Chapter 3

ANALOG SYSTEMS

(Version 2.1)

UNIVERSITÄT DUISBURG ESSEN

Fachgebiet
Nachrichtentechnische Systeme

NTS

3.1 A Short introduction to Network Functions

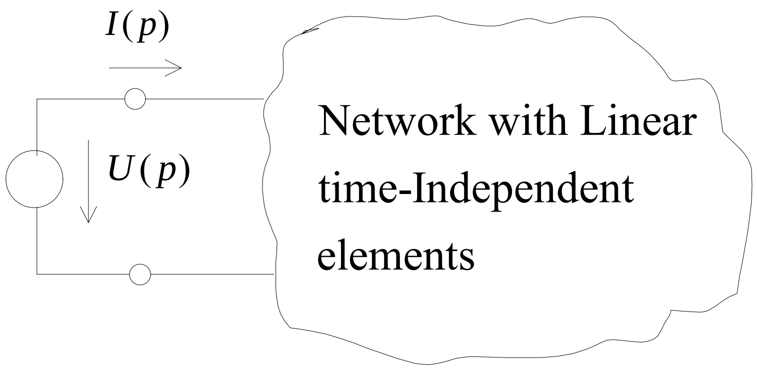
Network function: the mathematical relation between the Laplace transform of the excitation and the answer, if one assumes:

- The energy-less initialization state of all the network elements (zero-state)
- All the network elements are linear and time-independent.

One speaks of:

- Two-terminal network function, impedance function, admittance function → excitation and answer are applied to the same terminal.
- System function → excitation and answer are applied to different terminal. Notation of system function: $H_L(p)$

3.1 A Short introduction to Network Functions

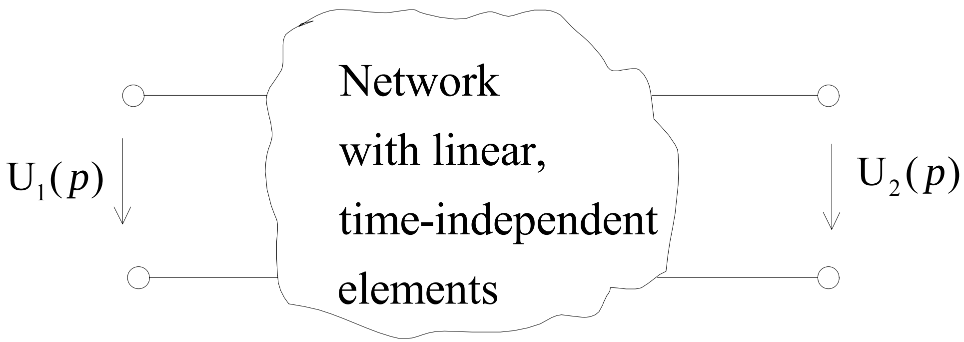


$U(p) = \text{Excitation}$

$I(p) = \text{Answer}$

$$Z(p) = \frac{U(p)}{I(p)}$$

Impedance function of a network



$U_1(p) = \text{Excitation}$

$U_2(p) = \text{Answer}$

$$H_L(p) = \frac{U_2(p)}{U_1(p)}$$

Definition of system function of a network

3.1 A short introduction to Network Functions

Some important properties of network function:

- Generally a LTI network can be described in rational fractional functions in p:

$$N(p) = \frac{P(p)}{Q(p)}$$ (all the coefficients of *P(p)* and *Q(p)* are real and constant)

- Or by the representation of the network function by its magnitude and its phase results:

$$N(p) = |N(p)|\, e^{j\varphi(p)} \;, \text{ where } |N(p)| : \text{magnitude and } \varphi(p) : \text{ phase}$$

3.1 A short introduction to Network Functions

If $p = j \cdot \omega$ is applied:

- $|N(j\omega)| = |N(-j\omega)| \;\rightarrow\; |N(j\omega)|$ is an even function
- $\varphi(-\omega) = \varphi(\omega) \qquad \rightarrow \; \varphi(\omega)$ is an odd function
- The system function is derived from quantities of the same dimension:

$$H_L(p) = \frac{U_2}{U_1}$$

Additional notes:

$$H_L(j\omega) = e^{-a(\omega) - jb(\omega)} = |H_L(j\omega)|\, e^{j\sphericalangle H_L(j\omega)} \quad \text{with}$$

$$H_L(j\omega) = \frac{U_2(j\omega)}{U_1(j\omega)} \Rightarrow a(\omega) = 20\log\left|\frac{U_1(j\omega)}{U_2(j\omega)}\right| dB \text{ is the damping ratio}$$

and $\qquad b(\omega) = -\varphi(\omega) = -\sphericalangle H_L(j\omega)$ is the damping angle

Chapter 3

ANALOG SYSTEM

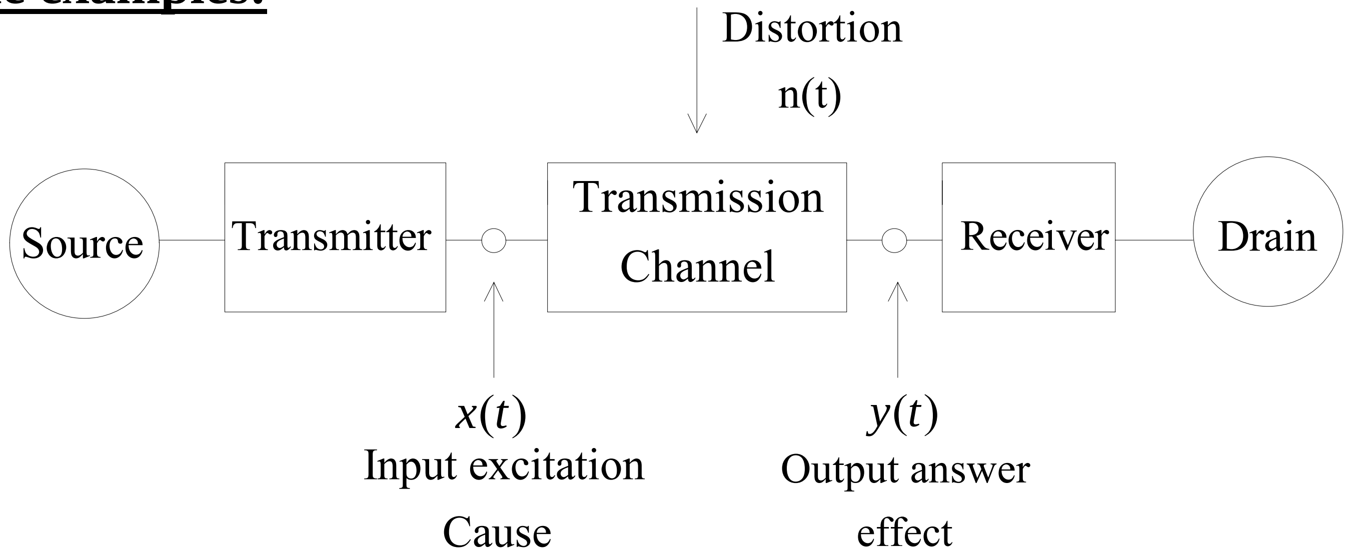
3.2 Basic Properties of a System

3.2.1 Definition of a System and General Remarks

One shall speak of a “system” if:

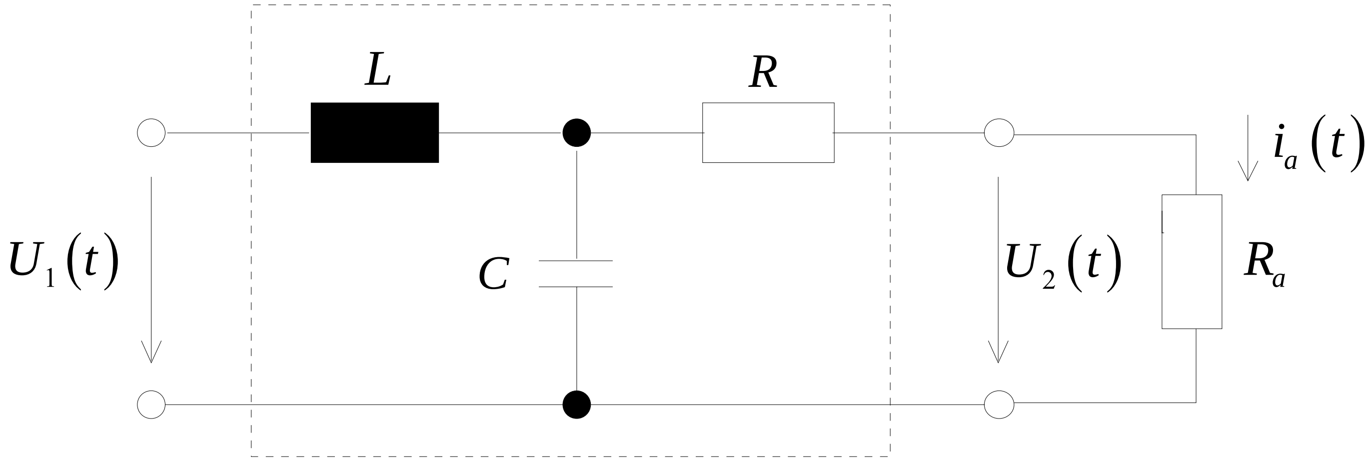
- a mathematical representation of such a circuit is given as an “input-output relation”.

Some examples:



Example of General transmission system

3.2.1 Definition of a System and General Remarks

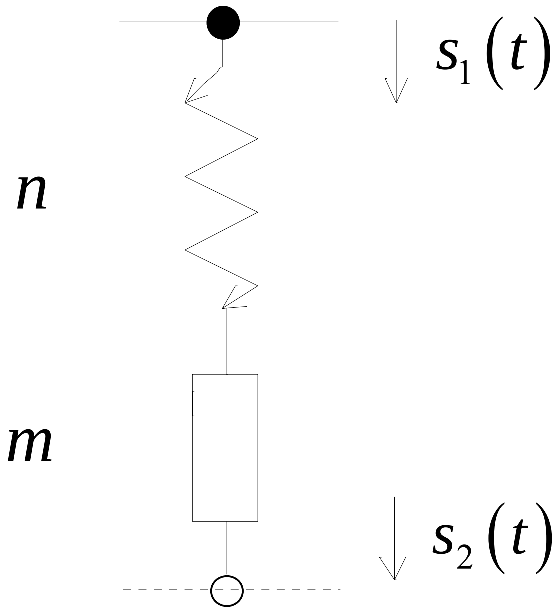


Input-Output connections:

$$u_1(t) \rightarrow u_2(t) \quad \text{or} \quad u_1(t) \rightarrow i_a(t)$$

Example of electrical system (Four-terminal network)

3.2.1 Definition of a System and General Remarks



Mechanical Example

Path

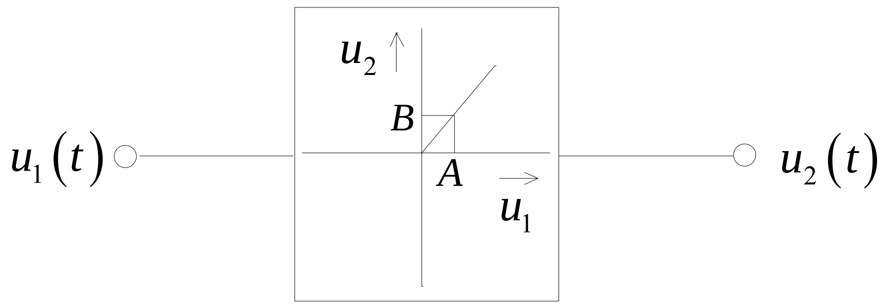
Force, Acceleration

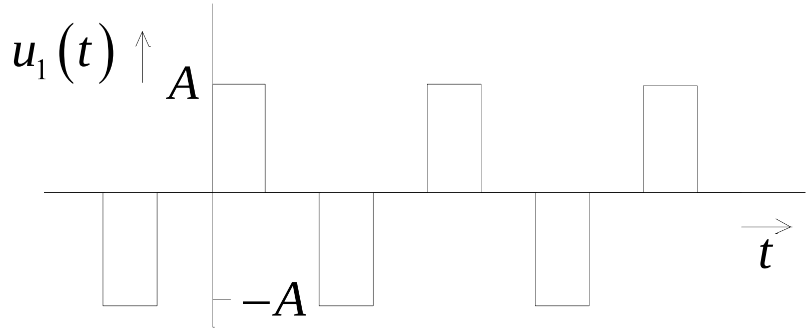
Input-Output Connection:

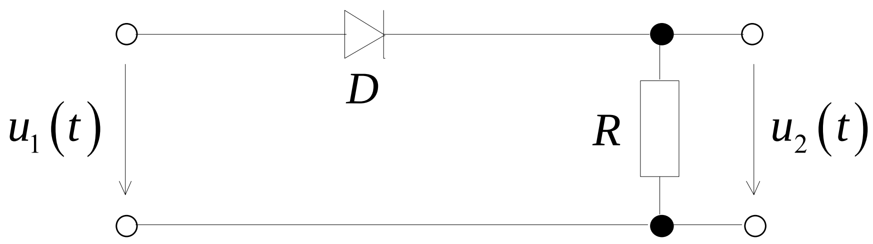
$$s_1(t) \rightarrow s_2(t)$$

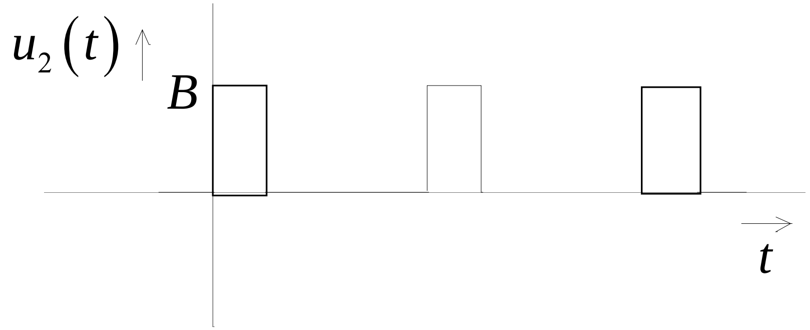
Example of Mechanical system

3.2.1 Definition of a System and General Remarks









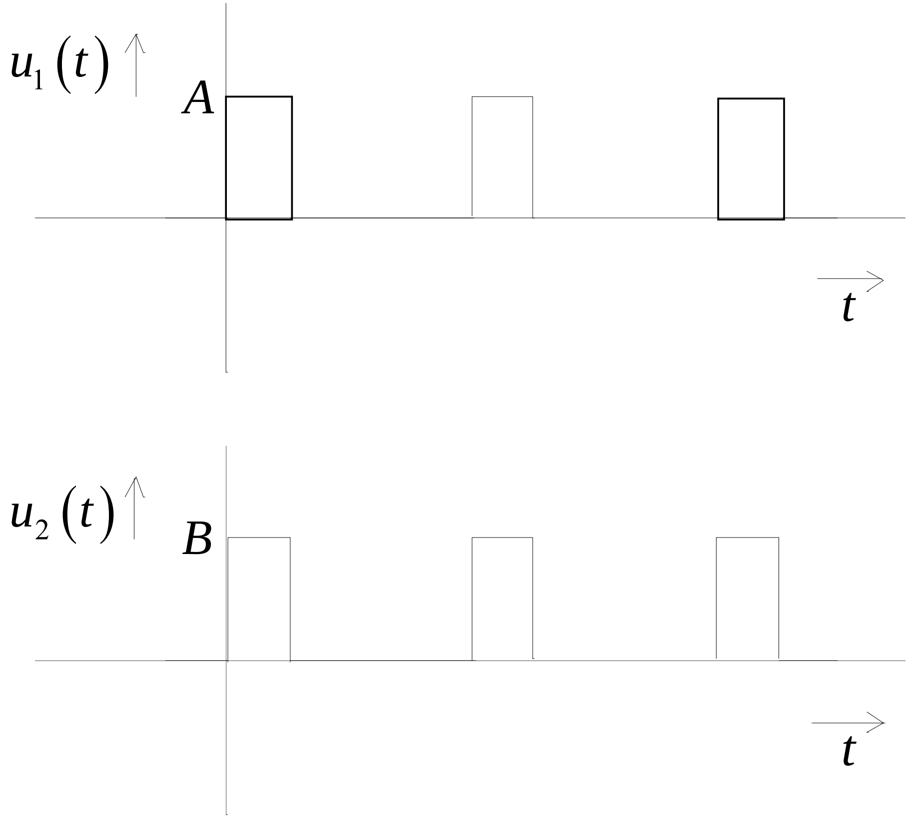
Example of a half-way rectifier

Rectifier input-output relation with an example input

3.2.1 Definition of a System and General Remarks

- Now 2 questions occur:
  - Is a system by a special input-output relation uniquely and completely represented ?
  - Under which circumstances is a benefit given from using the systems point of view ?

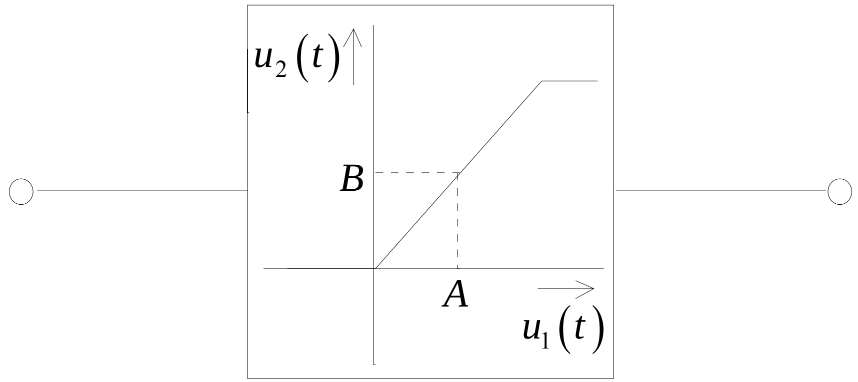
→ The uniqueness of the input-output relation of a system is not always given. It depends on the properties of the chosen input signal



Unipolar input signal and appropriate output signal of the rectifier

3.2.1 Definition of a System and General Remarks

Within the system view, it will be important to use suitable input functions for uniquely specifying the system properties. Here is an example for that:

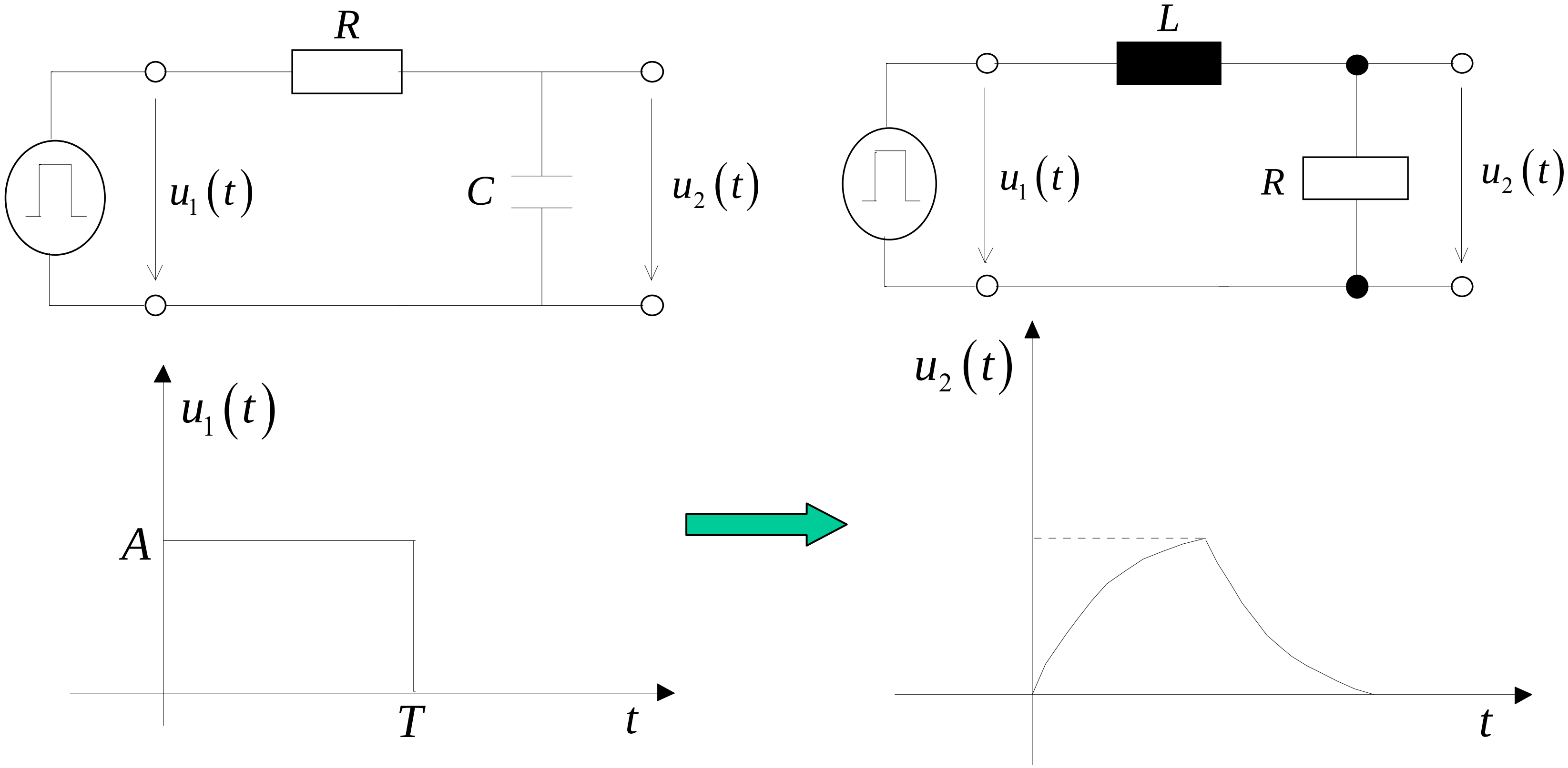


Characteristics curve of a rectifier with saturation

If a bipolar impulse chain is used as an input for the rectifier, it can not be figured out whether the system has a saturation or not !

3.2.1 Definition of a System and General Remarks

The systems which are set up completely different can generally exhibit the same mathematical description. Here is an example:



Different Circuits with the same Input-Output relation

3.2.1 Definition of a System and General Remarks

Benefits from the system theory:

1. Unique characterization of LTI system by analysis of the answer of the system at a certain standard input function is possible. Mathematically it leads to a LTI transform in time domain:

$$s(t) \Rightarrow g(t) \quad \text{or} \quad g(t) = T[s(t)]$$

2. Unique characterization of LTI system in frequency domain by the transfer function is possible → it helps in simplifying the calculation.

3.2.2 Basic System Properties

1. Time-invariance:

A system $s(t) \rightarrow g(t)$ is called time-invariant if it follows: $s(t-\tau) \rightarrow g(t-\tau)$

$\longrightarrow$ The system's reactions are always the same, independent from any delays at the input.

