Chapter 4

Discrete Systems

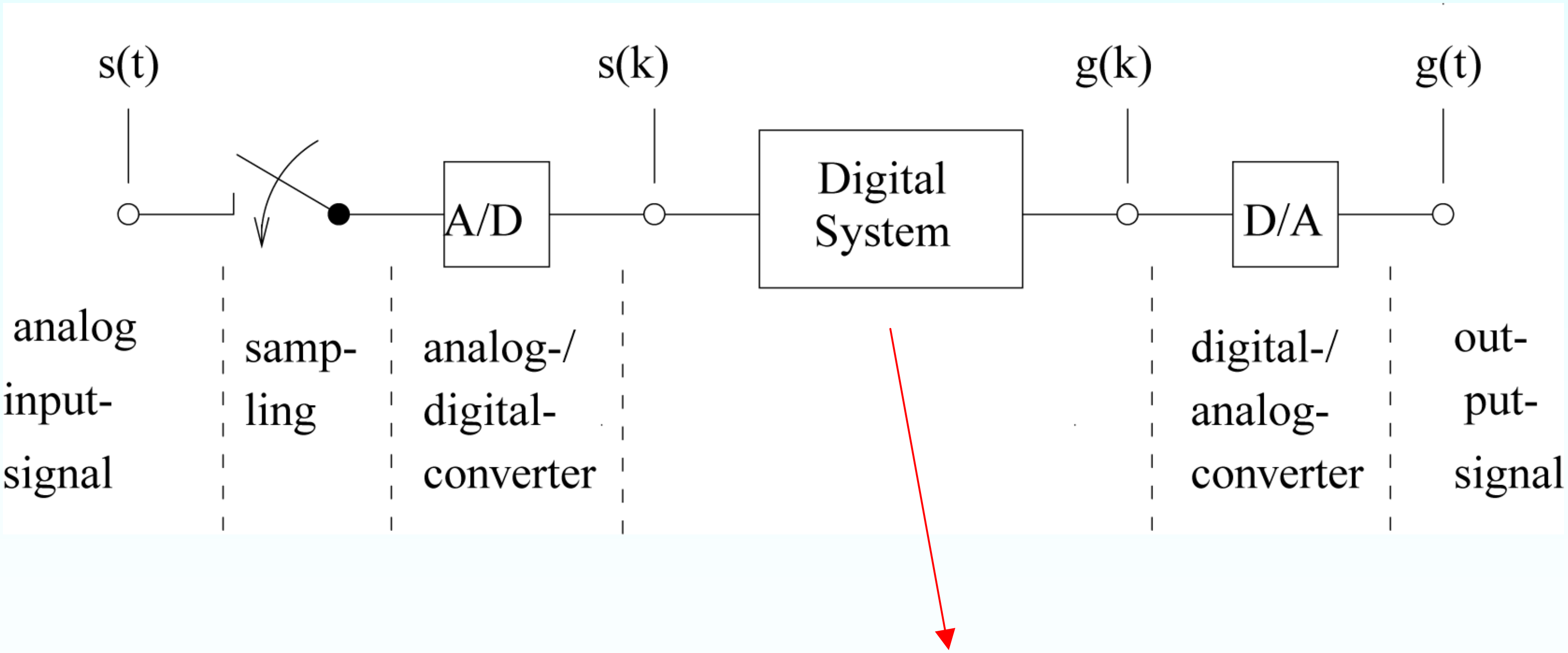
UNIVERSITÄT DUISBURG ESSEN

Fachgebiet
Nachrichtentechnische Systeme

NTS

4.1 Introduction

Graphik shows general scheme of a digital signal processing system
(used in cound cards, digital sound processors, precise filtering applications etc.).
Note: In general additional low-pass filters are needed!

