of the noise contribution in $r(t)$ can be calculated as

$$\begin{aligned} \tilde{\sigma}_N^2 &= \mathrm{E}\left\{[g_1 n_1(t) + g_2 n_2(t)]\,[g_1 n_1(t) + g_2 n_2(t)]^*\right\} \\ &= \mathrm{E}\left\{g_1 g_1^* n_1(t) n_1^*(t) + g_2 g_2^* n_2(t) n_2^*(t) + g_1 g_2^* n_1(t) n_2^*(t) + g_1^* g_2 n_1^*(t) n_2(t)\right\} \\ &= |a_1|^2 \mathrm{E}\{n_1(t)n_1^*(t)\} + |a_2|^2 \mathrm{E}\{n_2(t)n_2^*(t)\} \\ &= |a_1|^2 \sigma_N^2 + |a_2|^2 \sigma_N^2 \\ &= \left(|a_1|^2 + |a_2|^2\right) \sigma_N^2 \,. \end{aligned} \quad (5)$$

1.3 From (1), (2) and (3) one can obtain that the instantaneous SNR γ_i for the i-th branch can be expressed as

$$\begin{aligned} \gamma_i &= \frac{\mathrm{E}\left\{[g_i h_i x(t)]\,[g_i h_i x(t)]^*\right\}}{\mathrm{E}\left\{[g_i n_i(t)]\,[g_i n_i(t)]^*\right\}} = \frac{\left(|a_i|^2\right)^2 \mathrm{E}\{x(t)x^*(t)\}}{|a_i|^2 \mathrm{E}\{n_i(t)n_i^*(t)\}} \\ &= \frac{|a_i|^2 \sigma_X^2}{\sigma_N^2} \,, \end{aligned}$$

where $i = 1$ and 2. Hence, the instantaneous SNR for each branch can be represented as

$$\gamma_1 = \frac{|a_1|^2 \sigma_X^2}{\sigma_N^2}, \quad \text{for the first branch,} \tag{6}$$

$$\gamma_2 = \frac{|a_2|^2 \sigma_X^2}{\sigma_N^2}, \quad \text{for the second branch.} \tag{7}$$

1.4 The relation between γ_{MRC} and γ_i with $i = 1$ and 2 is

$$\gamma_{\mathrm{MRC}} = \sum_{i=1}^{2} \gamma_i = \gamma_1 + \gamma_2 .$$

Since from (4) and (5), we can conclude that

$$\begin{aligned}
\gamma_{\mathrm{MRC}} &= \frac{\mathrm{E}\left\{\left[\left(|a_1|^2 + |a_2|^2\right) x(t)\right] \left[\left(|a_1|^2 + |a_2|^2\right) x(t)\right]^*\right\}}{\tilde{\sigma}_N^2} \\
&= \frac{\left(|a_1|^2 + |a_2|^2\right)^2 \sigma_X^2}{\left(|a_1|^2 + |a_2|^2\right) \sigma_N^2} \\
&= \frac{|a_1|^2 \sigma_X^2}{\sigma_N^2} + \frac{|a_2|^2 \sigma_X^2}{\sigma_N^2} \\
&\overset{(6),\,(7)}{=} \gamma_1 + \gamma_2 .
\end{aligned}$$

1.5 The outage probability can be calculated as

$$\begin{aligned}
P_{\mathrm{out}} &= \Pr\{\gamma_{\mathrm{MRC}} \leq \gamma_{\mathrm{s}}\} \\
&= 1 - \mathrm{e}^{-\frac{\gamma_{\mathrm{s}}}{\Gamma}} \sum_{i=1}^{2} \frac{1}{(i-1)!} \left(\frac{\gamma_{\mathrm{s}}}{\Gamma}\right)^{i-1} .
\end{aligned} \tag{8}$$

From the assumption of the problem, we know that

$$\frac{\gamma_{\mathrm{s}}}{\Gamma} = \frac{1}{10} = 0.1 . \tag{9}$$

Inserting (9) into (8) yields

$$\begin{aligned}
P_{\mathrm{out}} &= 1 - \mathrm{e}^{-0.1} \left[\left(\frac{1}{10}\right)^0 + \left(\frac{1}{10}\right)^1\right] \\
&\approx 1 - 0.9953 = 0.0047 = 0.47\% .
\end{aligned}$$