2. Additivity:

A system is called additivity if:

$$s_2(t) \rightarrow g_2(t)$$

$$s_1(t) \rightarrow g_1(t)$$

$$\longrightarrow s_1(t) + s_2(t) \rightarrow g_1(t) + g_2(t)$$

3. Homogeneity:

$$s(t) \rightarrow g(t) \Rightarrow as(t) \rightarrow ag(t)$$

3.2.2 Basic System Properties

4. Linearity:

The combination of both additivity and homogeneity results linearity.

$$as_1(t) + bs_2(t) \rightarrow ag_1(t) + bg_2(t)$$

If the excitation x(t) of the system is known, then the wanted answer y(t) can be determined in the following way:

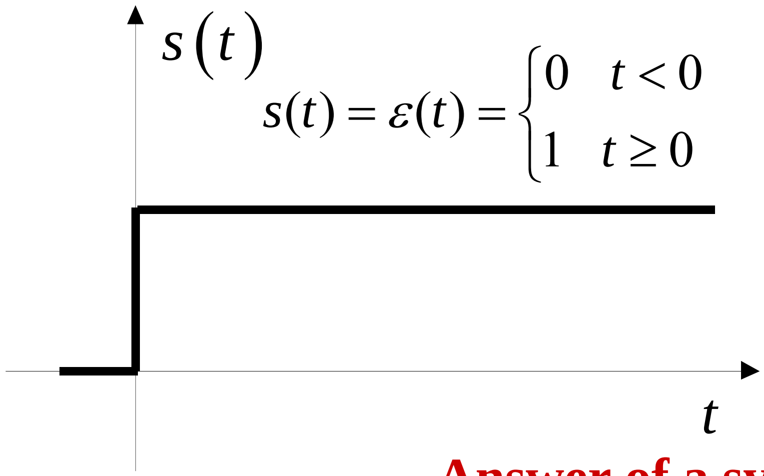
$$x(t) = \sum_{i=1}^{n} s_i(t) \text{ whereby } s_i(t) \rightarrow g_i(t)$$

then $y(t) = \sum_{i=1}^{n} g_i(t)$ because of the linearity

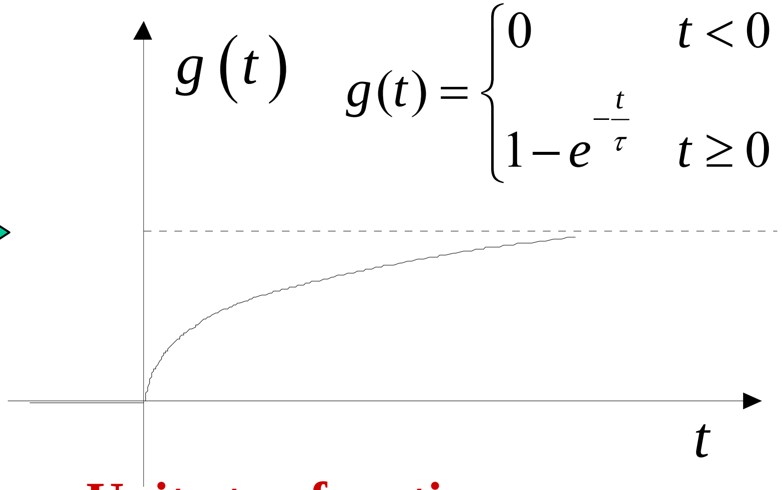
Example: let's assume an LTI system with its answer to the unit-step as followings:

$$s(t) = \varepsilon(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \xrightarrow{\text{answer}} g(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-\frac{t}{\tau}} & t \geq 0 \end{cases}$$

3.2.2 Basic System Properties

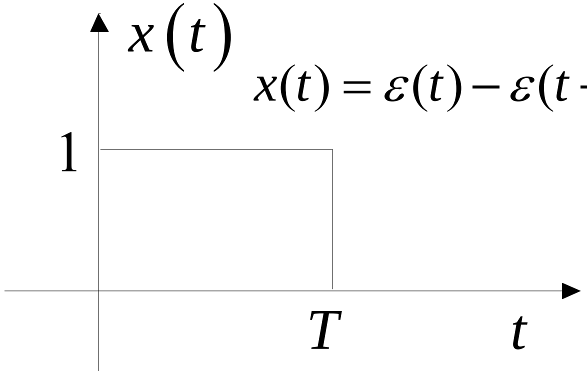


$$s(t) = \varepsilon(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



$$g(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-\frac{t}{\tau}} & t \geq 0 \end{cases}$$

Answer of a system to an Unit-step function



$$x(t) = \varepsilon(t) - \varepsilon(t - T)$$

?

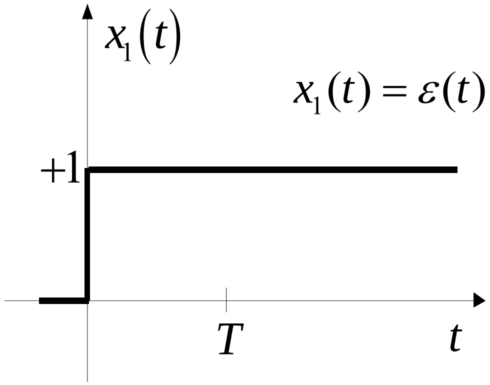
3.2.2 Basic System Properties

The solution goes by exploiting the linearity and time-invariance of the system:

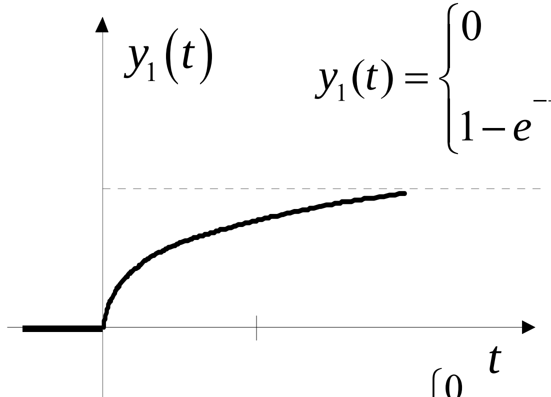
- First step: separation of the input function into 2 steps using rect-function
- Second step: find out the 2 answers for each rect-function
- Third step: superposition of the answer

In the following slide it shows the graphical determination of the system answer to a rectangular impulse

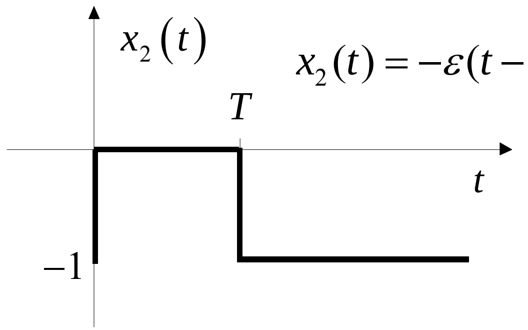
3.2.2 Basic System Properties



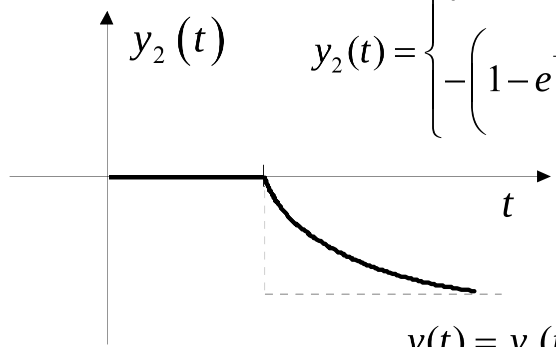
$$x_1(t) = \varepsilon(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



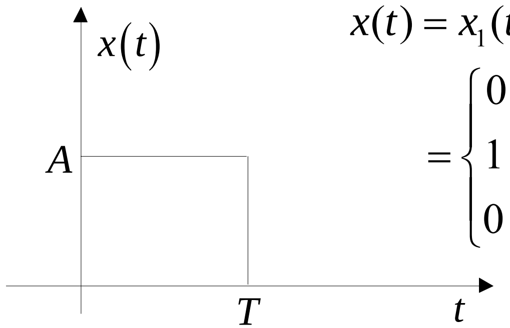
$$y_1(t) = \begin{cases} 0 & t < 0 \\ 1 - e^{-\frac{t}{\tau}} & t \geq 0 \end{cases}$$



$$x_2(t) = -\varepsilon(t-T) = \begin{cases} 0 & t < 0 \\ -1 & t \geq 0 \end{cases}$$

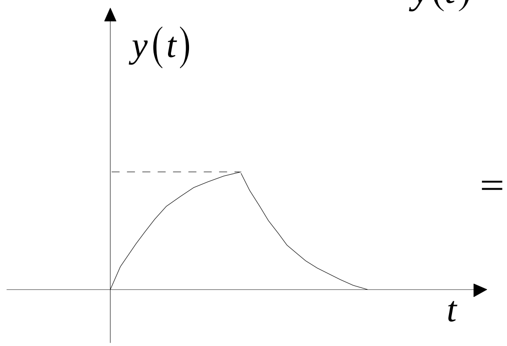


$$y_2(t) = \begin{cases} 0 & t < 0 \\ -\left(1 - e^{-\frac{t-T}{\tau}}\right) & t \geq 0 \end{cases}$$



$$x(t) = x_1(t) + x_2(t)$$

$$= \begin{cases} 0 & t<0 \\ 1 & 0 \leq t \leq T \\ 0 & t > T \end{cases}$$



$$y(t) = y_1(t) + y_2(t)$$

$$= \begin{cases} 0 & t<0 \\ 1 - e^{-\frac{t}{\tau}} & 0 \leq t \leq T \\ e^{-\frac{t}{\tau}} \cdot \left(e^{\frac{T}{\tau}} - 1\right) & t>T \end{cases}$$

3.2.3 Realisable Systems

1. Real systems:

   A system is called real when it follows:

   Real input ⟹ **Real output**

2. Causal system:

   A system is causal if the output signal g(t) until an arbitrary time depends only the input signal s(t) until this time.

   No effects before the cause

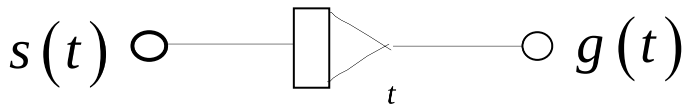
   $$s(t) \equiv 0 \quad \text{for} \quad t < t_v \longrightarrow g(t) \equiv 0 \quad \text{for} \quad t < t_v$$

3. Stable system:

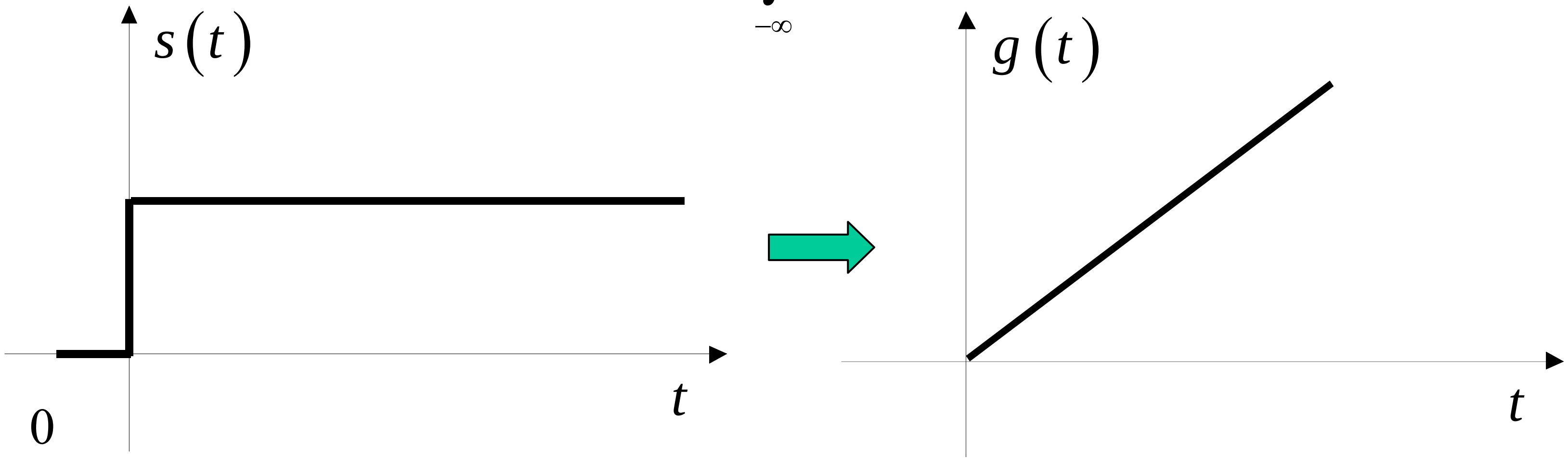
   A system $s(t) \rightarrow g(t)$ is stable in the BIBO sense (Bounded Input Bounded Output) if: $|s(t)| < M_1 < \infty \quad \forall t,$ then $|g(t)| < M_1 < \infty \quad \forall t$

3.2.3 Realisable Systems

An example of non-stable system: the ideal integrator



$$s(t) \quad \rightarrow \quad g(t) = \int_{-\infty}^{t} s(\tau)d\tau$$



A system is asymptotically stable if a damped input signal causes a damped output signal

$$\lim_{t \to +\infty} s(t) = 0 \quad \rightarrow \quad \lim_{t \to +\infty} g(t) = 0$$

3.2.3 Realisable Systems

4. Memory less and dynamic systems:

If for anytime t, the value of the output signal g(t) of a system depends exclusively on the value of the input signal s(t) at the same time t, then the system is called memory-less

Chapter 3

ANALOG SYSTEM

3.3 Analog Linear Time-Invariant System

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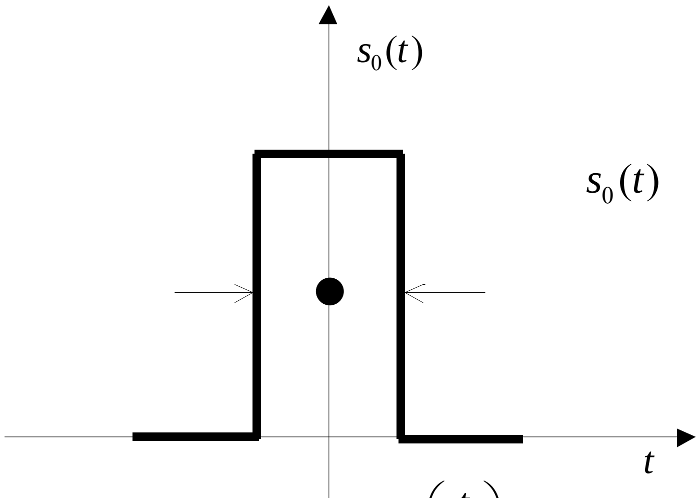
3.3.1 Convolution Integral and Impulse Response

A LTI system is denoted as:

$$s(t) \quad \rightarrow \quad g(t) = T\left[s(t)\right]$$

Derivation of general form of T[s(t)]:

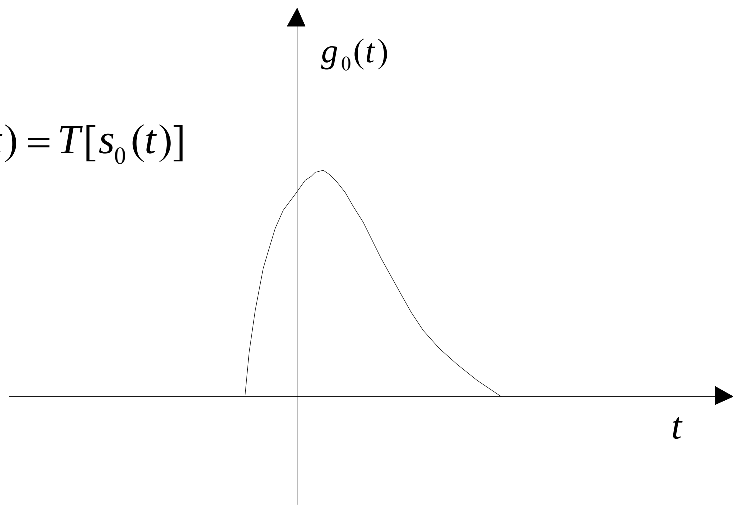
Let's assume: a LTI system reacts to a small rectangular impulse as:



$$s_0(t) = rect\left(\frac{t}{T}\right)$$

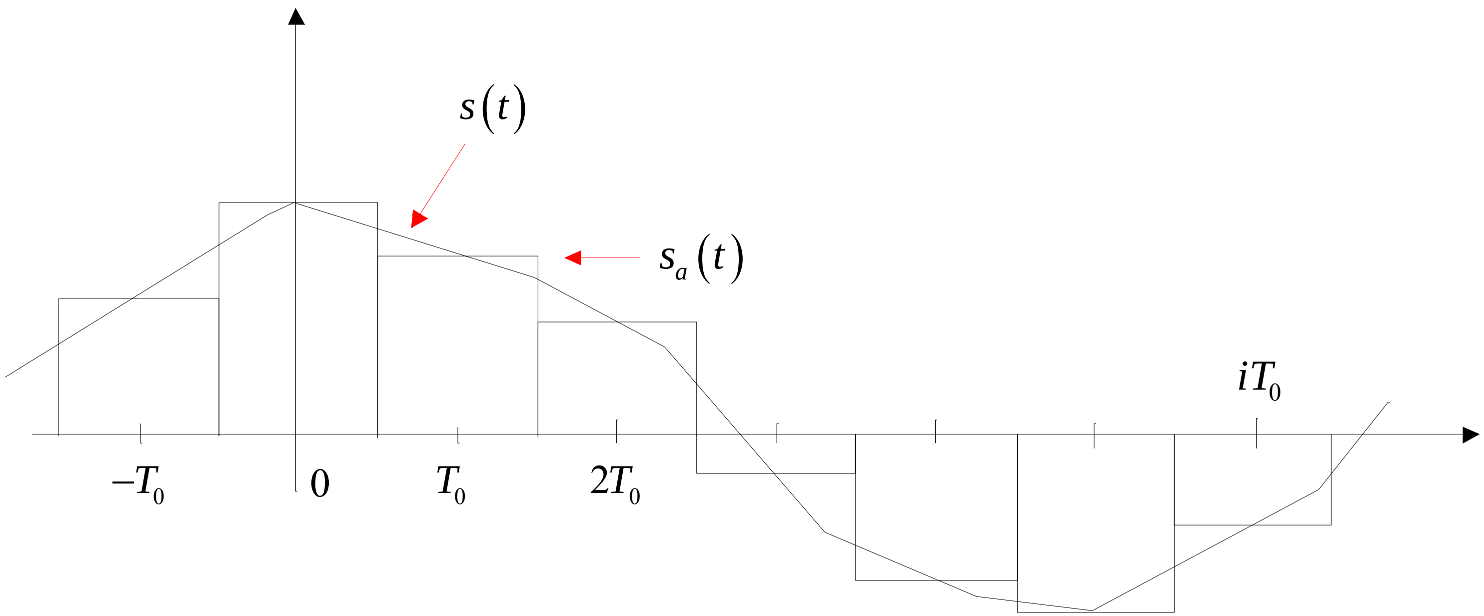
$$s_0(t) \quad \rightarrow \quad g_0(t) = T[s_0(t)]$$

$$\Rightarrow$$



3.3.1 Convolution Integral and Impulse Response

The answer of the system to an arbitrary input signal can be calculated approximately if the properties LTI are used:



Division of an input signal into a staircase signal

3.3.1 Convolution Integral and Impulse Response

Step 1: separate the input signal into staircase signal elements

$$s(t) \approx s_a(t) = \sum_{i=1}^{\infty} s(iT_0)s_0(t - iT_0) \quad \text{with} \quad i = 0, \pm 1, \pm 2, ..., \pm\infty$$

Step 2: because the system is time-invariant, it follows:

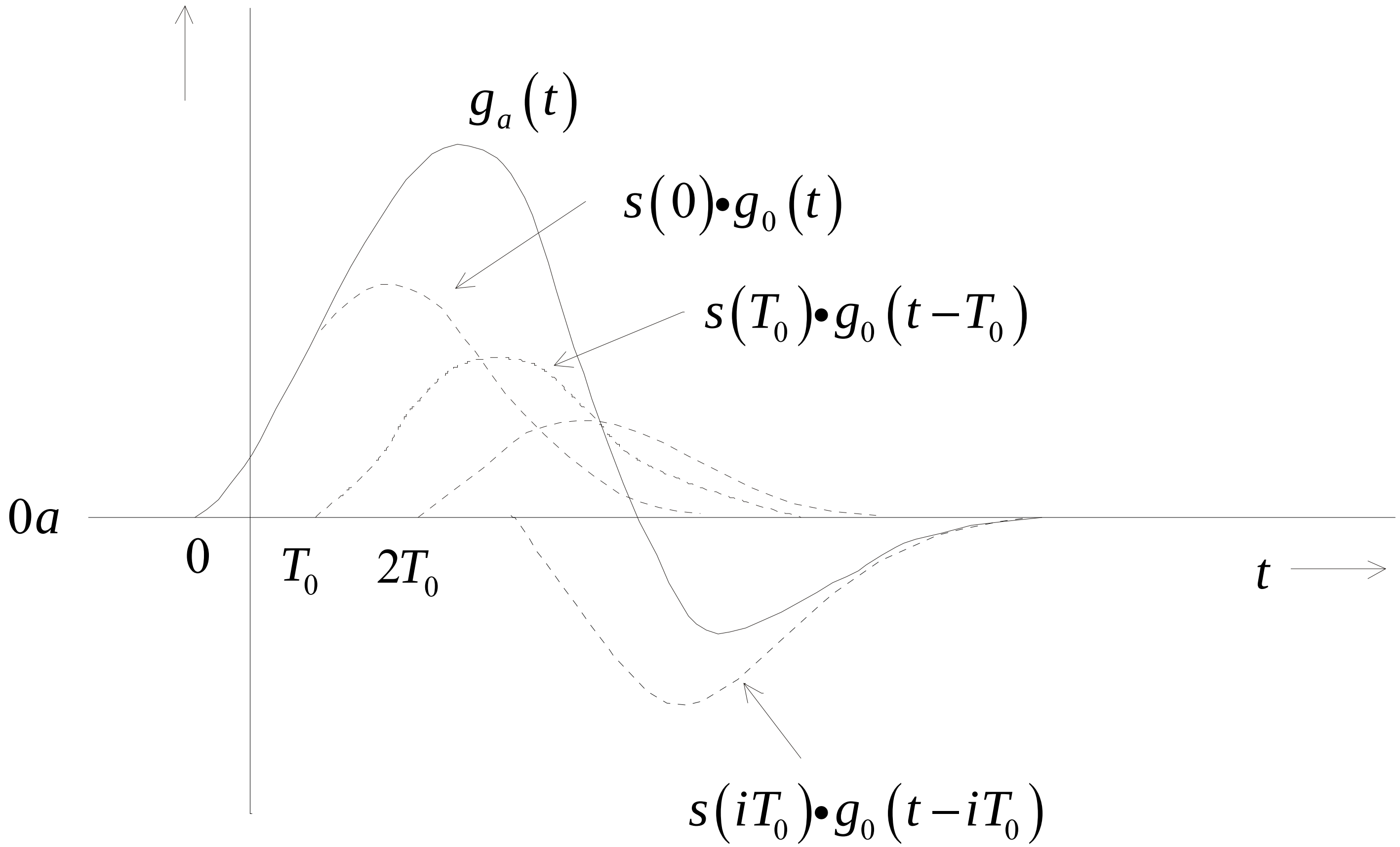
$$s(iT_0) \cdot s_0(t - iT_0) \rightarrow s(iT_0) \cdot g_0(t - iT_0)$$

Step 3: because the system is linear, one can use the superposition:

$$g(t) \approx g_a(t) = \sum_{i=1}^{\infty} s(iT_0) \cdot g_0(t - iT_0)$$

→ The smaller T_0 is, the more accurate the approximation of $g_a(t)$ is.