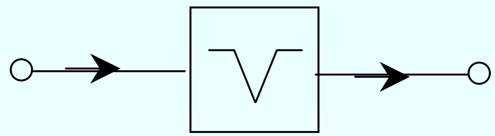


$$\{s(k)\} \rightarrow \{g(k)\} = T\left[\{s(k)\}\right] \quad \text{where} \quad k = -\infty \ldots +\infty$$

4.1 Introduction

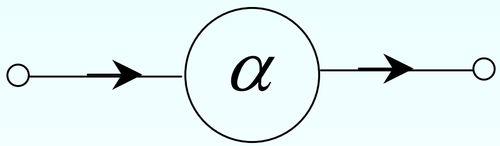
In practice, such a system is represented by a digital signal processor, essentially consisting of the following elements:

Delay:



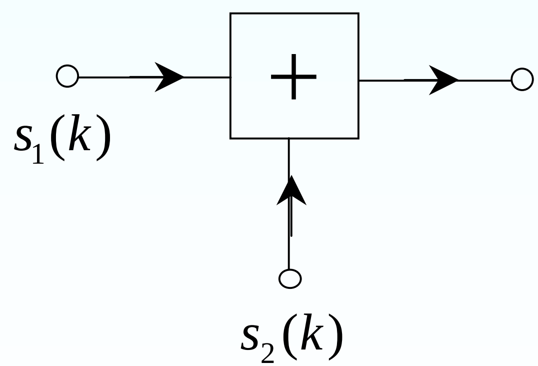
$$s(k-1) \quad \text{or} \quad s(k-1) = V[s(k)]$$

Multiplier:



$$\alpha \cdot s(k)$$

Adder:



$$s_1(k) + s_2(k)$$

4.1 Introduction

Example:

The relation of the example from section 2.3.5.1 with

$$g(k+2)+c_1 g(k+1)+g(k)=s(k+1)+s(k)$$

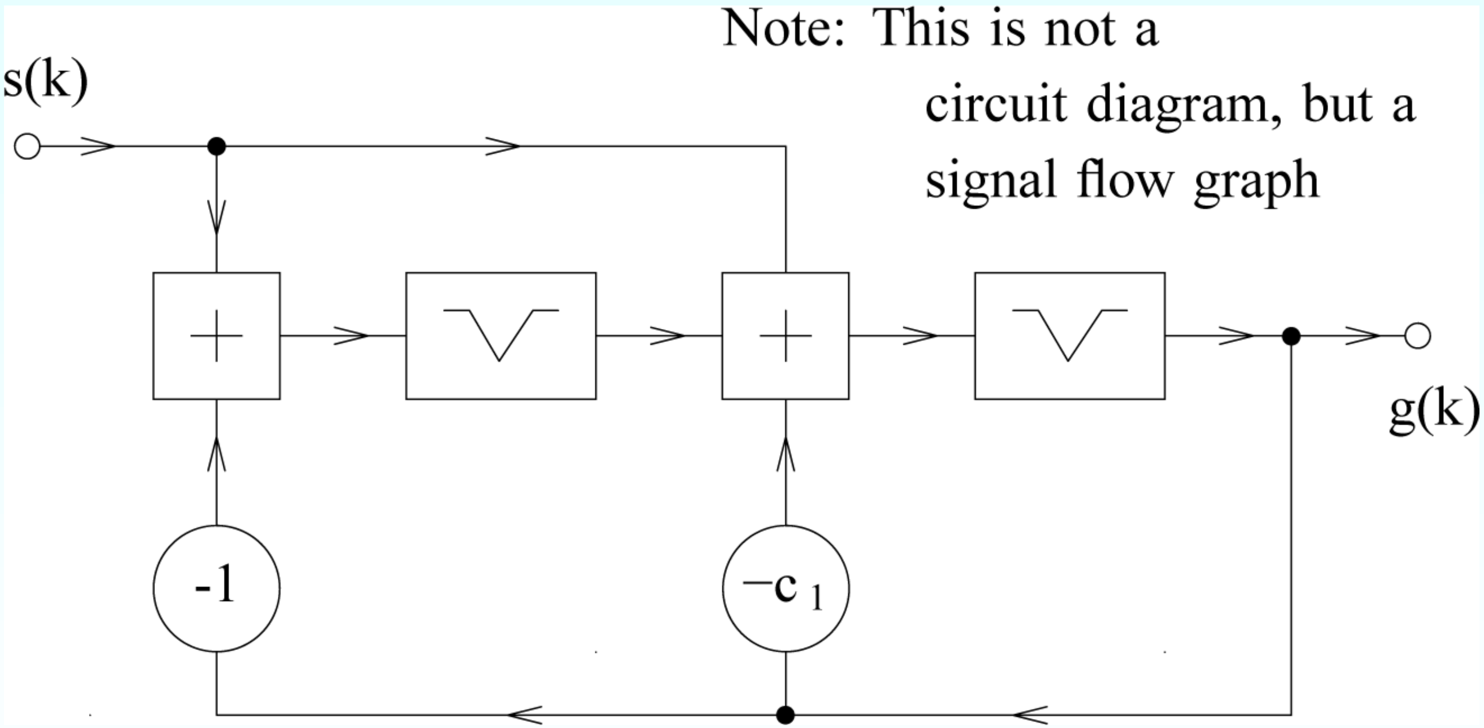
can be rewritten as:

$$g(k+2)=-c_1 g(k+1)-g(k)+s(k+1)+s(k)$$

$$g(k)=-c_1 g(k-1)-g(k-2)+s(k-1)+s(k-2)$$

The corresponding signal flow is represented in the following diagram:

4.1 Introduction



Signal flow diagram of the example of a digital filter

4.1 Introduction

The same basic properties as for analog filters can also be found for discrete systems:

1- Real values:

From real-valued {s(k)} follows that also {g(k)} is real-valued

2- Time-invariance:

$$\{s(k)\} \rightarrow \{g(k)\} \quad \text{gives} \quad \{s(k+\kappa)\} \rightarrow \{g(k+\kappa)\}$$

3- Linearity:

$$\{s_1(k)\} \rightarrow \{g_1(k)\}$$

$$\{s_\nu(k)\} \rightarrow \{g_\nu(k)\}$$

$$\{s_n(k)\} \rightarrow \{g_n(k)\}$$

$$\Rightarrow \quad \sum_{\nu=1}^{n} \{\alpha_\nu s_\nu(k)\} \rightarrow \sum_{\nu=1}^{n} \{\alpha_\nu g_\nu(k)\}$$

Causality and stability are here as important as for analog systems.

4.2 Linear, Time-Invariant Discrete Systems

4.2.1 Difference Equations

The most important class of discrete LTI-systems is the one, describable by an n-th order equation of the following kind:

$$\sum_{\alpha=0}^{n} a_\alpha s(k-\alpha) = \sum_{\beta=0}^{n} b_\beta g(k-\beta)$$

From this, one obtains:

$$g(k) = \frac{1}{b_0}\left[\sum_{\alpha=0}^{n} a_\alpha s(k-\alpha) - \sum_{\beta=1}^{n} b_\beta g(k-\beta)\right]$$

4.2.1 Difference Equations

Definition:

A discrete-time LTI-system is called recursive if the calculation of each output value $g(k)$ from the preceding output values $g(k - ß)$ with $ß > 0$ is performed.

Definition:

A causal digital LTI-system is called non-recursive if the calculation of each output value $g(k)$ is possible without the use of previously calculated output signals $g(k - ß)$ with $ß > 0$.

4.2.1 Difference Equations

Example:

Given is a second order discrete LTI-system, where *n* = *m* = *2*, thus:

$$g(k) = \frac{1}{b_0}\left[\sum_{\alpha=0}^{2} a_\alpha s(k-\alpha) - \sum_{\beta=1}^{2} b_\beta g(k-\beta)\right]$$

One can set $b_0 = 1$ here without any restrictions:

$$g(k) = a_0 s(k) + a_1 s(k-1) + a_2 s(k-2) - b_1 g(k-1) - b_2 g(k-2)$$

In the next slide it is specified: $b_{1,2} = d_{1,2}$!!