3.3.1 Convolution Integral and Impulse Response



Superposition of the single answers

3.3.1 Convolution Integral and Impulse Response

Now a limiting $T_0 \to 0$ provides a perfect approximation:

$$s(t) = \lim_{T_0 \to \infty} s_a(t) \quad (*)$$

For this, an extension is made as followings:

$$s_a(t) = \sum_{i=-\infty}^{\infty} s(iT_0) \cdot \frac{s_0(t - iT_0)}{T_0} \cdot T_0 \quad \Rightarrow \quad g_a(t) = \sum_{i=-\infty}^{\infty} s(iT_0) \cdot \frac{g_0(t - iT_0)}{T_0} \cdot T_0$$

The relation (*) is fulfilled if:

1. T_0 becomes an infinitesimal $d\tau$
2. $i \cdot T_0$ becomes the continous variable τ, so that $s(i \cdot T_0) \to s(\tau)$
3. $\dfrac{s_0(t - i \cdot T_0)}{T_0}$ becomes Dirac's delta function $\delta(t - \tau)$
4. $\dfrac{g_0(t - i \cdot T_0)}{T_0} \overset{!}{=} h(t - \tau)$ where $h(t)$ called the impulse response.

3.3.1 Convolution Integral and Impulse Response

With the „extraction-property" of Dirac's delta function, one obtains:

$$\lim_{T_0 \to \infty} s_a(t) = s(t) = \int_{-\infty}^{\infty} s(\tau)\delta(t-\tau)d\tau$$

$$\lim_{T_0 \to \infty} g_a(t) = g(t) = \int_{-\infty}^{\infty} s(\tau)h(t-\tau)d\tau$$

convolutional integral

<u>Some remarks:</u>

- convolutional integral is a general method for determing the response of an LTI system to any excitation.

- *h(t)*, the impulse response, is required to evaluate the convolutional integral and can be determined as following:

$$\delta(t) \to g(t) = \int_{-\infty}^{\infty} \delta(\tau)h(t-\tau)d\tau = h(t)$$

3.3.1 Convolution Integral and Impulse Response

• For causal systems with „No effect before the cause“ follows:

$$h(t) \equiv 0 \quad \text{for} \quad t < 0$$

→ The impulse response of a causal system is also a causal signal function

The convolutional integral for causal system can be described as:

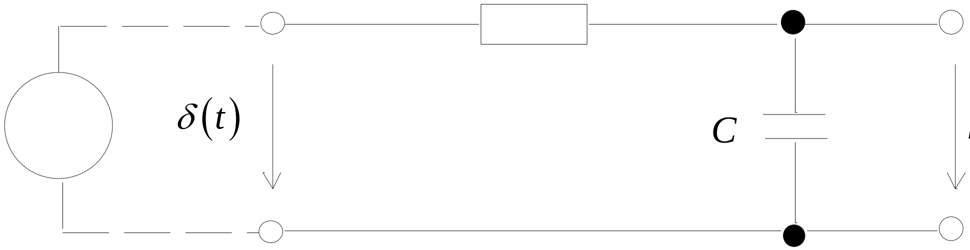
$$g(t) = \int_{-\infty}^{+\infty} s(\tau) \cdot h(t-\tau) d\tau \text{ with } h(t-\tau) \equiv 0 \quad \text{for } t-\tau < 0 \;\Leftrightarrow\; t < \tau$$

$$= \int_{-\infty}^{t} s(\tau) h(t-\tau) d\tau$$

3.3.1 Convolution Integral and Impulse Response

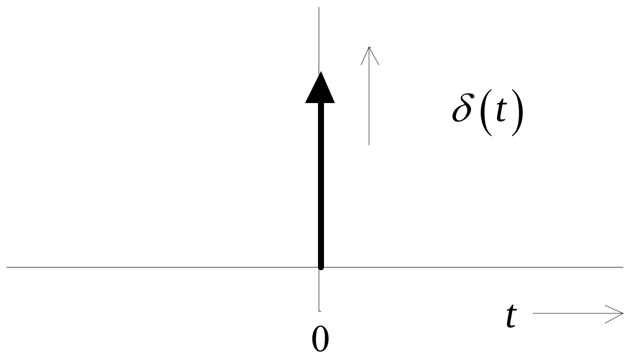
Example: given the following:

RC - circuit

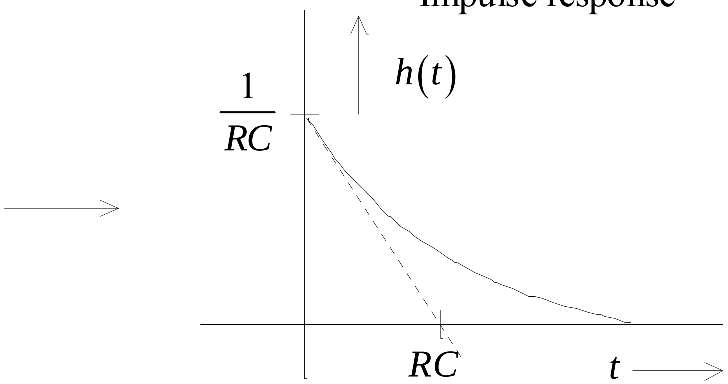


$$h(t) = \begin{cases} \frac{1}{RC} e^{-\frac{t}{RC}} & \text{for} \quad t \geq 0 \\ 0 & \text{for} \quad t < 0 \end{cases}$$

Dirac's delta function

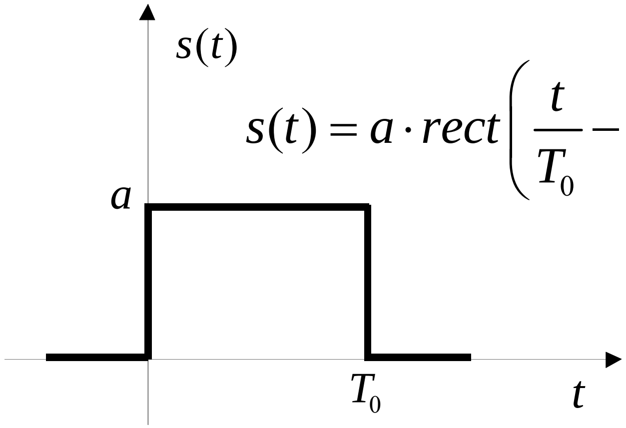


Impulse response



3.3.1 Convolution Integral and Impulse Response

Question: Reaction of the system to a rectangular impulse at the input



$$s(t) = a \cdot rect\left(\frac{t}{T_0} - \frac{1}{2}\right) = a \cdot rect\left(\frac{t - \frac{T_0}{2}}{T_0}\right) \Rightarrow ?$$

Solution: Evaluation of the convolution integral:

$$g(t) = \int_{-\infty}^{+\infty} s(\tau) \cdot h(t - \tau) d\tau$$

3.3.1 Convolution Integral and Impulse Response

It leads to the following relations and results:

1. $g(t) \equiv 0 \quad \text{for} \quad t < 0$

2. $g(t) = \int_0^t s(\tau) \cdot h(t-\tau)\,d\tau \quad \text{for} \quad 0 \le t \le T_0$

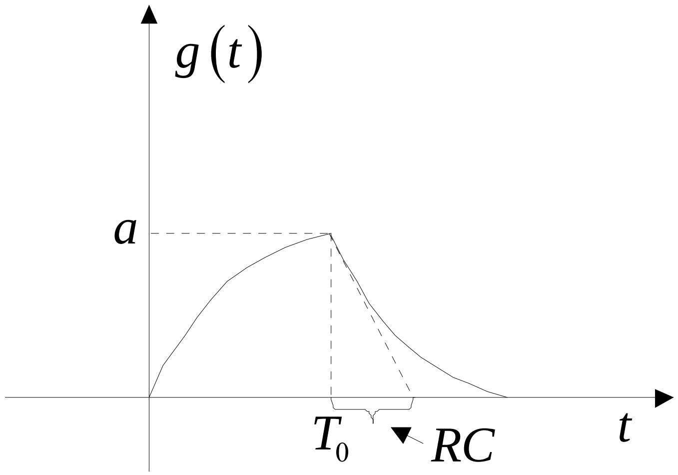
$$g(t) = \int_0^t a \frac{1}{RC} \cdot e^{-\left(\frac{t-\tau}{RC}\right)} d\tau = \frac{a}{RC} \cdot e^{-\left(\frac{t}{RC}\right)} \int_0^t e^{\left(\frac{\tau}{RC}\right)} d\tau$$

$$= \frac{a}{RC} \cdot e^{-\left(\frac{t}{RC}\right)} RC e^{\left(\frac{\tau}{RC}\right)} \Bigg|_0^t = a\left(1 - e^{-\left(\frac{t}{RC}\right)}\right)$$

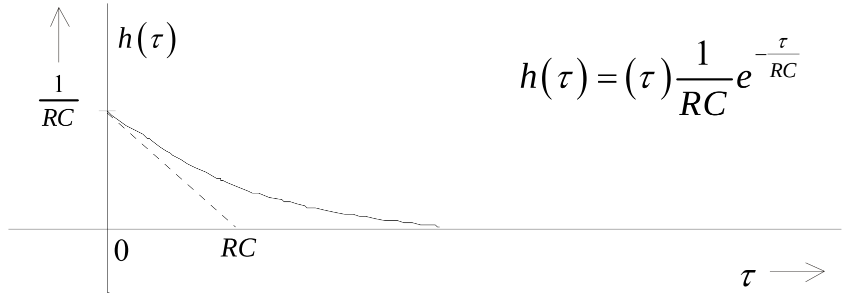
3. $g(t) = \int_0^{T_0} s(\tau) \cdot h(t-\tau)\,d\tau \quad \text{for} \quad t > T_0$

$$g(t) = \int_0^{T_0} a \cdot \frac{1}{RC} \cdot e^{-\left(\frac{t-\tau}{RC}\right)} d\tau = \frac{a}{RC} \cdot e^{-\left(\frac{t}{RC}\right)} \cdot \int_0^{T_0} e^{\left(\frac{\tau}{RC}\right)} d\tau$$

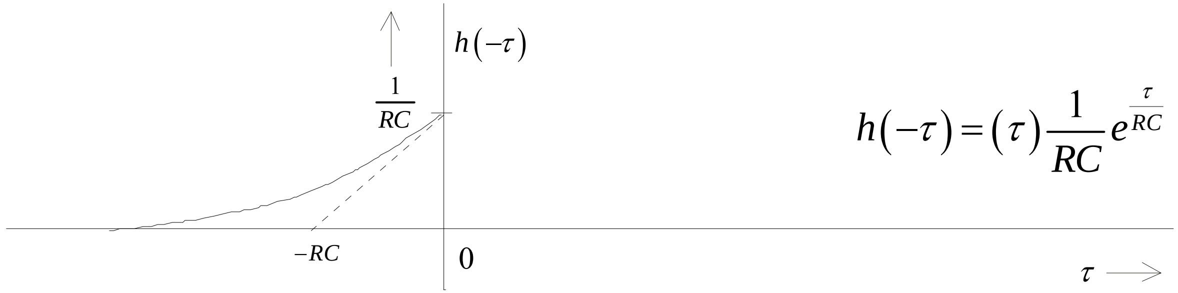
$$= a e^{-\left(\frac{t}{RC}\right)} e^{\left(\frac{\tau}{RC}\right)} \Bigg|_0^{T_0} = a e^{-\left(\frac{t}{RC}\right)} \cdot \left(-1 + e^{\left(+\frac{T_0}{RC}\right)}\right)$$



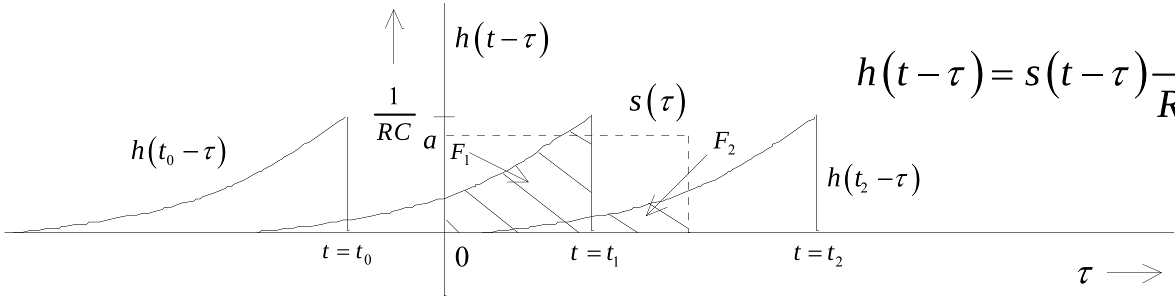
3.3.1 Convolution Integral and Impulse Response



$$h(\tau)=(\tau)\frac{1}{RC}e^{-\frac{\tau}{RC}}$$



$$h(-\tau)=(\tau)\frac{1}{RC}e^{\frac{\tau}{RC}}$$



$$h(t-\tau)=s(t-\tau)\frac{1}{RC}e^{-\frac{t-\tau}{RC}}$$

3.3.2 Step Response

The system response to any excitation s(t) can be determined as:

$$s(t) \to g(t) = s(t) * h(t)$$

Example: $\delta(t) \to h(t)$ Impulse response

Similarly, the response of an LTI-system to the unit step function gives:

$$s(t) = \varepsilon(t) \to g(t) = \varepsilon(t) * h(t) = \int_{-\infty}^{+\infty} \varepsilon(t-\tau) h(\tau) d\tau = \int_{-\infty}^{t} h(\tau) d\tau$$

In short: $w(t) = \int_{-\infty}^{t} h(\tau) d\tau$ **Step response**

Moreover: $h(t) = \frac{d}{dt} w(t)$

3.3.2 Step Response

Proof:

$$s(t) \to g(t) = \int_{-\infty}^{+\infty} s(\tau)h(t-\tau)d\tau$$

The partial integration gives: $\int_a^b u.dv = u.v \Big|_a^b - \int_a^b v.du$

$$s(\tau) = u(\tau) \quad \Rightarrow \quad \frac{du(\tau)}{d\tau} = s'(\tau) \quad \Rightarrow \quad du = s'(\tau)d\tau$$

$$h(t-\tau)d\tau = dv \quad \Rightarrow \quad \frac{dv}{d\tau} = h(t-\tau) \quad \text{where } \; t-\tau = x \;\to\; -d\tau = dx$$

$$\frac{dv}{dx} = -h(x) \quad \Rightarrow \quad v(x) = -\int h(x)dx = -w(x) = -w(t-\tau)$$

Thus:

$$s(t) \to g(t) = -s(\tau)w(t-\tau)\Big|_{-\infty}^{+\infty} + \int_{-\infty}^{+\infty} w(t-\tau)s'(\tau)d\tau$$

3.3.3 The Transfer Function *H(ω)*

Let the exponential function be the excitation of a LTI-system:

$$s(t) = s_0 e^{j\omega t} = s_0 \cos \omega t + j s_0 \sin \omega t$$

The response of any LTI-system is given by the convolution of impulse response and input signal:

$$g(t) = s_0 e^{j\omega t} * h(t) = \int_{-\infty}^{+\infty} h(\tau) \cdot s_0 e^{j\omega(t-\tau)} d\tau$$

$$= s_0 e^{j\omega t} \int_{-\infty}^{+\infty} h(\tau) \cdot e^{-j\omega\tau} d\tau = s_0 e^{j\omega t} \cdot H(\omega)$$

where:

$$H(\omega) = \int_{-\infty}^{+\infty} h(t) \cdot e^{-j\omega t} dt$$

⟶ **Transfer function** of the system:
The Fourier transform of the systems impulse response

3.3.3 The Transfer Function *H(ω)*

By applying the Fourier transform to the convolution as shown before, one gets:

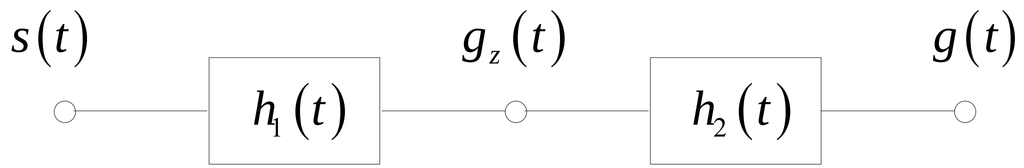
$$g(t) = s(t) * h(t) = \int_{-\infty}^{+\infty} s(\tau)h(t-\tau)d\tau$$

$$\mathcal{F} \; \circ\!\!-\!\!\bullet$$

$$G(\omega) = S(\omega)H(\omega) \quad \Leftrightarrow \quad \boxed{H(\omega) = \frac{G(\omega)}{S(\omega)} \quad \forall S(\omega) \neq 0}$$

<u>Effectless systems:</u>

Let‘s assume a combined system as follows:



Cascade of two LTI systems

3.3.3 The Transfer Function *H(ω)*

One obtains:

$$g_z(t) = s(t) * h_1(t)$$

$$g(t) = g_z(t) * h_2(t) \quad \Rightarrow \quad g(t) = [s(t) * h_1(t)] * h_2(t)$$

$$\mathcal{F} \; \circ\!\!-\!\!\bullet$$

$$G(\omega) = S(\omega) H_1(\omega) H_2(\omega)$$

Theorem: When connecting effectless systems in a chain, the transfer functions are multiplied.

Moreover:

The properties of the convolution product lead to the following equations:

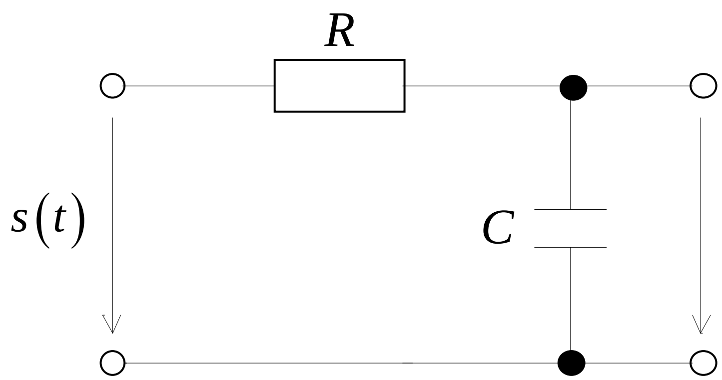
$$g(t) = [s(t) * h_1(t)] * h_2(t)$$

$$g(t) = s(t) * [h_1(t) * h_2(t)] \quad \overset{\mathcal{F}}{\circ\!\!-\!\!\bullet} \quad G(\omega) = S(\omega)[H_1(\omega) H_2(\omega)] = S(\omega) H_{tot}(\omega)$$

3.3.3 The Transfer Function $H(\omega)$

Example of a simple filter as a LTI-system:

Given is the impulse response of the following system:



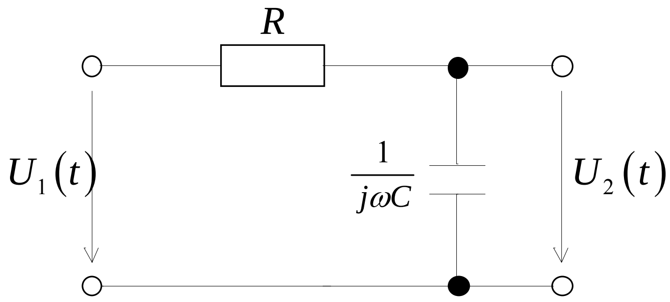
$$g(t) \Rightarrow h(t) = \begin{cases} \frac{1}{RC} \cdot e^{-\frac{t}{RC}} & for \quad t \geq 0 \\ 0 & for \quad t < 0 \end{cases}$$

$$H(\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt = \frac{1}{RC} \int_{0}^{+\infty} e^{-\frac{t}{RC}} e^{-j\omega t} dt$$

$$= \frac{1}{RC} \int_{0}^{+\infty} e^{-\left(\frac{1}{RC} + j\omega\right)t} dt = \frac{1}{RC} \cdot \frac{-1}{\frac{1}{RC} + j\omega} e^{-\left(\frac{1}{RC} + j\omega\right)t} \Bigg|_{0}^{+\infty} = \frac{1}{1 + j\omega RC}$$

3.3.3 The Transfer Function *H(ω)*

The same result can be obtained by using network analysis: $H(\omega) = \dfrac{U_2(\omega)}{U_1(\omega)}$



RC-Circuit as complex voltage divider

$$\frac{U_2(\omega)}{U_1(\omega)} = \frac{\dfrac{1}{j\omega C}}{R + \dfrac{1}{j\omega C}} = \frac{1}{1 + j\omega RC} = H(\omega)$$

This method (determination of transfer function based on network analysis) works for any LTI network!

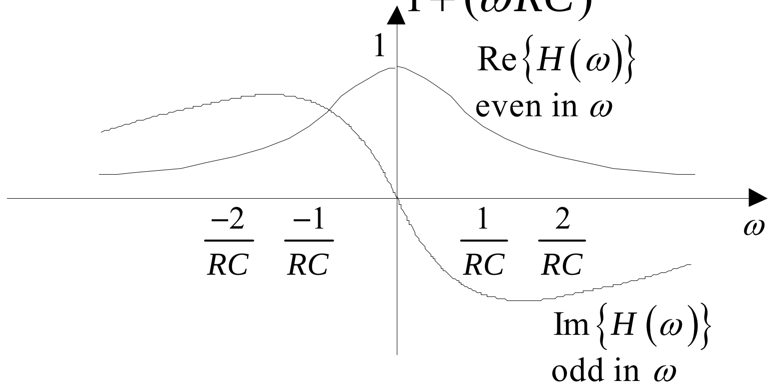
3.3.3 The Transfer Function $H(\omega)$

Real and Imaginary part of the transfer function of an RC circuit:

Example: $$H(\omega) = \frac{1}{1 + j\omega RC}$$

In general: $$H(\omega) = \mathrm{Re}\{H(\omega)\} + j\,\mathrm{Im}\{H(\omega)\} = H_R(\omega) + jH_X(\omega)$$

$$\mathrm{Re}\{H(\omega)\} = \frac{1}{1+(\omega RC)^2} = H_R(\omega) \qquad \mathrm{Im}\{H(\omega)\} = -\frac{\omega RC}{1+(\omega RC)^2} = H_X(\omega)$$



Real and imaginary part of transfer function of an RC-circuit (low-pass filter)

Another representation for the transfer function: $H(\omega) = |H(\omega)|\,e^{j\varphi(\omega)} = A(\omega)e^{j\varphi(\omega)}$

$|H(\omega)|$ Magnitude of the transfer function

$\varphi(\omega)$ Phase of the transfer function

3.3.3 The Transfer Function *H(ω)*

The magnitude and phase of the transfer function of an RC-Circuit:

Example:

$$|H(\omega)| = \sqrt{(\mathrm{Re}\{H(\omega)\})^2 + (\mathrm{Im}\{H(\omega)\})^2} = \frac{1}{\sqrt{1+(\omega RC)^2}}$$

$$\varphi(\omega) = \arctan\left(\frac{\mathrm{Im}\{H(\omega)\}}{\mathrm{Re}\{H(\omega)\}}\right) = -\arctan(\omega RC)$$

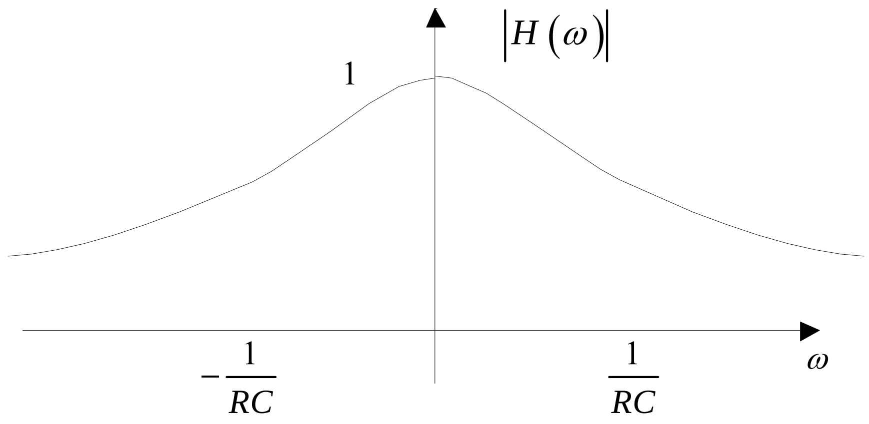
Because h(t) is real-valued function, one gets:

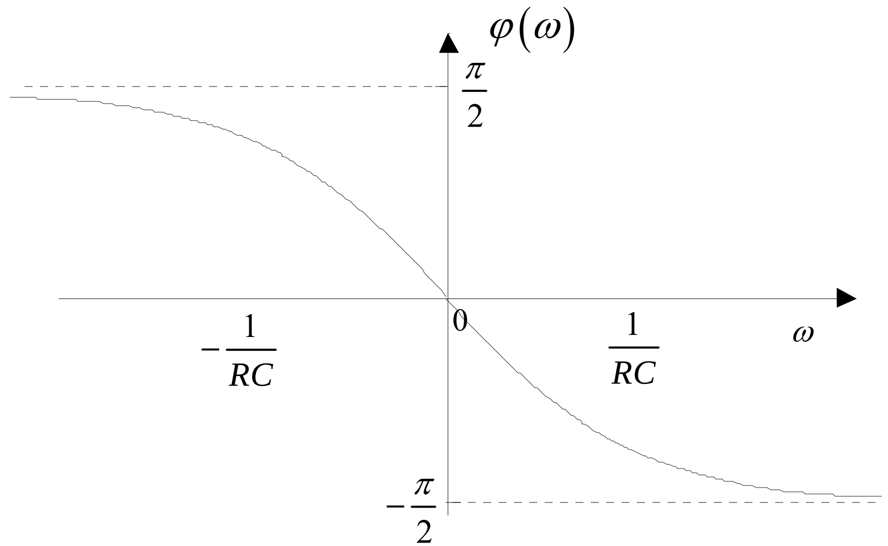
$$H(-\omega) = H^*(\omega)$$

$$|H(\omega)| = |H(-\omega)|$$

$$\varphi(-\omega) = -\varphi(\omega)$$

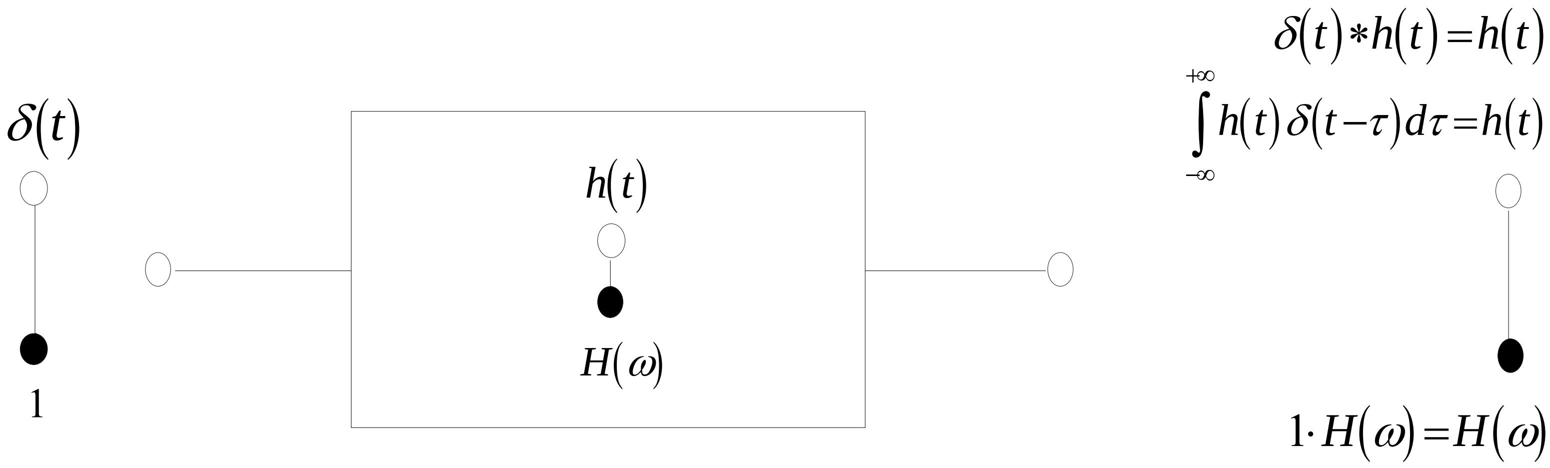
3.3.3 The Transfer Function $H(\omega)$





Magnitude and phase of an RC-Circuit (Low-pass filter)

3.3.4 Analog LTI-Systems, Describable by Differential Equations



Description of a LTI system with h(t) and H(ω)

3.3.4 Analog LTI-Systems, Describable by Differential Equations

A certain class of analog LTI system can also be described by finite differential equations as follows:

$$\sum_{\nu=0}^{m} a_\nu \frac{d^\nu}{(dt)^\nu} s(t) = \sum_{\mu=0}^{n} c_\mu \frac{d^\mu}{(dt)^\mu} g(t)$$

These kinds of system are LTI if the coefficient a_ν and c_μ are constants.

The Fourier transform of both sides of the equation above yields:

$$\sum_{\nu=0}^{m} a_\nu (j\omega)^\nu S(\omega) = \sum_{\mu=0}^{n} c_\mu (j\omega)^\mu G(\omega)$$

or $$H(\omega) = \frac{G(\omega)}{S(\omega)} = \frac{\sum_{\nu=0}^{m} a_\nu (j\omega)^\nu}{\sum_{\mu=0}^{n} c_\mu (j\omega)^\mu} \longrightarrow h(t) = F^{-1}\left\{\frac{G(\omega)}{S(\omega)}\right\} = F^{-1}\left\{\frac{\sum_{\nu=0}^{m} a_\nu (j\omega)^\nu}{\sum_{\mu=0}^{n} c_\mu (j\omega)^\mu}\right\}$$

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

Short introduction about Low-pass and Band-pass system:

Most transmission channels are limited to a certain frequency range due to:

- natural property of the channel (cable, radio,...)
- usage of filters to separate the information content and the noise

→ A system view is useful to:

- describe the basic properties of filters, transmission system or their effect on given signals
- simplify the design of transmission and other systems

Low-pass and band-pass filters are most important system for theory and practice

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

There are other ways of writing the transfer function of filters:

$$H(\omega) = |H(\omega)| e^{j\varphi(\omega)} = A(\omega) e^{-jb(\omega)}$$

Hereby, it follows:

1. <u>Damping ratio:</u> $a(\omega) = -20 \log A(\omega) \;\; dB$
2. <u>Damping angle:</u> $b(\omega) = -\varphi(\omega)$
3. <u>Phase delay:</u> $\tau_{Ph}(\omega) = -\frac{1}{\omega} \varphi(\omega)$

→ is the time delay of a cosine signal corresponding to a specific phase φ

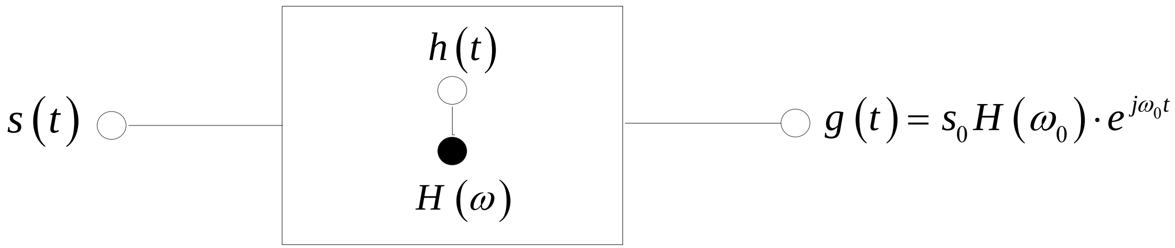
3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

So that the signal output holds:

$$g(t) = s_0 A(\omega_0) e^{j\varphi(\omega_0)} e^{j\omega_0 t} = s_0 A(\omega_0) e^{j\omega_0\left(t + \frac{\varphi(\omega_0)}{\omega_0}\right)}$$

The signal output is amplified by the factor $A(\omega_0)$ and is delayed about the phase delay:

$$\tau_{Ph}(\omega_0) = -\frac{1}{\omega_0}\varphi(\omega_0)$$



To the explanation of the phase delay

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

4. Envelope delay or group delay:

$$\tau_{Gr}(\omega) = -\frac{d}{d\omega}\varphi(\omega)$$

For the explanation, a narrow-band signal is provided in term of amplitude modulated cosine

$$s(t) = s_s(t)\cos\omega_0 t$$

with the following characteristics is observed:

- $|S_s(\omega)| = 0$ for $|\omega| > \omega_g$ with $\omega_g \ll \omega_0$
- This signal will be applied to a linear filter with constant amplitude and linear damping angle:

Linear filter: $s(t)$ ○—— $H(\omega) = A(\omega)\cdot e^{-jb(\omega)}$ ——○ $g(t)$

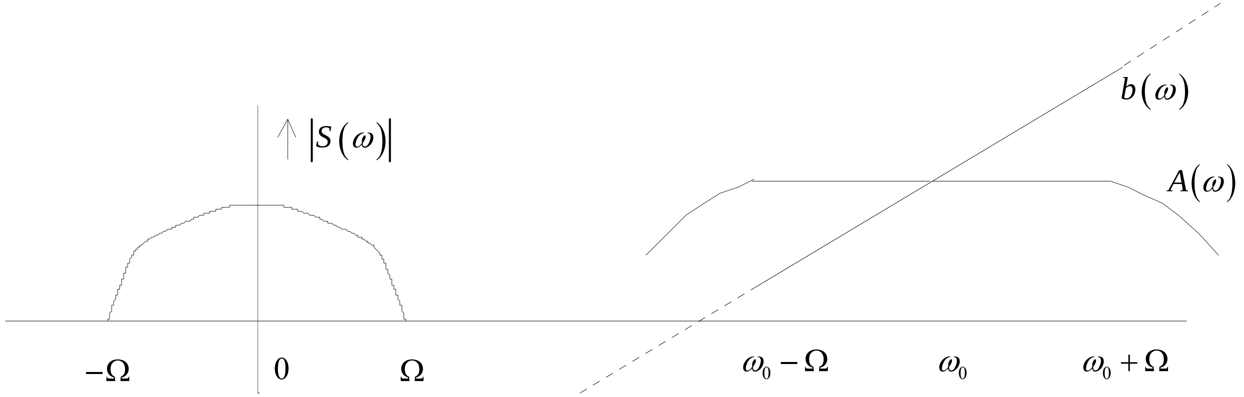
3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

$$A(\omega) = A(\omega_0) \quad \text{for} \quad |\omega_0 - \omega_g| < |\omega| < |\omega_0 + \omega_g|$$

$$\varphi(\omega) = -b(\omega) = \varphi(\omega_0) + \frac{d}{dt}\varphi(\omega)\Big|_{\omega_0} .(\omega - \omega_0) = -\omega_0 \tau_{Ph}(\omega_0) - (\omega - \omega_0)\tau_{Gr}(\omega_0)$$

For this case, it can be shown:

$$g(t) = s_s(t - \tau_{Gr}) \cos \omega_0 (t - \tau_{Ph})$$

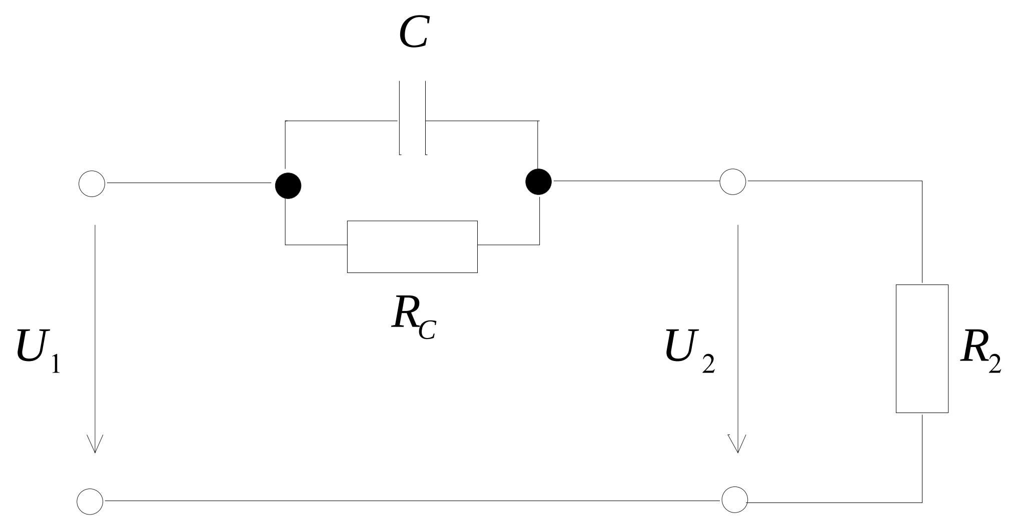


Magnitude of an input signal and magnitude and damping angle of the transmission system

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

Note: Negative phase retardation and envelope delays are no contradictions to the reality because both are defined for the steady-state condition of the system.

Example:



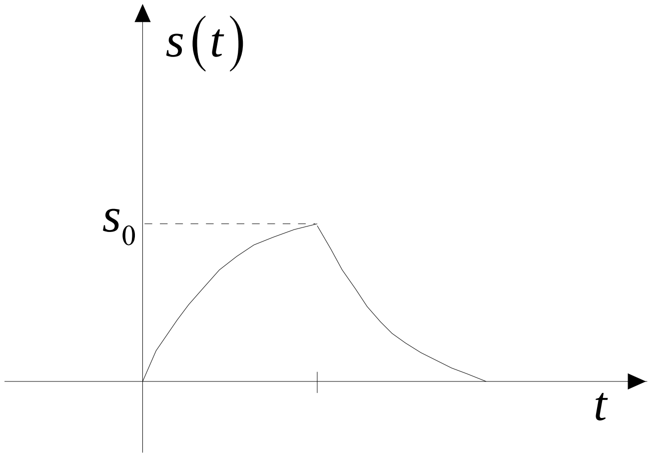
Realizable system with partly negative envelope delay

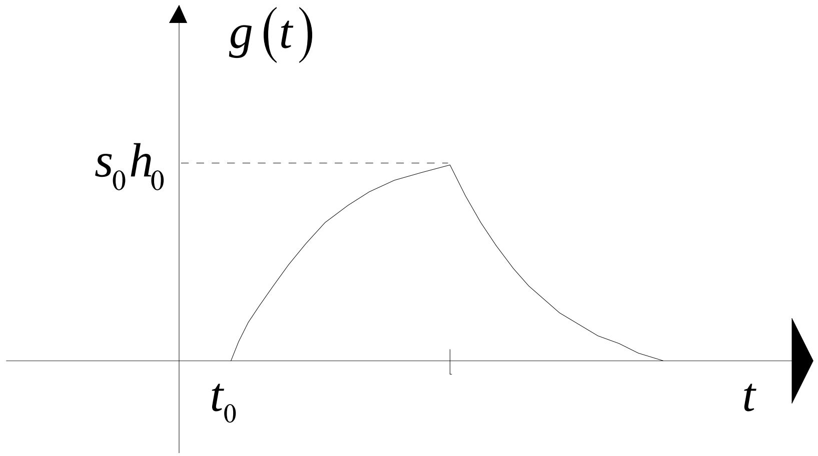
3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

Distortion-less system:

→ is a system or a filter which does not change the form of the input signal

$s(t) \rightarrow g(t) = h_0 s(t - t_0)$ where h_0 and t_0 are real and constant





Example: Input and output of a distortion-less system

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

<u>Properties of a distortion-less system:</u>

1. $s(t) \rightarrow g(t) = s(t) * h(t) = h_0 s(t - t_0)$
2. $H(\omega) = h_0 e^{-j\omega t_0}$: transfer function
3. This means: $A(\omega) = h_0$ and $\varphi(\omega) = -\omega . t_0$

   thus: $\tau_{Ph} = \tau_{Gr} = t_0$ ⟶ frequency-independent

All deviations from these properties are called linear distortions:

when
- $A(\omega) \neq$ const. : amplitude distortions
- $\varphi(\omega) \neq -\omega t_0$: phase distortions

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3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

A low-pass filter is a filter which has the property

$$A(\omega) = |H(\omega)| = 0 \quad \text{for} \quad |\omega| > \omega_g = 2\pi f_g$$

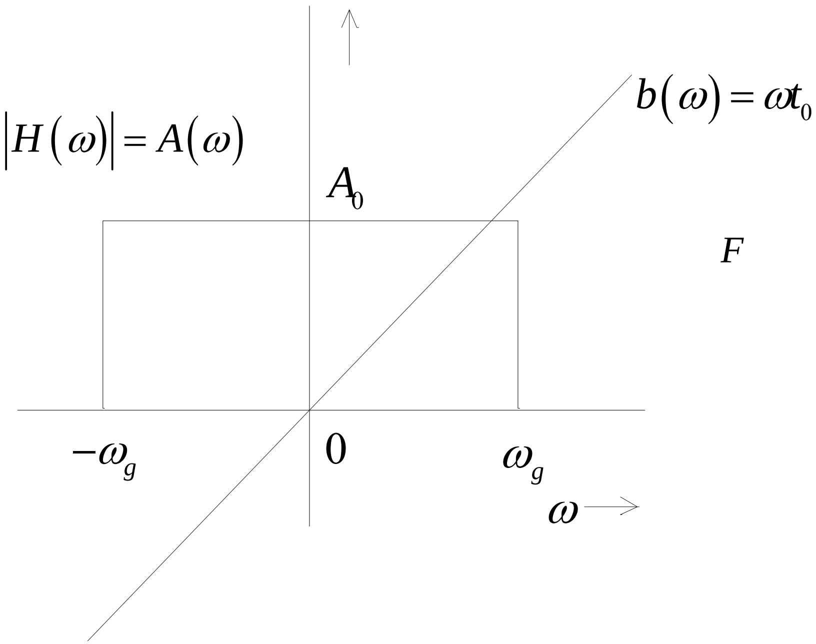
The ideal Low-pass filter:

The transfer function: $$H(\omega) = A_0 rect\left(\frac{\omega}{2\omega_g}\right) e^{-j\omega t_0}$$

The impulse response: $$h(t) = \frac{A_0 \omega_g}{\pi} si(\omega_g (t - t_0))$$

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

$$H(\omega) = A_0 rect\left(\frac{\omega}{2\omega_g}\right) \cdot e^{-j\omega t_0}$$



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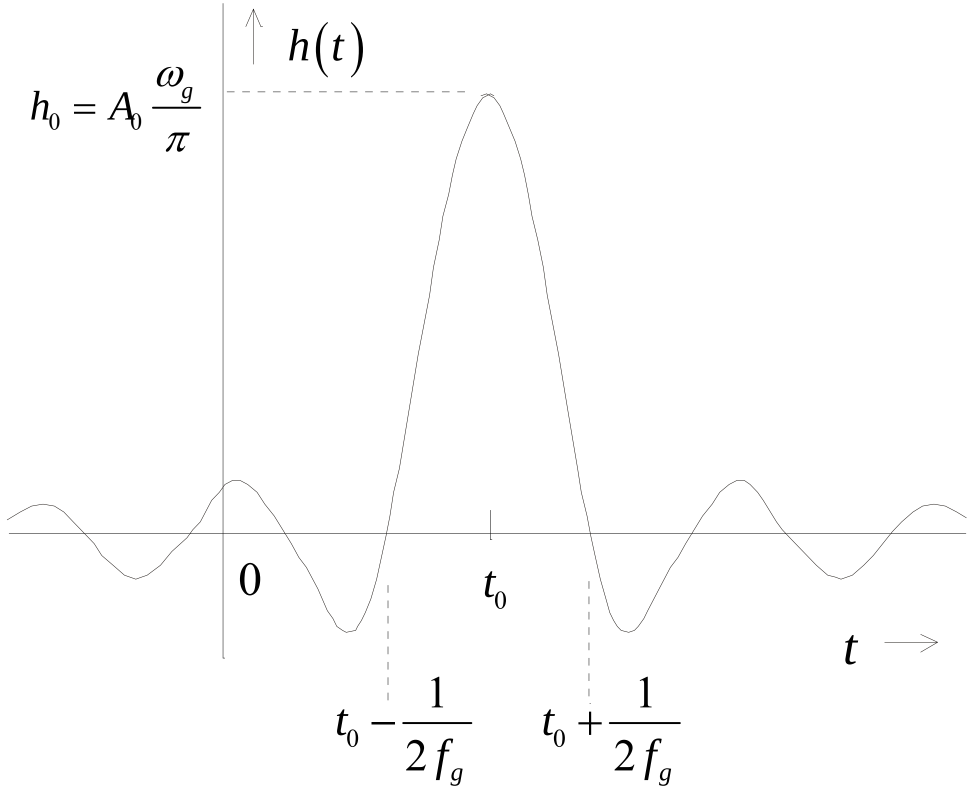