4.2.1 Difference Equations

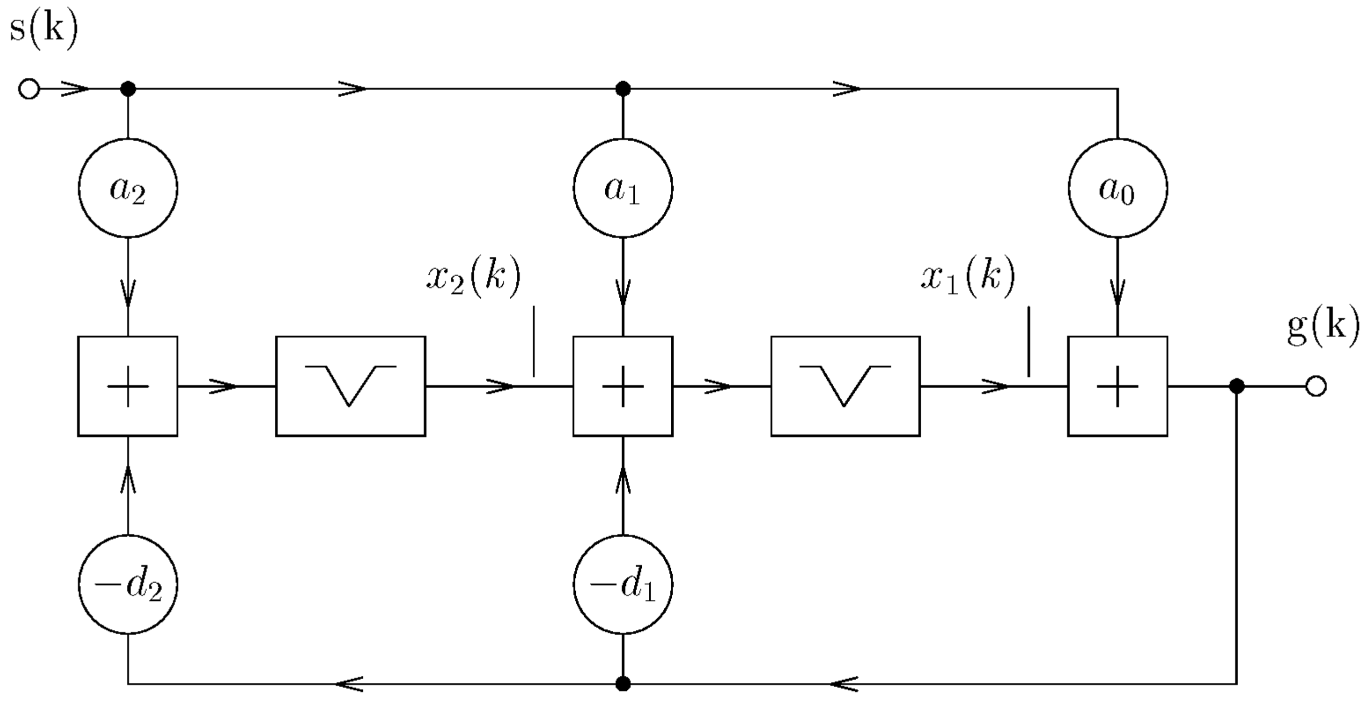


Figure 4.4: Structure of a 2nd Order (IIR-)Filter

4.2.2 The Discrete Impulse Response

Definition:

The impulse response of a discrete system is the response $g(k)$ of the system to $s(k) = \gamma_0(k)$

This special sequence to be observed at the output is denoted by: $h(k)$

The answer of the system to any causal excitation $s(k)$ with $s(k) \equiv 0 \; for \; k < 0$ thus is:

$$g(k) = \sum_{\nu=0}^{+\infty} h(\nu) \cdot s(k-\nu) = h(k) * s(k)$$

→ Discrete convolution

The causality provides: $h(k) \equiv 0 \quad \text{for} \quad k < 0,$ thus

$$s(k) = g(k) \equiv 0 \quad \text{for} \quad k < 0$$

4.2.3 The Discrete Transfer Function $H_z(z)$

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4.2.3 The Discrete Transfer Function $H_z(z)$

If {s(k)} as input signal for a discrete system is used with

$$s(k) = Ue^{kpT} = Uz^k \quad \text{where } k = 0 \ldots \infty$$

and with $p = \sigma + j\omega$ being the complex frequency, one gets:

$$g(k) = h(k) * s(k) = \sum_{\nu=0}^{+\infty} h(\nu) \cdot s(k-\nu)$$

$$= \sum_{\nu=0}^{+\infty} h(\nu) \cdot Uz^{k-\nu} = Uz^k \underbrace{\sum_{\nu=0}^{+\infty} h(\nu) \cdot z^{-\nu}}_{Z\{h(\nu)\} = H_z(z)} = Uz^k \cdot H_z(z)$$

<u>Discrete transfer function.</u>

4.2.3 The Discrete Transfer Function $H_z(z)$

* According to z-transform properties it holds:

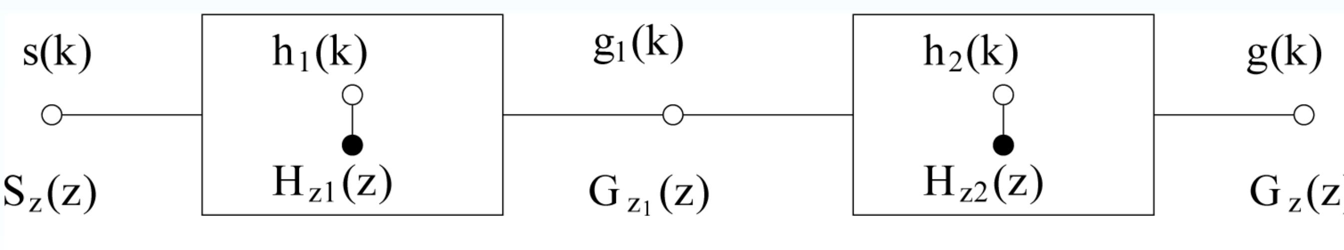
$$g(k) = \sum_{\nu=0}^{+\infty} h(\nu) \cdot s(k-\nu) = h(k) * s(k) \quad ○\!\!-\!\!\!-\!\!● \quad G_z(z) = H_z(z) \cdot S_z(z)$$

$$\Leftrightarrow \quad H_z(z) = \frac{G_z(z)}{S_z(z)}$$

* The discrete transfer function is the z-transform of the impulse response {h(k)}

$$H_z(z) = \sum_{\nu=0}^{+\infty} h(\nu) \cdot z^{-\nu}$$

* For a chain of two discrete LTI-systems it holds:



4.2.3 The Discrete Transfer Function $H_z(z)$

With the equations:

$$g_1(k) = h_1(k) * s(k) \quad \text{and} \quad g(k) = g_1(k) * h_2(k)$$

it follows: $g(k) = g_1(k) * h_2(k) = \left[h_1(k) * s(k)\right] * h_2(k)$

or $g(k) = h_1(k) * h_2(k) * s(k) \quad \circ\!\!-\!\!\bullet \quad G_z(z) = \underbrace{H_{z1}(z) \cdot H_{z2}(z)}_{H_z(z)} \cdot S_z(z)$

$$G_z(z) = H_z(z) \cdot S_z(z) \longleftarrow H_z(z) = H_{z1}(z) H_{z2}(z)$$

4.2.3 The Discrete Transfer Function $H_z(z)$

Considering a difference equation, this relation can be described in the z-domain:

$$\sum_{\alpha=0}^{n} a_\alpha s(k-\alpha) = \sum_{\beta=0}^{n} b_\beta g(k-\beta)$$

The z-transform of both sides of the equation is then:

$$\sum_{\alpha=0}^{n} a_\alpha S_z(z) \cdot z^{-\alpha} = \sum_{\beta=0}^{n} b_\beta G_z(z) \cdot z^{-\beta}$$