The ideal Low-pass filter in the frequency and time domain

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

The step response:

$$\omega(t) = \int_{-\infty}^{t} h(\tau)d\tau = h_0 \int_{-\infty}^{t} si(\omega_g(\tau - t_0))d\tau$$

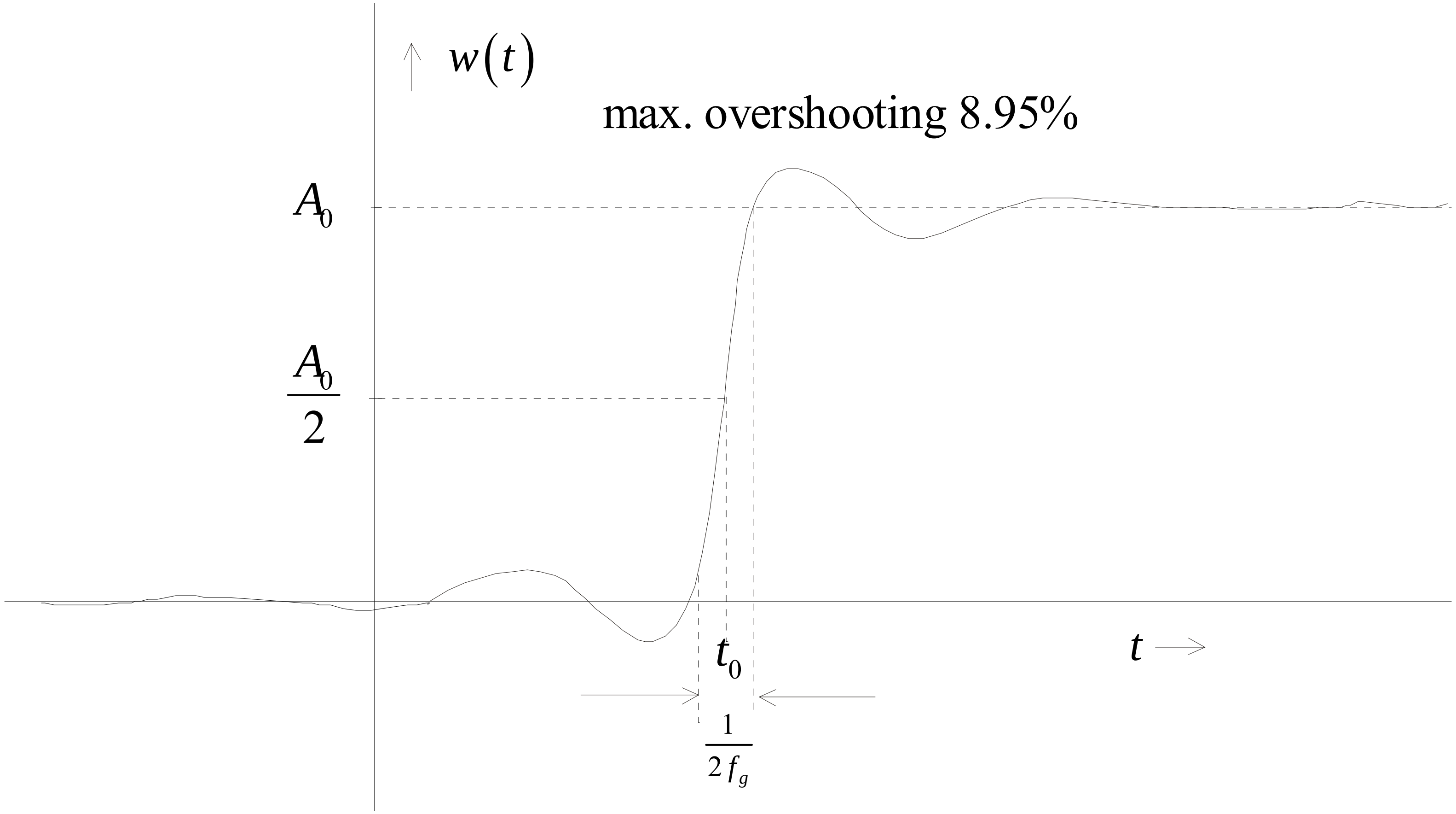
$$= h_0 \left[\int_{-\infty}^{t_0} si(\omega_g(\tau - t_0))d\tau + \int_{t_0}^{t} si(\omega_g(\tau - t_0))d\tau \right] \quad \text{(Si func. is symmetric)}$$

Using integral sine Si(x):

$$Si(x) = \int_{0}^{x} si(\xi)d\xi \quad \text{and} \quad Si(-x) = Si(x) \quad \text{and} \quad Si(-\infty) = \frac{\pi}{2}$$

One gets: $\omega(t) = \frac{1}{2} A_0 \left[1 + \frac{2}{\pi} Si(\omega_g(t - t_0)) \right]$

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems



Step response of a Low-pass filter with Overshooting

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

- Overshooting: 8.95 % of the step amplitude.
- Rise time of the low-pass:

$$\text{Step size } A_0 = \text{ rise time } t_e \cdot \text{ maximum gradient}$$

$$\longrightarrow \quad A_0 = t_e \frac{d}{dt} w(t)\bigg|_{t=t_0} = t_e h(t)\bigg|_{t=t_0} = t_e \frac{A_0 \omega_g}{\pi}$$

So the rise time becomes:

$$t_e = \frac{\pi}{\omega_g} = \frac{1}{2 f_g}$$

$\longrightarrow$ reverse proportional to cut-off frequency

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

The ideal Band-pass filter:

is defined by the following relation:

$$H(\omega) = A_0 \left[rect\left(\frac{\omega-\omega_0}{\Delta\omega}\right) + rect\left(\frac{\omega+\omega_0}{\Delta\omega}\right)\right] e^{-j\omega t_0}$$

- Non-zero in a finite band $\Delta\omega$
- This range does not include $\omega = 0$

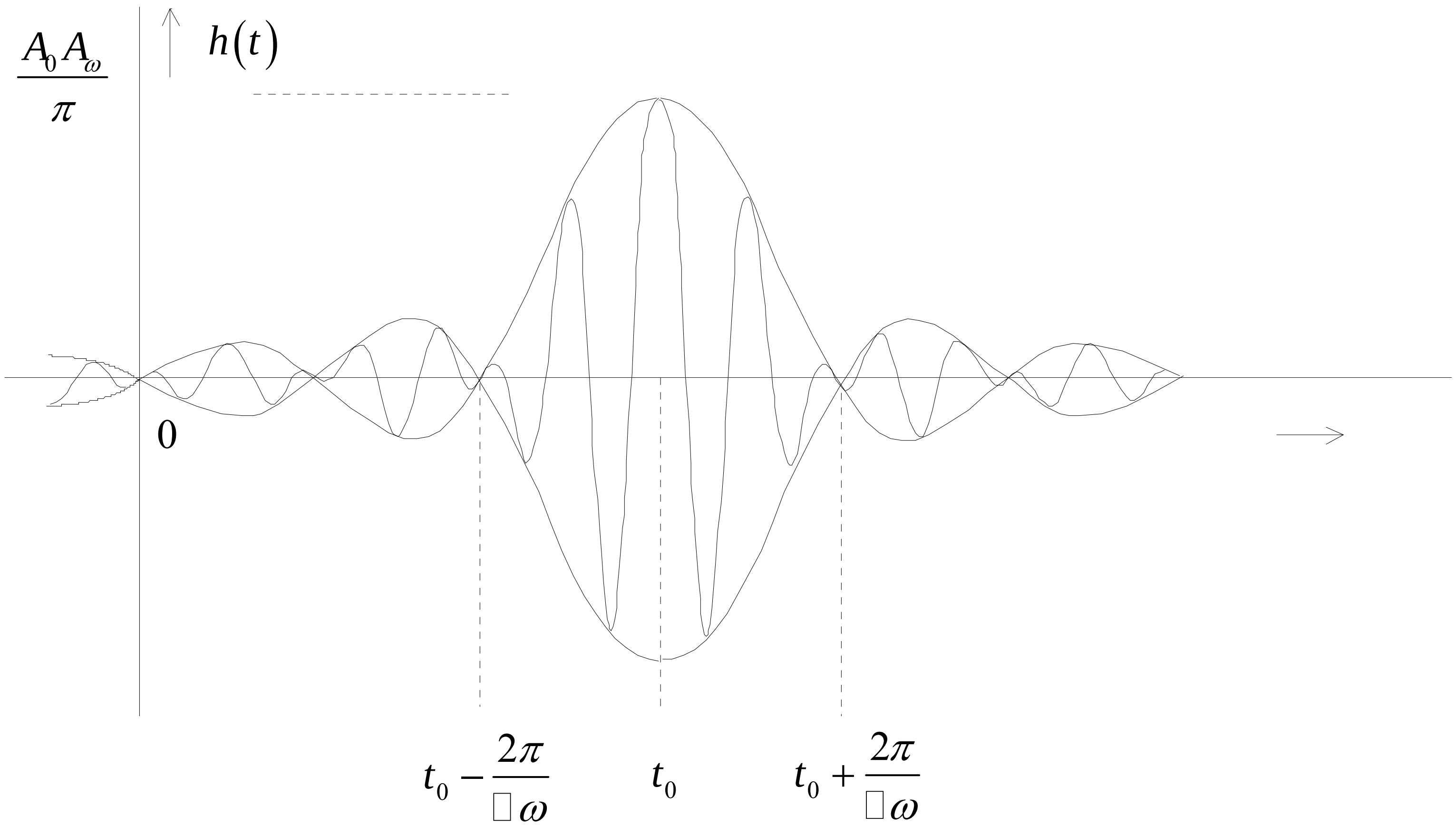
$\mathcal{F}$ ●—○

$$h(t) = A_0 \left[\frac{\Delta\omega}{2\pi} si\left(\frac{\Delta\omega}{2} t\right) e^{j\omega_0 t} + \frac{\Delta\omega}{2\pi} si\left(\frac{\Delta\omega}{2} t\right) e^{-j\omega_0 t}\right] * \delta(t-t_0)$$

$$= \frac{A_0 \Delta\omega}{2\pi} si\left(\frac{\Delta\omega}{2}(t-t_0)\right)\left[e^{j\omega_0 (t-t_0)} + e^{-j\omega_0 (t-t_0)}\right]$$

$$= \frac{A_0 \Delta\omega}{\pi} si\left(\frac{\Delta\omega}{2}(t-t_0)\right)\cos(\omega_0 (t-t_0))$$

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems



Impulse response of a Band-pass filter

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

If one compares the impulse response of the ideal band-pass and the ideal low-pass, one can gets:

$h(t) \quad = \quad h_T(t) \times \text{complex envelope} \quad \text{with}$

$h(t)$: impulse response of the band-pass

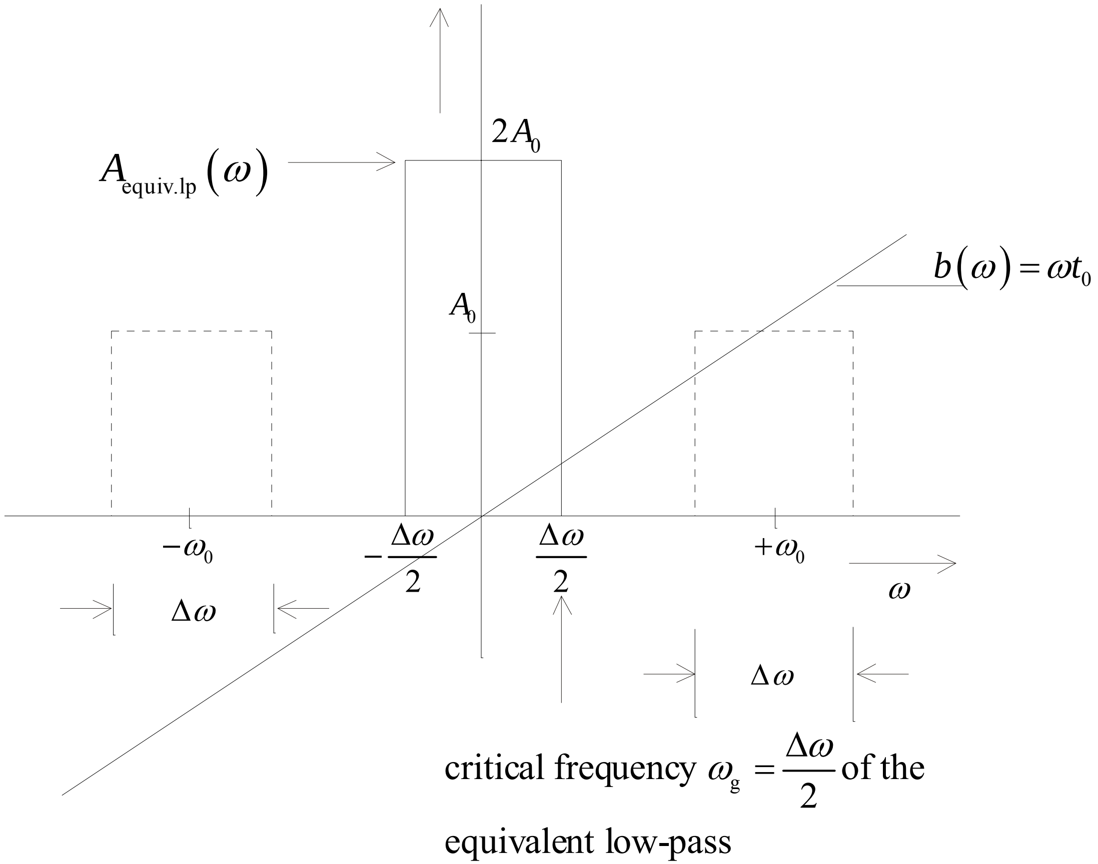
$h_T(t)$: equivalent low-pass impulse response

$$h(t) = \underbrace{2\frac{A_0 \frac{\Delta\omega}{2}}{\pi} si\left[\frac{\Delta\omega}{2}(t-t_0)\right]}\cos\omega_0(t-t_0)$$

$$= \qquad h_T(t) \quad \cdot \quad \cos\omega_0(t-t_0) \quad \text{with the following relations:}$$

$$H^0(\omega) = 2A_0 rect\left(\frac{\omega-\omega_0}{\Delta\omega}\right)e^{-j\omega t_0} \quad \text{and} \quad H_T(\omega) = 2A_0 rect\left(\frac{\omega}{\Delta\omega}\right)e^{-j\omega t_0}$$

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems



Symmetrical Band-pass and appropriate equivalent Low-pass

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

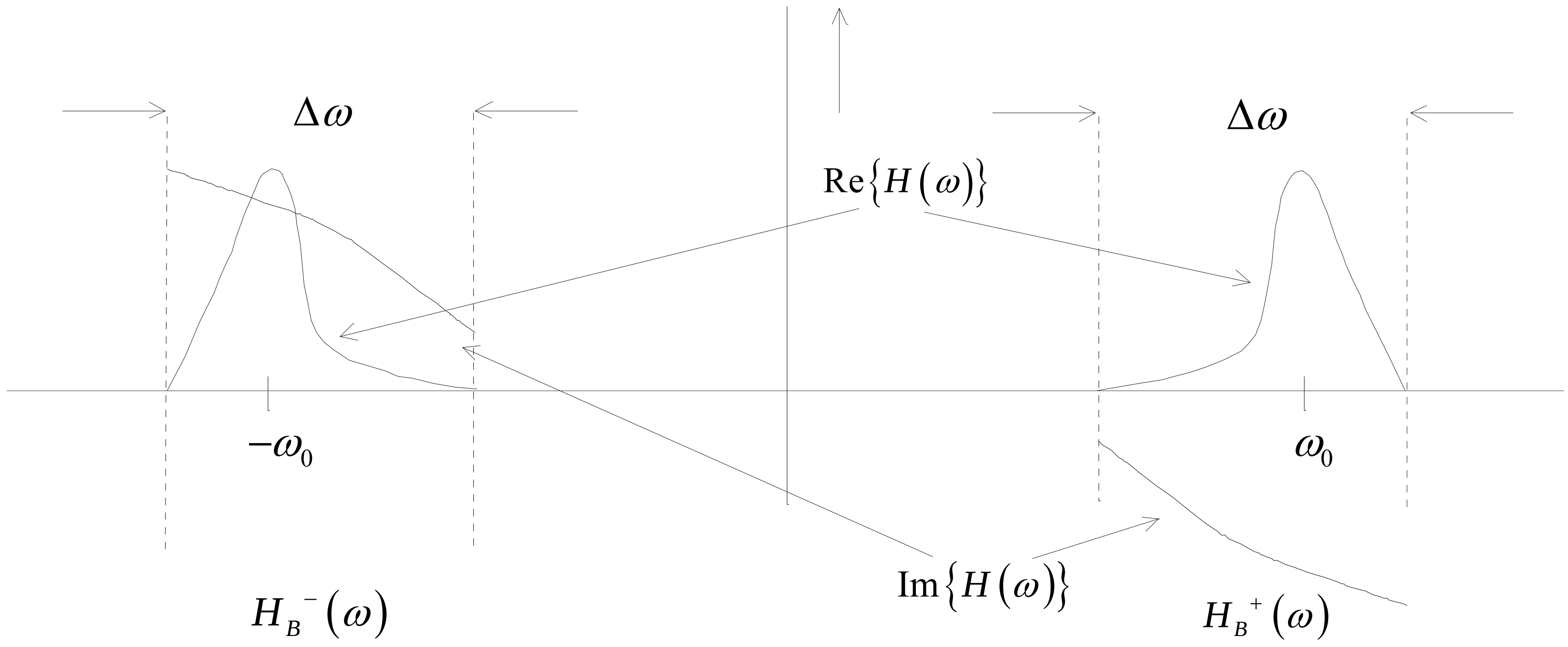
The Non-symmetrical Band-pass:

The impulse response of a band-pass (and all other LTI-Systems) must be a real-valued function of time and has the following properties:

$$\mathrm{Re}\{H(\omega)\} = \mathrm{Re}\{H(-\omega)\} \text{ : is an even function in } \omega$$

$$\mathrm{Im}\{H(\omega)\} = -\mathrm{Im}\{H(-\omega)\} \text{ : is an odd function in } \omega$$

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems



Spectrum of a Non-symmetrical Band-pass

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

For the following considerations, it gives:

$$H(\omega) = H^{+}(\omega) + H^{-}(\omega) \quad \text{with}$$

$$H^{+}(\omega) = H(\omega) \cdot \varepsilon(\omega) \quad \text{and}$$

$$H^{-}(\omega) = H(\omega) \cdot \varepsilon(-\omega)$$

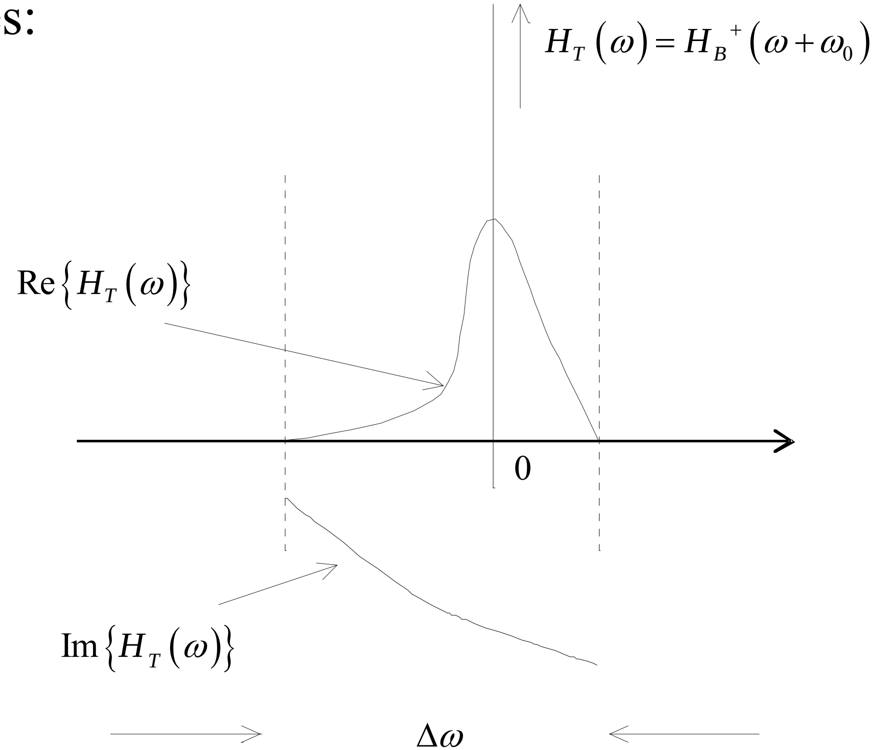
Because of its properties:

$$H_T(\omega) = 2H^{+}(\omega + \omega_0)$$

$$H_T(\omega) = \mathrm{Re}\{H_T(\omega)\} + j\,\mathrm{Im}\{H_T(\omega)\}$$

$$\mathcal{F} \;\bullet\!\!-\!\!\circ$$

$$h_T(t) = u(t) + j \cdot v(t)$$



Equivalent low-pass signal of the non-symmetrical band-pass

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

Based on the formulas above, one can get the following relations:

$$H_T(\omega) = 2H^+(\omega + \omega_0) = 2\left[H^-(-\omega - \omega_0)\right]^*$$

or

$$H^+(\omega) = \frac{1}{2}H_T(\omega - \omega_0) = \frac{1}{2}H^0(\omega)$$

$$H^-(\omega) = \frac{1}{2}H_T(-\omega - \omega_0)$$

with: $H(\omega) = H^+(\omega) + H^-(\omega)$

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

The impulse response of the non symmetrical band-pass is given by:

$$h(t) = \mathrm{Re}\left\{h_T(t)e^{j\omega_0 t}\right\} = u(t)\cos\omega_0 t - v(t)\sin\omega_0 t$$
$$= \sqrt{u^2(t)+v^2(t)}\cos(\omega_0 t + \varphi_T(t)) = \left|h_T(t)\right|\cos(\omega_0 t + \varphi_T(t))$$

$h_T(t)$: equivalen low-pass or complex envelope

$u(t)$: in-phase component

$v(t)$: quadrature component

The impulse response of the general band-pass is an amplitude and angle-modulated cosine signal.

All relations between $s(t), s^0(t), s_T(t)$ and its Fourier Transforms

also hold for $h(t), h^0(t),\ h_T(t)$ and its Fourier Transforms! (see chap. 2, S.95-100)

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

Transmission of Band-pass signals through Band-pass filters:

For any LTI-system with an arbitrary input y(t), the following holds:

$$y(t) \rightarrow g(t) = y(t) * h(t) \quad \circ\!\!\overset{\mathcal{F}}{\longrightarrow}\!\!\bullet \quad G(\omega) = Y(\omega) \cdot H(\omega)$$

Representing band-pass signal by means of equivalent low-pass signal:

$$G(\omega) = \frac{1}{2} G_T(\omega - \omega_0) + \frac{1}{2} G_T{}^*(-\omega - \omega_0)$$

$$= \left[\frac{1}{2} Y_T(\omega - \omega_0) + \frac{1}{2} Y_T{}^*(-\omega - \omega_0) \right] \cdot \left[\frac{1}{2} H_T(\omega - \omega_0) + \frac{1}{2} H_T{}^*(-\omega - \omega_0) \right]$$

$$= \frac{1}{4} Y_T(\omega - \omega_0) \cdot H_T(\omega - \omega_0) + \frac{1}{4} Y_T{}^*(-\omega - \omega_0) \cdot H_T{}^*(-\omega - \omega_0)$$

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

If one assumes, that $Y_T(\omega) = 0 \; and \; H_T(\omega)$ for $\omega \leq -\omega_0$
(means: $y(t)$ and $h(t)$ are narrow-banded), then:

$$G_T(\omega) = \frac{1}{2} Y_T(\omega) H_T(\omega) \quad \bullet\!\!\overset{\mathcal{F}}{-\!\!-\!\!-}\!\!\circ \quad g_T(t) = \frac{1}{2} y_T(t) * h_T(t)$$

and thus:

$$g(t) = \mathrm{Re}\left\{ g_T(t) e^{j\omega_0 t} \right\} = \mathrm{Re}\left\{ \frac{1}{2} \left[y_T(t) * h_T(t) \right] e^{j\omega_0 t} \right\}$$

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

All-pass filters:

is an LTI system with the property: $|H(\omega)| = A(\omega) = A_0$

All-pass filters with linear phase:

$$H(\omega) = A_0 e^{-j\omega t_0} \quad \bullet\!\overset{\mathcal{F}}{—}\!\circ \quad h(t) = A_0 \delta(t - t_0)$$

are:

- ideal delay elements for $t_0 > 0$
- ideal predictors for $t_0 < 0$, because of:

$$s(t) \to g(t) = s(t) * A_0 \delta(t - t_0) = A_0 s(t - t_0)$$

Usage: often used to correct the phase of transmission channels and filters.

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems

Causal Low-pass filters:

can be derived from the non-causal, ideal low-pass by multiplying the impulse response with a suitable rectangular function:

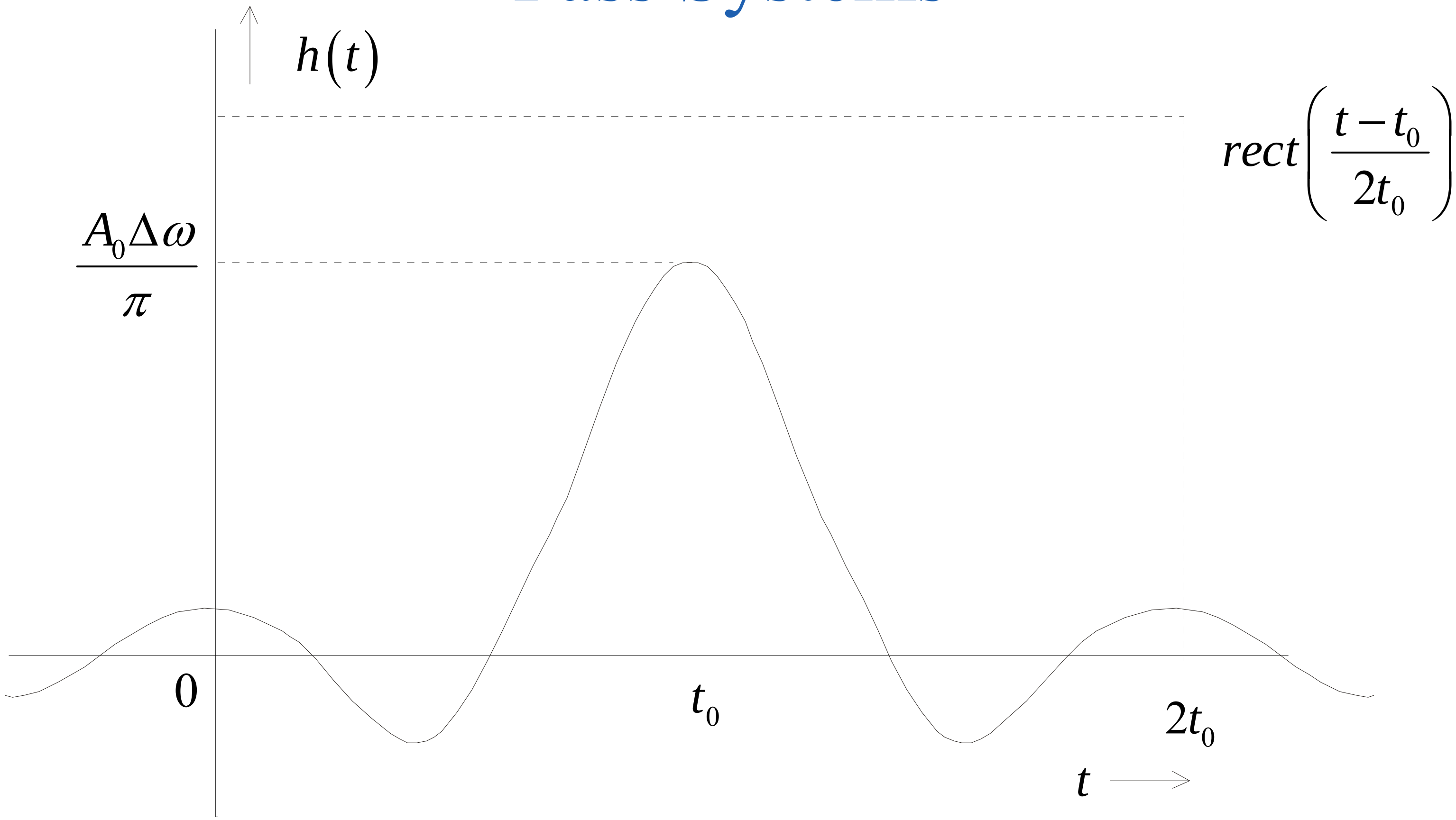
$$h_k(t) = h(t).rect\left(\frac{t-t_0}{2t_0}\right) = \frac{A_0\omega_g}{\pi} si(\omega_g(t-t_0))rect\left(\frac{t-t_0}{2t_0}\right)$$

$$= \frac{A_0\omega_g}{\pi} si(\omega_g t)rect\left(\frac{t}{2t_0}\right) * \delta(t-t_0)$$

$\mathcal{F}$ ○—●

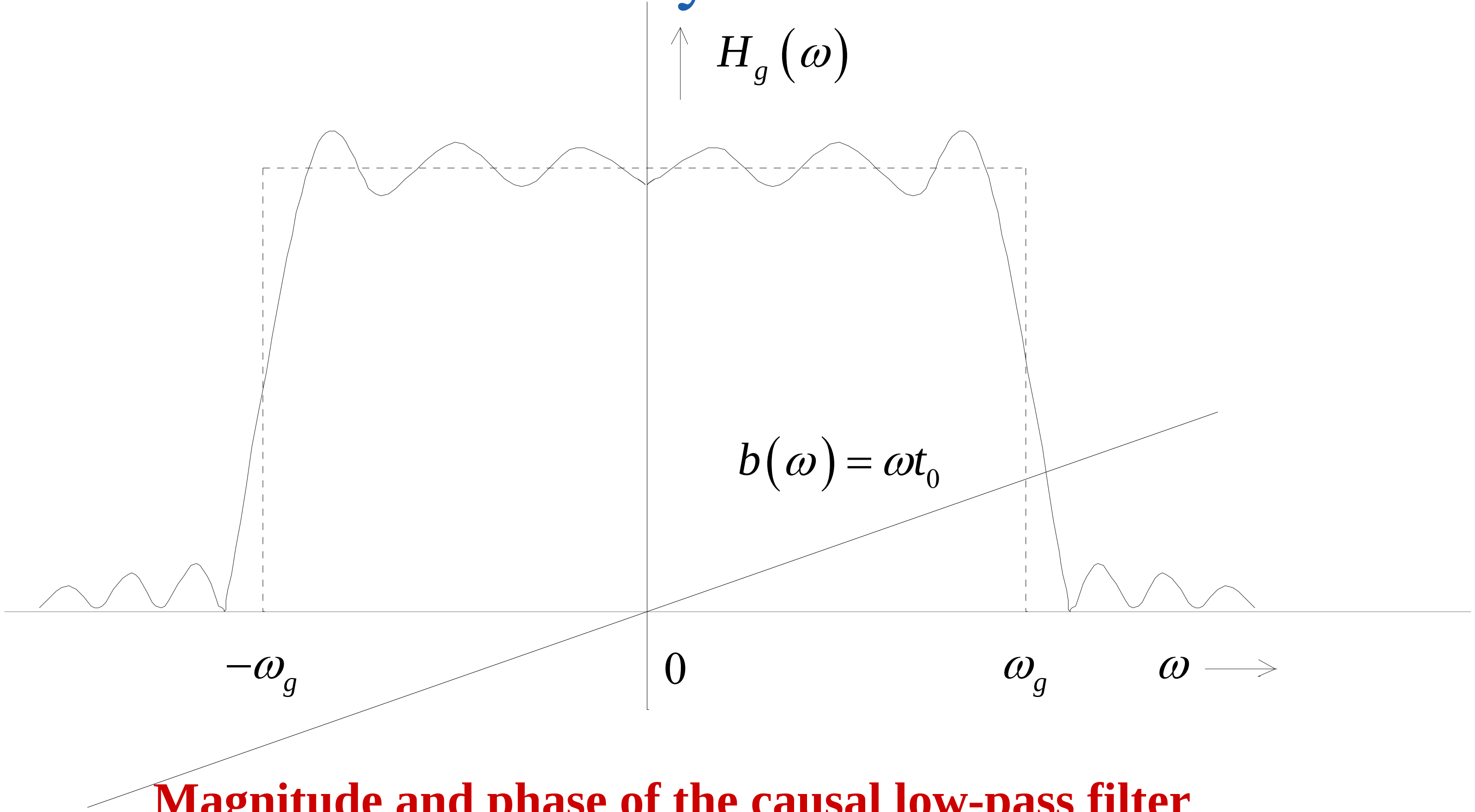
$$H_K(\omega) = \frac{1}{2\pi} A_0 rect\left(\frac{\omega}{2\omega_g}\right) * 2t_0.si(\omega t_0)e^{-j\omega t_0} = |H_K(\omega)|e^{-j\omega t_0}$$

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems



Multiplication of the impulse response of the ideal low-pass with a rectangular function

3.3.5 Causal, Analog Low-Pass and Band-Pass Systems



Magnitude and phase of the causal low-pass filter

→ The causal low-pass filter lost the shape of the ideal low-pass filter

3.3.6 The System function $H_L(p)$

Operating the excitation of the following input signal:

$$s(t) = s_0 e^{pt} = s_0 e^{\sigma t} e^{j\omega t} \quad \text{where} \quad p = \sigma + j\omega$$

One gets:

$$g(t) = s(t) * h(t) = s_0 e^{pt} * h(t) = s_0 \int_{-\infty}^{+\infty} h(\tau) e^{p(t-\tau)} d\tau$$

$$= s_0 e^{pt} \int_{-\infty}^{+\infty} h(\tau) e^{-p\tau} d\tau = s_0 e^{pt} I$$

➡ **The difference is I**

For a causal system, it obtains:

With $h(t) \equiv 0$ for $t < 0$

$$g(t) = s_0 e^{pt} \int_{0}^{\infty} h(\tau) e^{-p\tau} d\tau = s_0 e^{pt} H_L(p)$$

3.3.6 The System function $H_L(p)$

1. $H_L(p)$ is called system function
2. $H_L(p)$ is the Laplace transform of the impulse response of the causal LTI system.

$$H_L(p) = \int_0^\infty h(t) e^{-pt} dt$$

3. Because of the restriction to causality, it follows for the convolution integral:

$$g(t) = \int_0^t s(\tau) h(t-\tau) d\tau$$

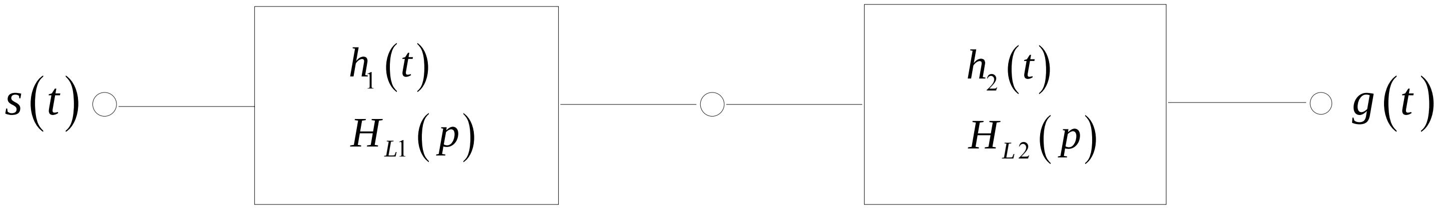
$$\mathscr{L} \;\circ\!\!-\!\!\bullet$$

$$G_L(p) = S_L(p) H_L(p) \quad \Leftrightarrow \quad H_L(p) = \frac{G_L(p)}{S_L(p)}$$

3.3.6 The System function $H_L(p)$

4. When combining 2 causal, effectless LTI-system (filters), one obtains:

$$H_L(p) = H_{L1}(p) \cdot H_{L2}(p)$$



Non-reactive combination of two causal LTI-system

5. $H(\omega)$ does not automatically equal the system function H_L(p) on the $j\omega\_axis$

When moving from $H_L(p)$ *to* $H(\omega)$, the following cases can occur:

Case 1:

Re $p = \beta_2 < 0$, $H_L(p)$ jω-axis is covered by convergence area ensuring:

$$H(\omega) = H_L(j\omega)$$

3.3.6 The System function $H_L(p)$

Case 2:

$$\operatorname{Re} p = \beta_1 > 0 \;\Rightarrow\; \text{the } j\omega - \text{axis is outside of the convergence area}$$

$$\rightarrow H(\omega) \text{ does not exist}$$

Note: for a causal system where the impulse response is integrable:

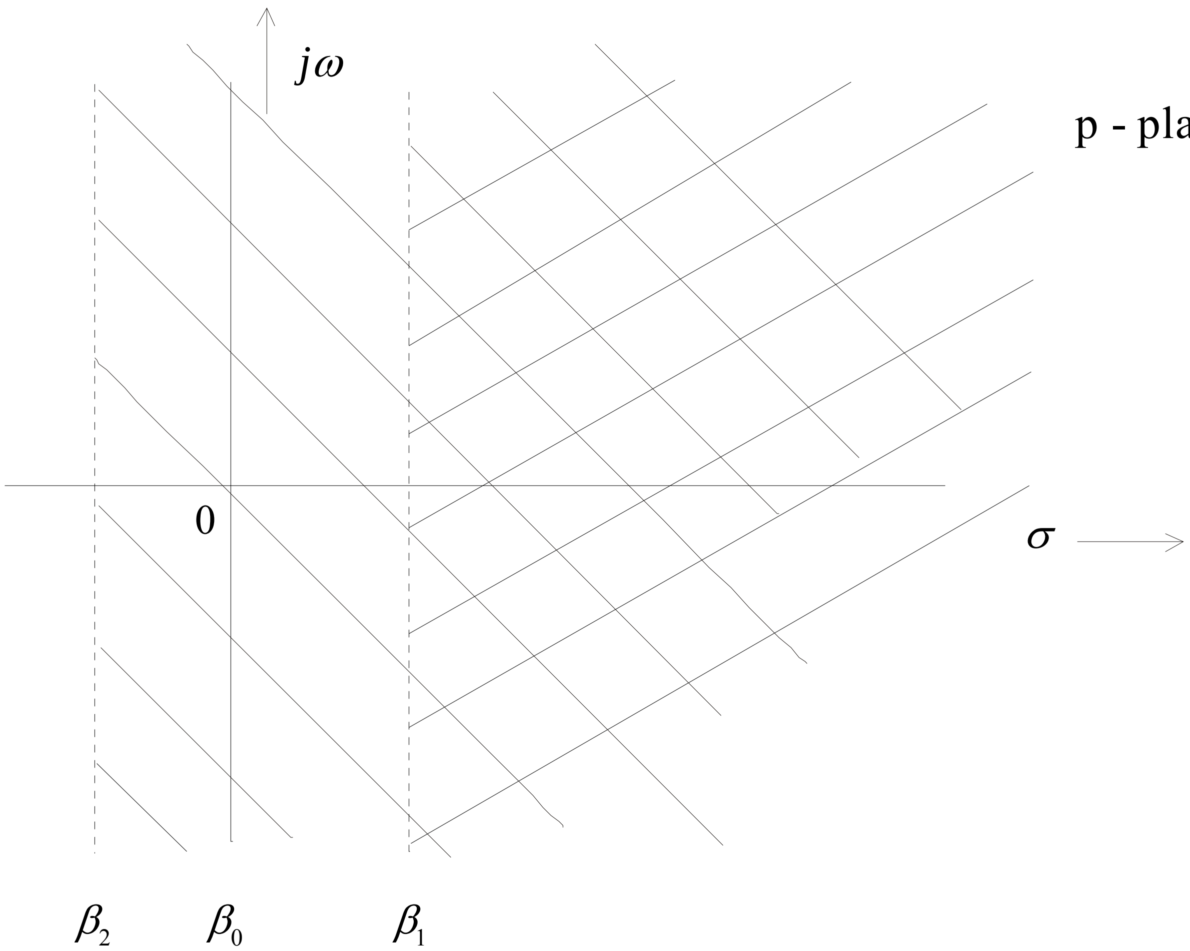
$$\int_0^\infty |h(t)|\, dt < \infty \longrightarrow \text{such case does not occur}$$

Case 3:

$$\operatorname{Re} p = \beta_0 = 0$$

$$\longrightarrow H(\omega) = H_L(j\omega) + \text{ function with suitable Dirac's delta functions}$$

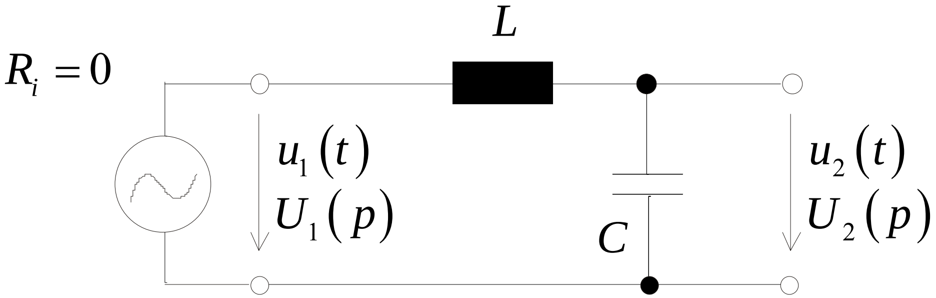
3.3.6 The System function $H_L(p)$



Convergence areas of the Laplace transform

3.3.6 The System function $H_L(p)$

Example: Given is the following network



The corresponding system can be derived as:

$$H_L(p) = \frac{U_2(p)}{U_1(p)} = \frac{\frac{1}{pC}}{pL + \frac{1}{pC}} = \frac{\omega_0^{\,2}}{p^2 + \omega_0^{\,2}} \quad \longrightarrow \quad h(t) = \omega_0 \sin(\omega_0 t) \cdot \varepsilon(t)$$

$$\longrightarrow \quad H(\omega) = \frac{\omega_0}{2\pi} \left\{ \left[j\pi\delta(\omega + \omega_0) - j\pi\delta(\omega - \omega_0) \right] * \left[\frac{1}{j\omega} + \pi\delta(\omega) \right] \right\}$$

$$= \frac{j\omega_0}{2} \left\{ \left[\delta(\omega + \omega_0) - \delta(\omega - \omega_0) \right] * \frac{1}{j\omega} + \pi \left[\delta(\omega + \omega_0) - \delta(\omega - \omega_0) \right] \right\}$$

3.3.6 The System function $H_L(p)$

where: $$\delta(\omega \pm \omega_0) * \frac{1}{j\omega} = \frac{1}{j}\int_{-\infty}^{+\infty} \delta(v \pm \omega_0)\frac{dv}{\omega - v} = \frac{1}{j}\frac{1}{\omega \pm \omega_0}$$

Simplification results:

$$H(\omega) = \frac{\omega_0}{2j}\left[\frac{1}{j\omega - j\omega_0} - \frac{1}{j\omega + j\omega_0}\right] + \frac{\omega_0 \pi}{2j}\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right]$$

The corresponding system function:

$$H_L(p) = \frac{\omega_0{}^2}{p^2 + \omega_0{}^2} = \frac{\omega_0{}^2}{(p - j\omega_0)\cdot(p + j\omega_0)} = \frac{\omega_0}{2j}\cdot\left[\frac{1}{p - j\omega_0} - \frac{1}{p + j\omega_0}\right]$$

$$H(\omega) = H_L(j\omega) + \frac{\omega_0 \pi}{2j}\left[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)\right]$$

$$H(\omega) = H_L(j\omega) + \pi\sum_{n=1}^{m} a_n \delta(\omega - \omega_n)$$

3.3.6 The System function $H_L(p)$

When moving from $H(\omega)$ *to* $H_L(p)$, one can conclude from the existance $H(\omega)$:

- If $H_L(p)$ exists for $\operatorname{Re} p = 0$ and $H(\omega)$ is represented as an analytical function of ω, $H_L(p)$ can be written as:

$$H_L(p) = H\left(\frac{p}{j}\right) \qquad \operatorname{Re} p \geq 0$$

3.3.7 Network Function as System Function

With networks consisting of LTI elements, the network function N(p) is a rational fraction function in p with constant, real-valued coefficients:

$$H_L(p) = N(p) = \frac{\sum_{i=0}^{m} a_i p^i}{\sum_{i=0}^{n} b_i p^i} = \frac{a_m p^m + \ldots + a_0}{b_n p^n + \ldots + b_0}$$