Rewriting this formula gives:

$$H_z(z) = \frac{G_z(z)}{S_z(z)} = \frac{\sum_{\alpha=0}^{n} a_\alpha \cdot z^{-\alpha}}{\sum_{\beta=0}^{n} b_\beta \cdot z^{-\beta}}$$

By means of an index conversion →

$$H_z(z) = \frac{\sum_{\mu=0}^{m} d_\mu \cdot z^{\mu}}{\sum_{\nu=0}^{n} c_\nu \cdot z^{\nu}}$$

where $n \geq m$

4.2.3 The Discrete Transfer Function $H_z(z)$

So this second form of the discrete transfer function $H_z(z)$ is a rational function of the variable z:

$$H_z(z) = \frac{\text{Numerator polynom in z}}{\text{Denominator ploynom in z}} = \frac{P(z)}{Q(z)}$$

A third form of the discrete transfer function is based on the zeros of the numerator and the denominator polynom:

$$H_z(z) = \frac{d_m}{c_n} \cdot \frac{\prod_{\mu=1}^{m} (z - z_{0\mu})}{\prod_{\nu=1}^{n} (z - z_{\infty\mu})}$$

Roots of the of the numerator or the zeros

Roots of the denominator or the poles

4.2.4 General Properties of Hz(z)

4.2.4 General Properties of $H_z(z)$

1. Properties of system coefficients and of poles and zeros:

The coefficients respectively are real constants. The zeros and poles are either real or conjugated complex:

$$z_{0_{\mu 1}} = \left| z_{0_{\mu 1}} \right| e^{j\psi_{0_{\mu 1}}} \quad \text{where } z_{0_{\mu 2}} = z^*_{0_{\mu 1}} = \left| z_{0_{\mu 1}} \right| e^{-j\psi_{0_{\mu 1}}}$$

$$z_{\infty_{\nu 1}} = \left| z_{\infty_{\nu 1}} \right| e^{j\psi_{\infty_{\nu 1}}} \quad \text{where } z_{\infty_{\nu 2}} = z^*_{\infty_{\nu 1}} = \left| z_{\infty_{\nu 1}} \right| e^{-j\psi_{\infty_{\nu 1}}}$$

2. Stability: (BIBO: bounded input bounded output criterion)

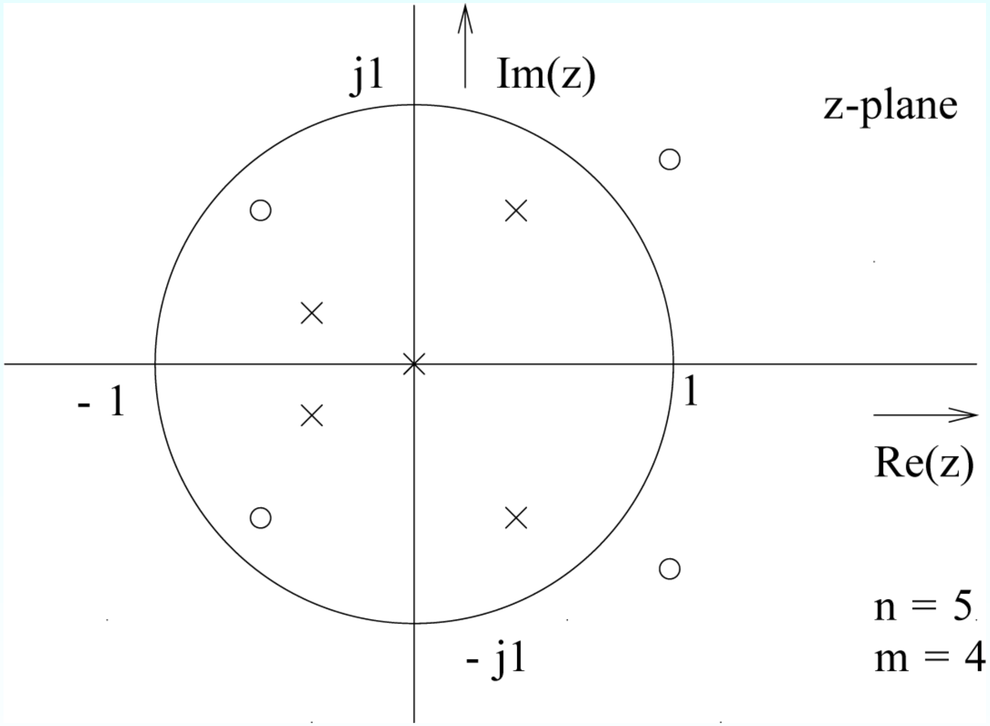
A discrete system obviously is stable, if

any bounded input signal $|s(k)| < M_1 < \infty \quad \forall k$ causes

a bounded output signal $|g(k)| < M_2 < \infty$

4.2.4 General Properties of $H_z(z)$

BIBO stability is given, if the following relation holds: $\left| z_{\infty_\nu} \right| < 1 \quad \forall \nu$



Pole- zero plot of a real, causal and stable system

4.2.5 Behaviour of $H_z(z)$ on the unit circle

If we observe the $H_Z(z)$ for any point z on the unit circle with $|z| = 1$ and $z = e^{j\omega T}$

we obtain the so-called frequency response.

$$H_z(e^{j\omega T}) = \frac{\sum_{\alpha=0}^{n} a_\alpha e^{-j\alpha\omega T}}{\sum_{\beta=0}^{n} b_\beta e^{-j\beta\omega T}} = \frac{d_m}{c_n} \frac{\prod_{\mu=1}^{m} (e^{j\omega T} - z_{0_\mu})}{\prod_{\nu=1}^{n} (e^{j\omega T} - z_{\infty_\nu})}$$

→ periodic function with $\omega T = 2\pi$ or $\omega = \frac{2\pi}{T}$

In short: $H_z(e^{j\omega T}) = H_a(\omega)$

4.2.5 Behaviour of $H_z(z)$ on the unit circle

In general :

$$G_z(z) = H_z(z) S_z(z) \quad \Rightarrow \quad G_z(e^{j\omega T}) = H_z(e^{j\omega T}) S_z(e^{j\omega T})$$

A normalized representation using $\omega T = \Omega$ *or* $F = \Omega \cdot 2\pi$, leads to:

$$H_z(e^{j\omega T}) = H_z(e^{j\Omega}) = H_{Na}(\Omega) = \frac{b_m}{c_n} \frac{\prod_{\mu=1}^{m} (e^{j\Omega} - z_{0_\mu})}{\prod_{\nu=1}^{n} (e^{j\Omega} - z_{\infty_\nu})}$$

Magnitude $\left| H_a(\Omega) \right|$ and its phase $\varphi_a(\Omega)$ can be rewritten as:

$$H_{Na}(\Omega) = \left| H_{Na}(\Omega) \right| e^{j\varphi_{Na}(\Omega)}$$

4.2.5 Behaviour of $H_z(z)$ on the unit circle

For the distance of each zero or each pole to the unit circle, it holds:

$$\left|e^{j\Omega}-z\right| = \left|\cos\Omega + j\sin\Omega - |z|\cos\psi - |z|\cdot j\sin\psi\right|$$

$$= \sqrt{\left(\cos\Omega - |z|\cos\psi\right)^2 + \left(\sin\Omega - |z|\sin\psi\right)^2}$$

$$= \sqrt{\cos^2\Omega - 2\cos\Omega|z|\cos\psi + |z|^2\cos^2\psi + \sin^2\Omega - 2\sin\Omega|z|\sin\psi + |z|^2\sin^2\psi}$$

$$= \sqrt{1 - 2|z|\cos(\Omega-\psi) + |z|^2}$$

with $z = |z| e^{j\psi}$

For the corresponding angle of the connection of each zero or each pole to the unit circle, it holds:

$$\angle\left(e^{j\Omega}-z\right) = \arctan\frac{\sin\Omega - |z|\sin\psi}{\cos\Omega - |z|\cos\psi}$$