3.3.8 Pole Zero plots

If the roots of the numerator polynomial p_0 and the denominator polynomial p_∞, one gets:

$$H_L(p) = \frac{a_m \prod_{i=1}^{\mu} (p - p_{0i})^{r_{0i}}}{b_n \prod_{i=1}^{\nu} (p - p_{\infty i})^{r_{\infty i}}}$$

zeros (numerator factors), poles (denominator factors)

where:

$$\sum_{i=1}^{\mu} r_{0i} = \text{Order of the numerator polynom } = m$$

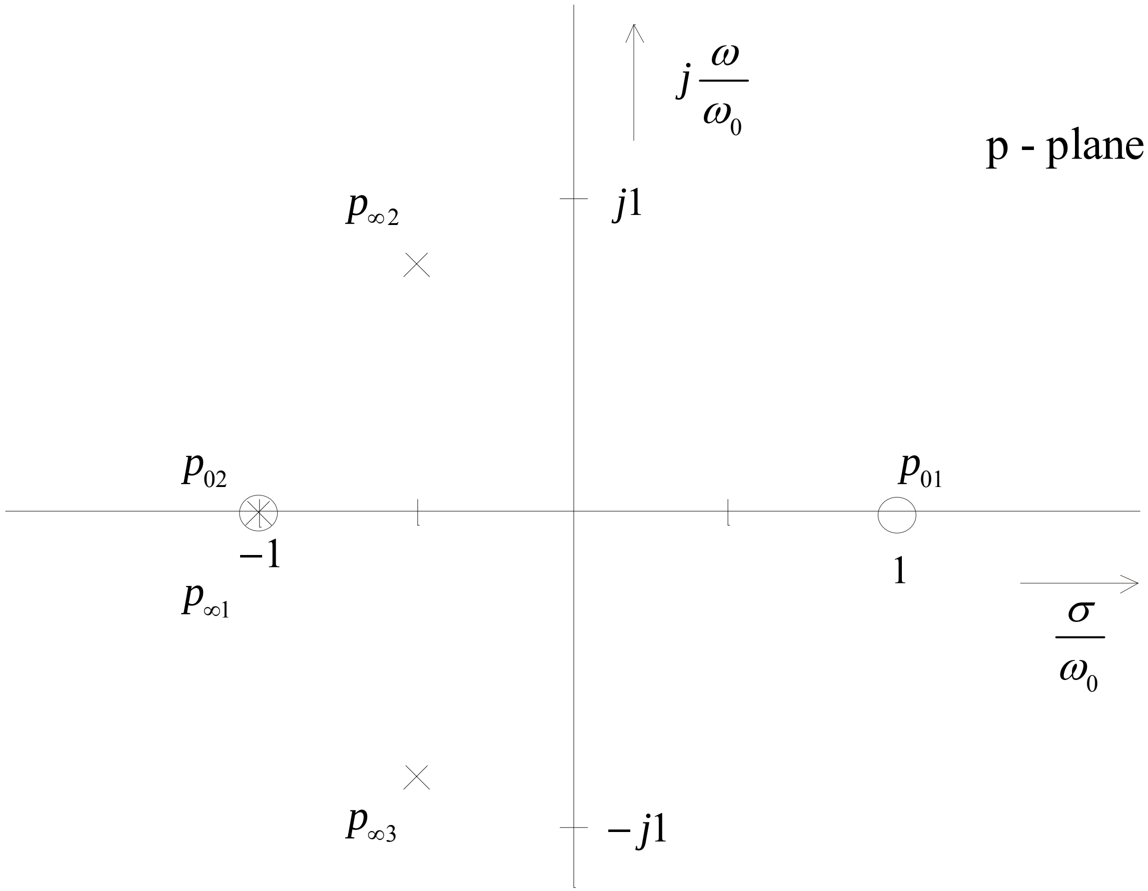
$$\sum_{i=1}^{\upsilon} r_{\infty i} = \text{Order of the denominator polynom } = \text{n}$$

Real constant: $K = \frac{a_m}{b_n}$

With poles and zeroes, the transmission properties are fully described:

$$G_L(p) = H_L(p) S_L(p) \quad \bullet\!\!\overset{\mathscr{L}}{—\!\!—}\!\!\circ \quad g(t) = h(t) * s(t) \quad \text{(apart from K)}$$

3.3.8 Pole Zero plots



Example of Pole-Zero plot

3.3.8 Pole Zero plots

The representation of pole-zero diagram can be used to determine the magnitude and phase and shows the influence of any zero and pole clearly:

$$(p - p_{0i}) = |p - p_{0i}| e^{j\varphi_{0i}} = |p - p_{0i}| e^{j\angle(p - p_{0i})}$$

and $$(p - p_{\infty i}) = |p - p_{\infty i}| e^{j\varphi_{\infty i}}$$

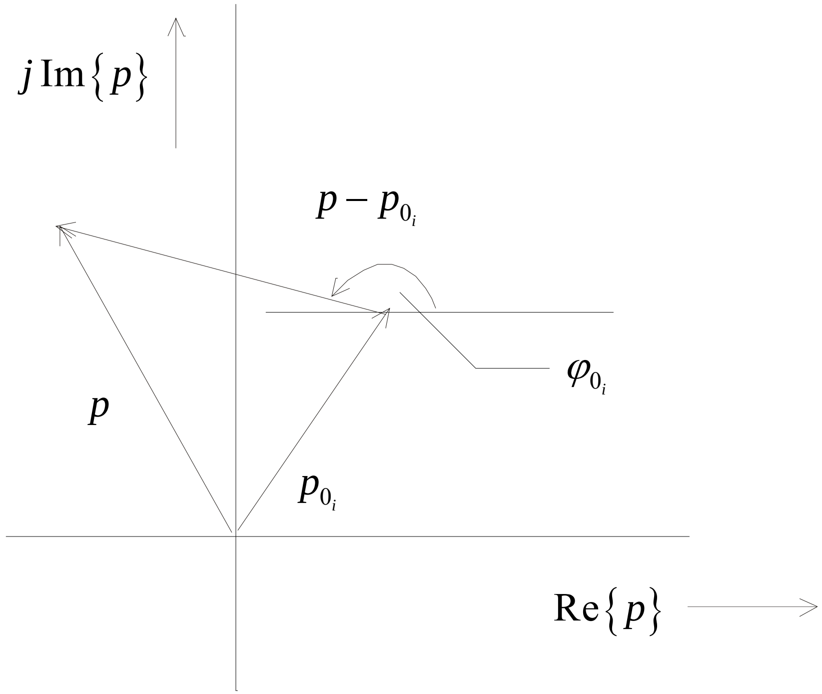
Thus:

$$|H_L(p)| = |K| \frac{\prod_{i=1}^{\mu} |p - p_{0i}|^{r_{0i}}}{\prod_{i=1}^{\nu} |p - p_{\infty i}|^{r_{\infty i}}}$$ **magnitude**

$$\varphi(p) = \arctan K + \sum_{i=1}^{\mu} r_{0i}\varphi_{0i}(p) - \sum_{i=1}^{\nu} r_{\infty i}\varphi_{\infty i}(p)$$ **phase**

where: $$\arctan K = \begin{cases} 0 & K > 0 \\ \pm\pi & K < 0 \end{cases}$$

3.3.8 Pole Zero plots

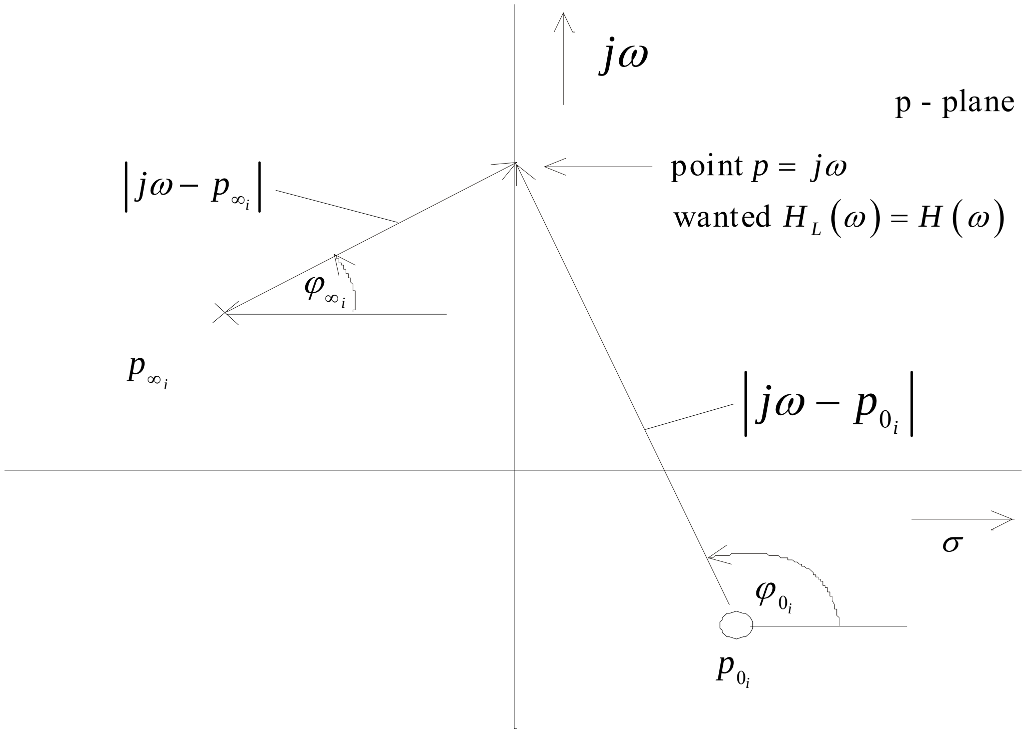


Method to determine the influence of a Zero and a Pole from that Pole-zero plot

3.3.8 Pole Zero plots

With the representation for:

$$H_L(p) = |H_L(p)| \cdot e^{j\varphi(p)}$$ the influence of poles and zeros becomes clear



Method to determine the Influence of Poles and Zeros on the Magnitude and phase to a point on jw_axis

3.3.8 Pole Zero plots

• Poles and zeros with the same distance to the regarded point cancel each other

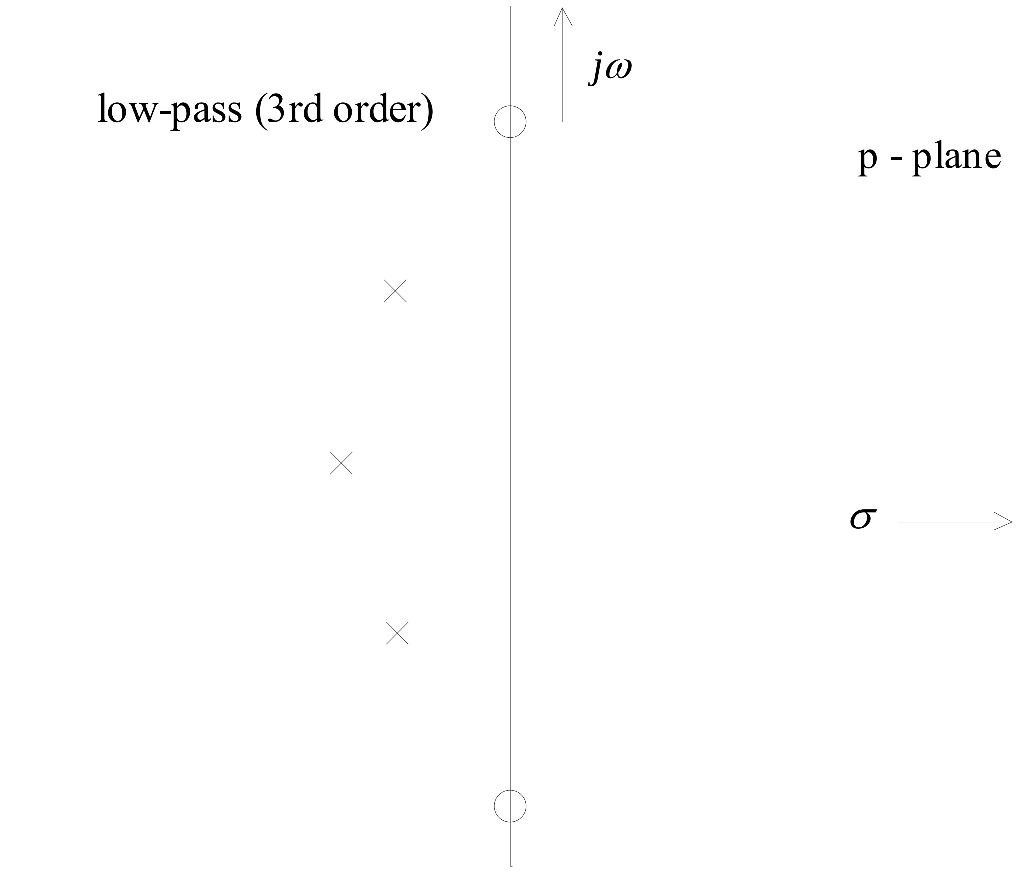
• The system function is mainly influenced by the corresponding pole and zero with the closest distance to p.

•A quick estimation for the transfer function is important for technical applications:

$$H_L(j\omega) = H(\omega) = A(\omega)e^{j\varphi(\omega)}$$

3.3.8 Pole Zero plots

Low-pass system: **Number of poles > number of zeros**



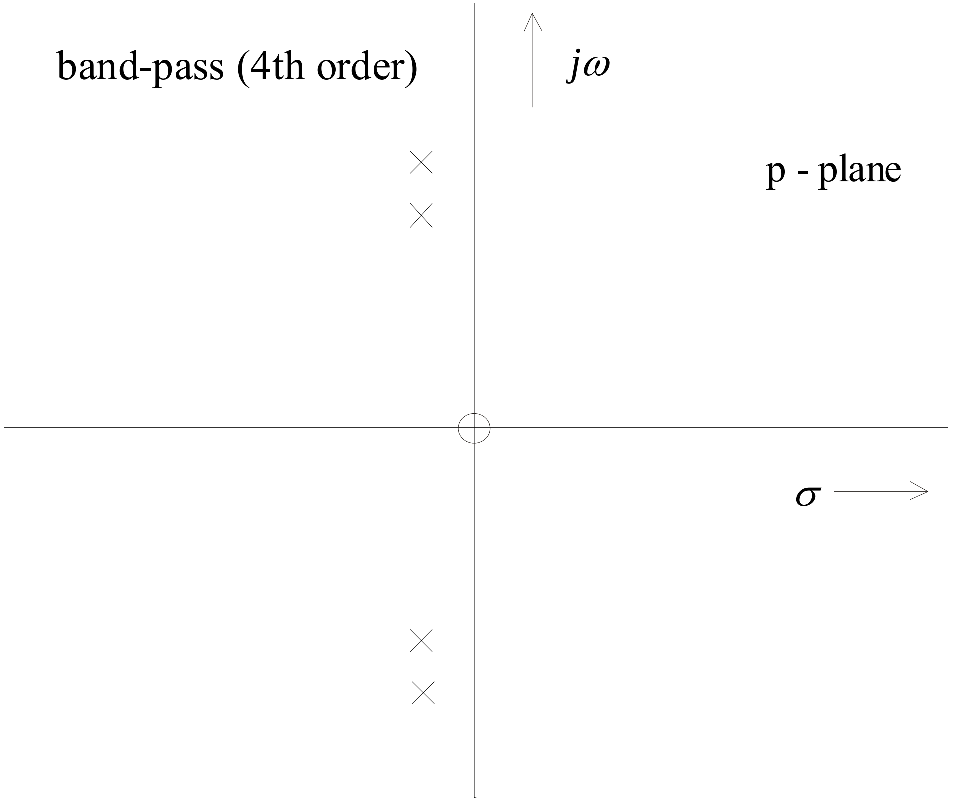
because

$$A(\omega)\Big|_{\omega \to \pm\infty} = 0$$

3.3.8 Pole Zero plots

Band-pass:

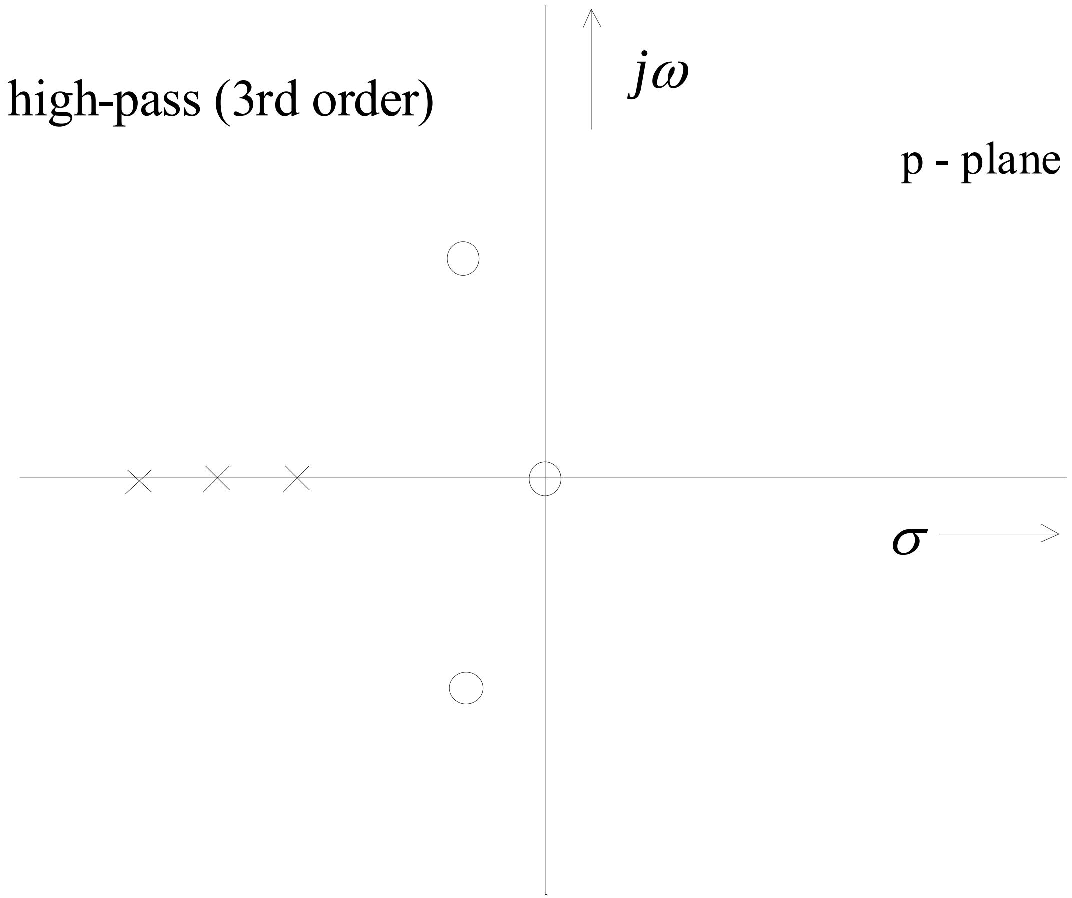
- $A(\omega)\Big|_{\omega \to \pm\infty}$ *and* $A(0) = 0$
- One or more zeros at the origin.
- No. of poles > No. of zeros



3.3.8 Pole Zero plots

High-pass: $\lim_{\omega \to \pm\infty} A(\omega) = K$

- At least one zero at origin
- No. poles = No. zeros



3.3.8 Pole Zero plots

Band-stop:

$$\lim_{\omega \to \pm\infty} A(\omega) \neq 0 \quad and \quad \lim_{\omega \to \omega_0} A(\omega) = 0$$

- one zero or more at center frequency
- same number of poles and zeros
- No zero at origin

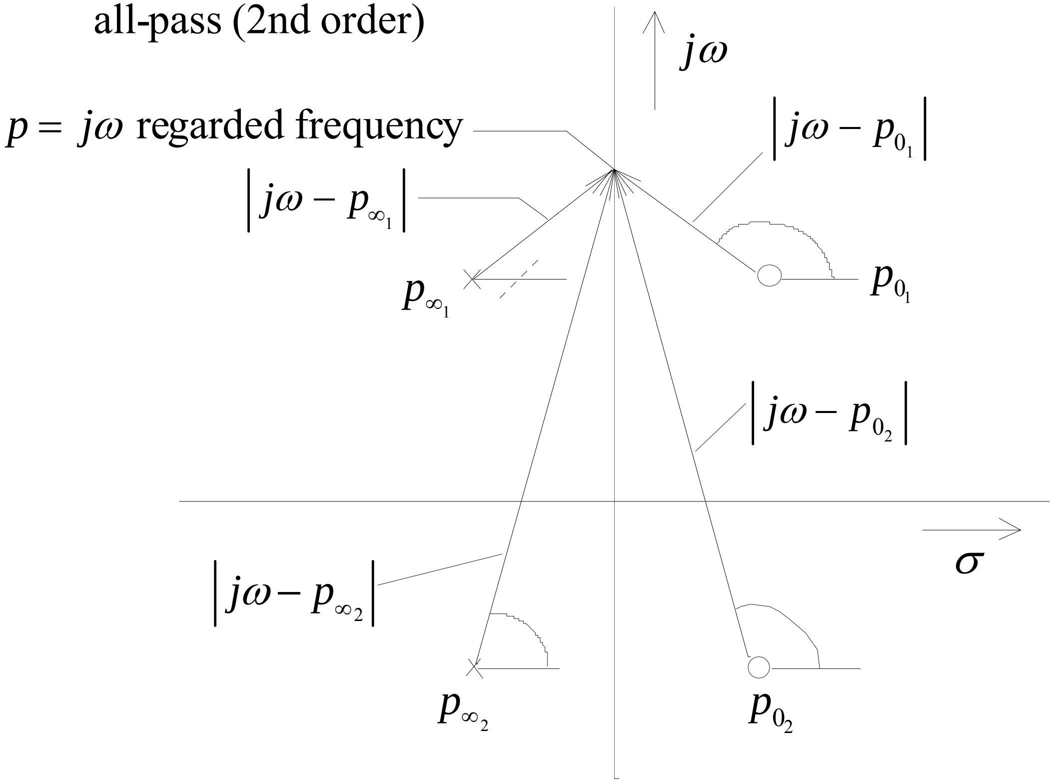
3.3.8 Pole Zero plots

All-pass:

$$|H_L(j\omega)| = |H(\omega)| = A(\omega) = |K| \frac{|j\omega - p_{01}||j\omega - p_{02}|}{|j\omega - p_{\infty 1}||j\omega - p_{\infty 2}|} = |K|$$

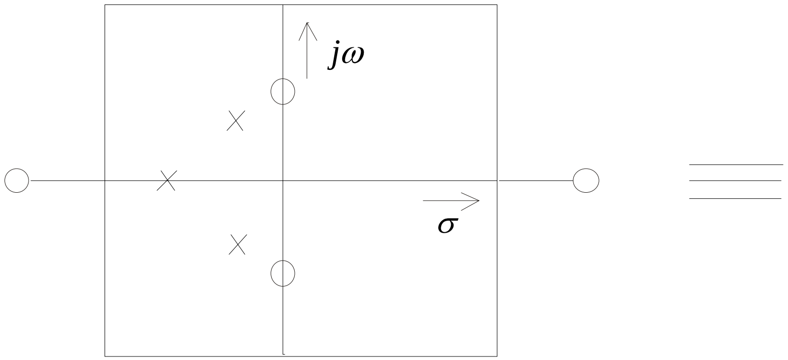
- No. poles = No. zeros
- Symmetry of pole/zero locations concern to jω-axis

all-pass (2nd order)



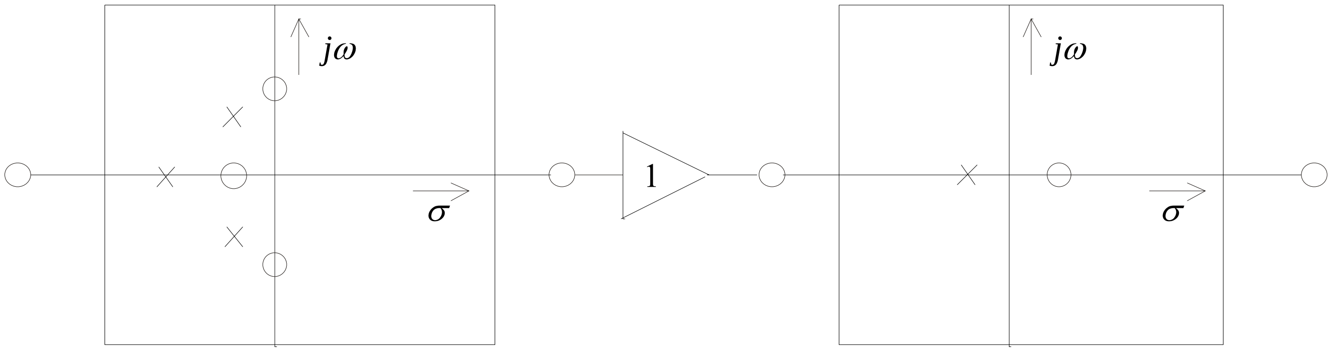
3.3.8 Pole Zero plots

System with all-pass

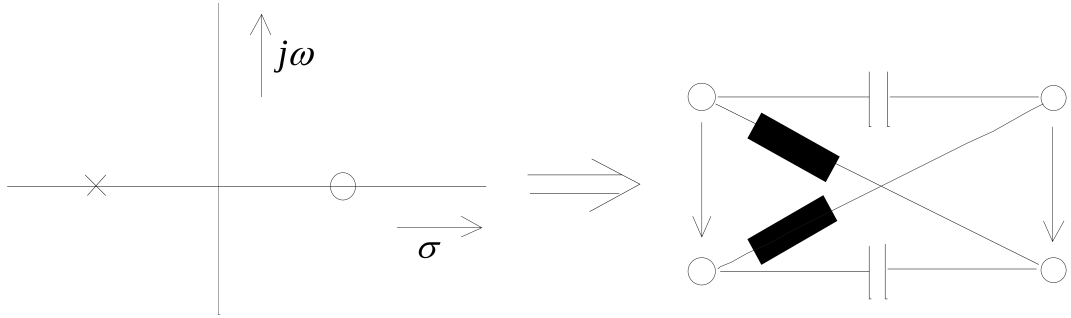


Minimum-phase-system

All-pass



3.3.8 Pole Zero plots



All-pass as Lattice type Filter for Phase correction.

Causal and stable LTI-systems:

- Poles only located in the left p-half-plane and on the imaginary axis.
- Zeros can be located in both half-planes.

3.3.8 Pole Zero plots

The effect of the single poles and zeros on the transmission system can be overviewed:

$$H_L(\omega) = e^{-a(\omega)} e^{-jb(\omega)}$$

Normalization of p can be carried out as:

$$p = \sigma + j\omega \quad \rightarrow \quad p = \frac{p}{\omega_N} = \frac{\sigma}{\omega_N} + j\frac{\omega}{\omega_N} = \Sigma + j\Omega$$

$$H_{LN}(p) = K \frac{\omega_N^m \prod_{i=1}^{\mu} (P - P_{0i})^{r_{0i}}}{\omega_N^n \prod_{i=1}^{\nu} (P - P_{\infty i})^{r_{\infty i}}} = K\omega_N^{(m-n)} \frac{\prod_{i=1}^{\mu} (P - P_{0i})^{r_{0i}}}{\prod_{i=1}^{\nu} (P - P_{\infty i})^{r_{\infty i}}}$$

where $\sum_{i=1}^{\mu} r_{0i} = m$ and $\sum_{i=1}^{\nu} r_{\infty i} = n$

3.3.8 Pole Zero plots

A logarithmation gives:

$$\ln H_{LN}(j\omega) = -a_{N_p}(\Omega) - jb(\Omega)$$

with $a_{N_p}(\Omega) = -\ln(A(\Omega\omega_N))$ **: frequency normalized damping ratio**

One can obtain:

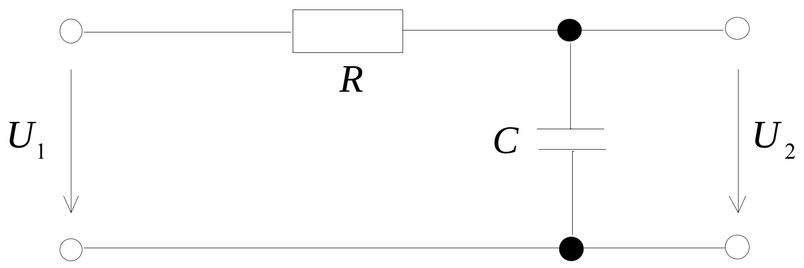
$$-a_{N_p}(\Omega) = \ln\left|K\omega_N^{m-n}\right| + \sum_{i=1}^{\mu} r_{0i} \ln\left|(P-P_{0i})^{r_{0i}}\right| - \sum_{i=1}^{\nu} r_{\infty,i} \ln\left|(P-P_{\infty i})^{r_{\infty i}}\right|$$

and for the **frequency normalised phase function**:

$$\varphi_N(\Omega) = \arctan K\omega_N^{m-n} + \sum_{i=1}^{\mu} r_{0i}\varphi_{N0i}(\Omega) - \sum_{i=1}^{\nu} r_{\infty i}\varphi_{N\infty i}(\Omega)$$

3.3.8 Pole Zero plots

Example: Given is the following circuit:



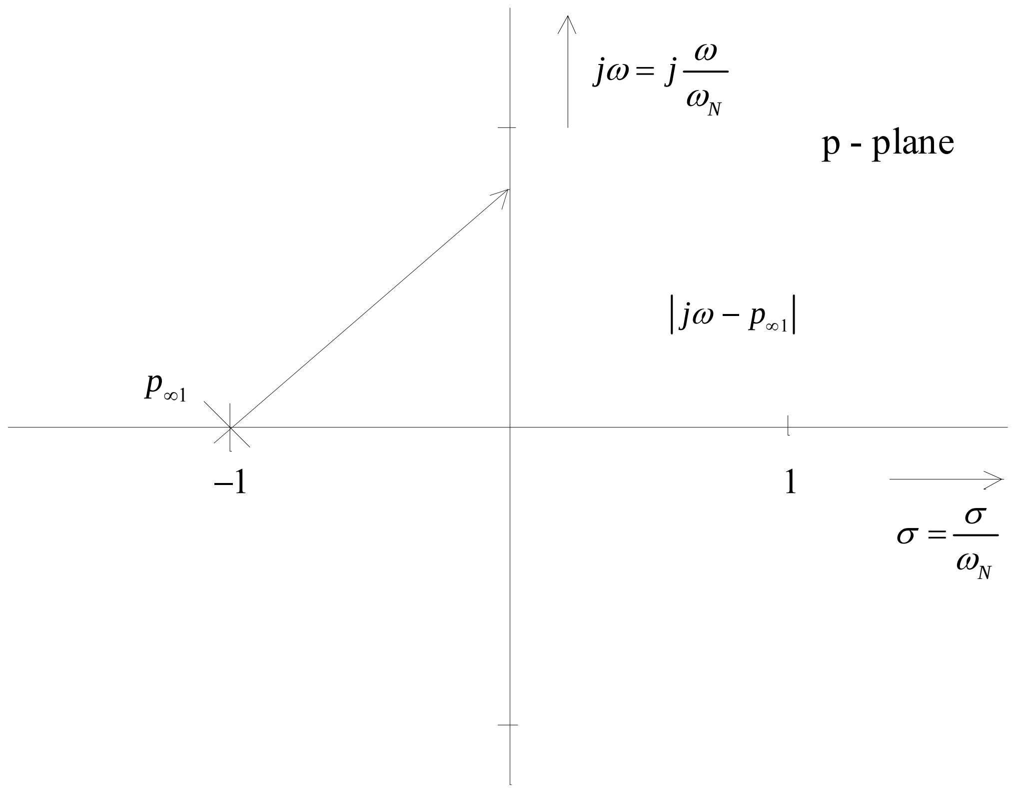
One gets:

$$H_L(p) = \frac{U_2}{U_1} = \frac{\frac{1}{pC}}{R + \frac{1}{pC}} = \frac{1}{pRC + 1} \quad \text{with} \quad p_{\infty 1} = -\frac{1}{RC}$$

A suitable normalization frequency is: $\omega_N = \frac{1}{RC}$

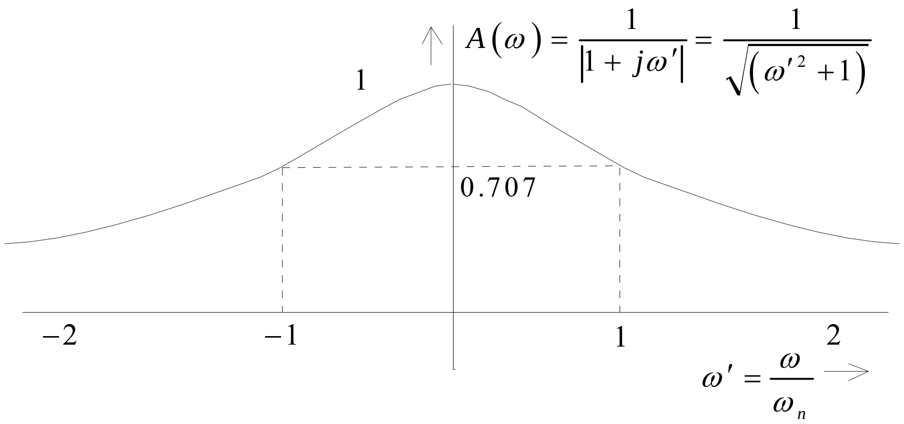
$$\Rightarrow H_L(P) = \frac{1}{\frac{p}{\omega_N} + 1} = \frac{1}{P + 1} \quad \text{and} \quad P_{\infty 1} = -1$$

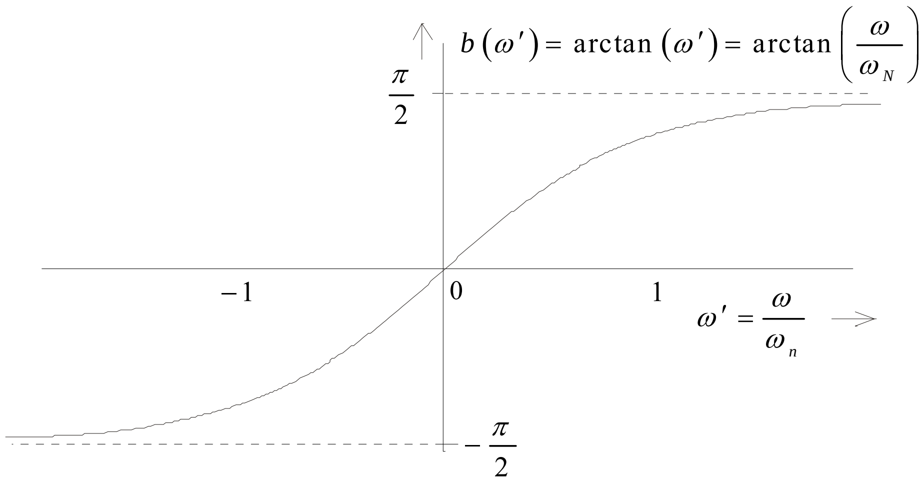
3.3.8 Pole Zero plots



Pole-Zero diagram of the RC-Circuit

3.3.8 Pole Zero plots

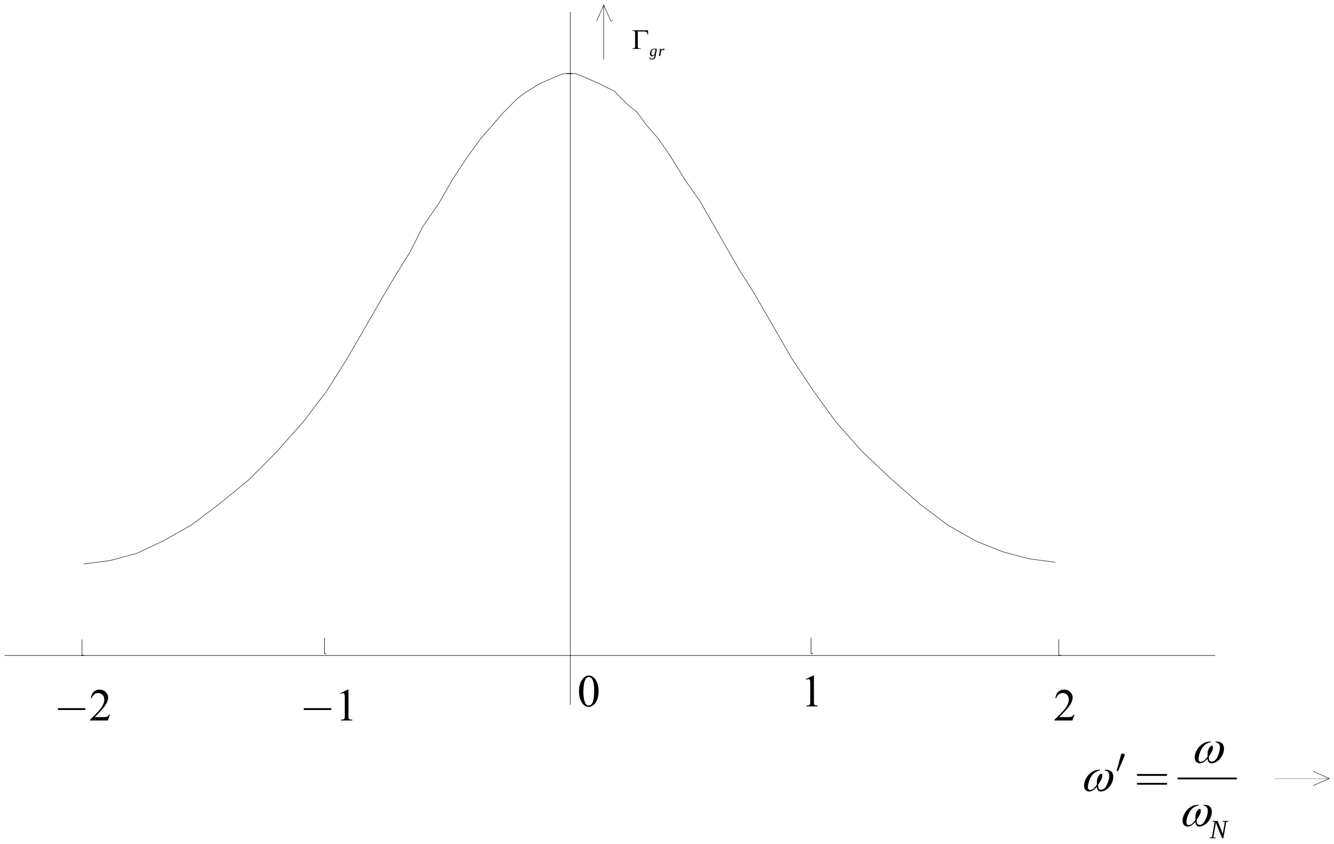




Magnitude and Phase of the RC-Circuit

3.3.8 Pole Zero plots

Appropriate envelope delay: $\tau_{GrN} = -\frac{d}{d\Omega}\varphi(\Omega) = \frac{1}{1+\Omega^2}$



Envelope delay of RC-Circuit

3.3.9 Minimum Phase systems

- An LTI system with zeros in the right half-plane is said to contain all-pass
- Such a system can be divided into:

❖ The all-pass which only effect on the phase

❖ The all-pass-less or minimum phase system which exhibits the same magnitude frequency response but with a smaller phase than the original system.

Transformation of non-minimum phase system into minimum phase:

$$\mathrm{Re}\left\{p_{0y}\right\} = -\mathrm{Re}\left\{p_{0x}\right\} \quad \text{and} \quad \mathrm{Im}\left\{p_{0y}\right\} = \mathrm{Im}\left\{p_{0x}\right\}$$

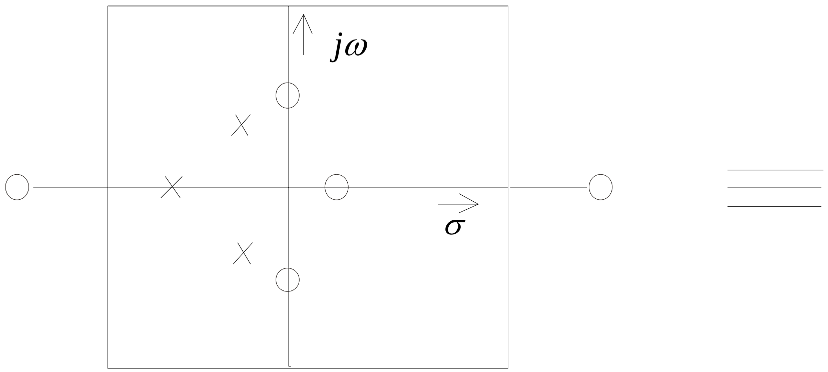
Theorem: LTI system without poles and zeros in the right half-plane are stable and of minimum phase.

Theorem: For minimum-phase systems, the phase $\mathrm{b}(\omega) = -\varphi(\omega)$

is fully prescribed by $A(\omega)$.

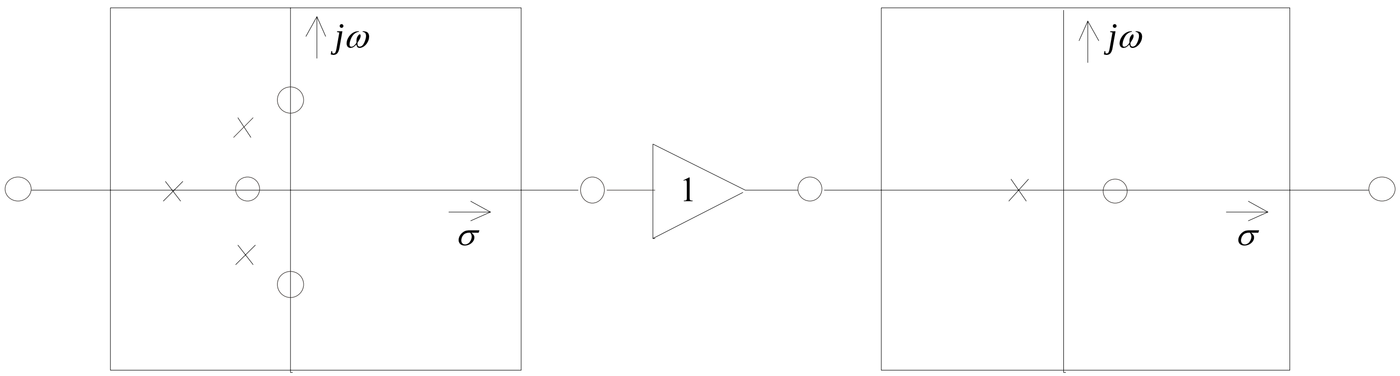
3.3.9 Minimum Phase Systems

System with all-pass



Minimum-phase-system

All-pass



3.3.10 Stability

An LTI system $s(t) \rightarrow g(t)$ is stable if:

$$|s(t)| < M_1 < \infty \longrightarrow |g(t)| < M_2 < \infty$$

From this, one can conclude:

$$s(t) \rightarrow g(t) = \int_{-\infty}^{+\infty} s(\tau)h(t-\tau)d\tau \quad \text{where} \quad |s(\tau)| < M_1 < \infty$$

$$|g(t)| = \left| \int_{-\infty}^{+\infty} s(\tau)h(t-\tau)d\tau \right| \leq \int_{-\infty}^{+\infty} |s(\tau)||h(t-\tau)|d\tau < M_1 \int_{-\infty}^{+\infty} |h(t-\tau)|d\tau$$

This results to $|g(t)| < M_2 < \infty$ if:

$$\int_{-\infty}^{+\infty} |h(t-\tau)|d\tau = \int_{-\infty}^{+\infty} |h(t)|d\tau < \infty \quad \text{for all t}$$

3.3.10 Stability

Reasoning: A system can be described by a fractional rational system function

$$H_L(p) = \frac{\sum_{i=0}^{m} a_i p^i}{\sum_{i=0}^{n} b_i p^i} = K \frac{\prod_{i=1}^{\mu} (p - p_{0i})^{r_{0i}}}{\prod_{i=1}^{\nu} (p - p_{\infty i})^{r_{\infty i}}}$$

or can be described by the following equation:

$$H_L(p) = K \cdot \sum_{i=1}^{\nu} \sum_{k=1}^{r_{\infty i}} \frac{A_{ik}}{(p - p_{\infty,i})^k} \qquad \text{(for n > m)}$$

or may have, in the simplest case, just single roots in the denominator polynomial:

$$H_L(p) = K \sum_{i=1}^{n} \frac{A_i}{p - p_{\infty,i}}$$

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The transform into the time domain results in the general case to

$$h(t) = K \cdot \left[\sum_{i=1}^{\nu} e^{p_{\infty,i} \cdot t} \cdot \sum_{k=1}^{r_{\infty,i}} \frac{A_{ik}}{(k-1)!} t^{k-1} \right] \cdot \varepsilon(t)$$

or for simple poles to:

$$h(t) = K \cdot \left[\sum_{i=1}^{n} A_i \cdot e^{p_{\infty,i} \cdot t} \right] \cdot \varepsilon(t)$$

1. In the general case, the impulse response consists of time function $e^{(p_{\infty i} \cdot t)} . t^{k-1}$ where the term $e^{(p_{\infty i} t)} t^{r_{\infty i}-1}$ have to be respected

   All these terms get smaller for a growing *t*, if:

$$\mathrm{Re}\left\{ p_{\infty i} \right\} = \sigma_{\infty i} < 0$$

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This means that the corresponding pole is located in the left p-half-plane:

$$\lim_{t \to \infty} e^{(p_{\infty i} t)} t^{r_{\infty i} - 1} = 0 \quad \text{for} \quad \mathrm{Re}\, p_{\infty i} < 0$$

Theorem: The poles of the system function of a stable causal network are all located in the left half-plane.

Theorem: The poles of the system function of a stable network can be located in the left half-plane and on the imaginary axis.

On the imaginary axis, they must be single. The appropriate denominator polynomial must have roots with negative real part or single roots with a real part of zero. This is the case of a special (conditional) stability!