4.2.5 Behaviour of $H_z(z)$ on the unit circle

Thus it results:

$$\left|H_{Na}(\Omega)\right| = \left|\frac{b_m}{c_n}\right| \cdot \frac{\prod_{\mu=1}^{m}\left|e^{j\Omega} - z_{0_\mu}\right|}{\prod_{\nu=1}^{n}\left|e^{j\Omega} - z_{\infty_\nu}\right|}$$

$$= \left|\frac{b_m}{c_n}\right| \cdot \frac{\prod_{\mu=1}^{m}\sqrt{1-2\left|z_{0_\mu}\right| \cdot \cos(\Omega - \psi_{0_\mu}) + \left|z_{0_\mu}\right|^2}}{\prod_{\nu=1}^{n}\sqrt{1-2\left|z_{\infty_\nu}\right| \cdot \cos(\Omega - \psi_{\infty_\nu}) + \left|z_{\infty_\nu}\right|^2}}$$

With $\frac{b_m}{c_n} > 0$ it follows:

$$\varphi_{Na}(\Omega) = \sum_{\nu=1}^{n} \arctan \frac{\sin\Omega - \left|z_{\infty_\nu}\right| \sin\Psi_{\infty_\nu}}{\cos\Omega - \left|z_{\infty_\nu}\right| \cos\Psi_{\infty_\nu}} - \sum_{\mu=1}^{m} \arctan \frac{\sin\Omega - \left|z_{0_\mu}\right| \sin\Psi_{0_\mu}}{\cos\Omega - \left|z_{0_\mu}\right| \cos\Psi_{0_\mu}}$$

4.2.5 Behaviour of $H_z(z)$ on the unit circle

For the derivative of the angle related to the connections of each pole or each zero it holds:

$$\frac{d}{d\Omega}\left(e^{j\Omega}-z\right)=\frac{d}{d\Omega}\arctan\frac{\sin\Omega-|z|\sin\psi}{\cos\Omega-|z|\cos\psi}=$$

$$\frac{1}{1+\frac{(\sin\Omega-|z|\sin\psi)^2}{(\cos\Omega-|z|\cos\psi)^2}}\cdot\frac{\cos\Omega(\cos\Omega-|z|\cos\psi)-(-\sin\Omega)(\sin\Omega-|z|\sin\psi)}{(\cos\Omega-|z|\cos\psi)^2}$$

$$=\frac{\cos^2\Omega-\cos\Omega|z|\cos\psi+\sin^2\Omega-\sin\Omega|z|\sin\psi}{(\cos\Omega-|z|\cos\psi)^2+(\sin\Omega-|z|\sin\psi)^2}$$

$$=\frac{1-|z|\cos(\Omega-\psi)}{\cos^2\Omega-2\cos\Omega|z|\cos\psi+|z|^2\cos^2\psi+\sin^2\Omega-2\sin\Omega|z|\sin\psi+|z|^2\sin^2\psi}$$

$$=\frac{1-|z|\cos(\Omega-\psi)}{1+|z|^2-2|z|\cos(\Omega-\psi)}$$

4.2.5 Behaviour of $H_z(z)$ on the unit circle

Thus the frequency normalised envelope delay results to:

$$\tau_{Nga}(\Omega) = \frac{d\varphi_{Na}(\Omega)}{d\Omega} = \sum_{\nu=1}^{n} \frac{1 - \left|z_{\infty_\nu}\right| \cdot \cos(\Omega - \Psi_{\infty_\nu})}{1 - 2\left|z_{\infty_\nu}\right| \cdot \cos(\Omega - \Psi_{\infty_\nu}) + \left|z_{\infty_\nu}\right|^2}$$

$$- \sum_{\mu=1}^{m} \frac{1 - \left|z_{0_\mu}\right| \cdot \cos(\Omega - \Psi_{0_\mu})}{1 - 2\left|z_{0_\mu}\right| \cdot \cos(\Omega - \Psi_{0_\mu}) + \left|z_{0_\mu}\right|^2}$$

Note: According to the formulas given above it follows:

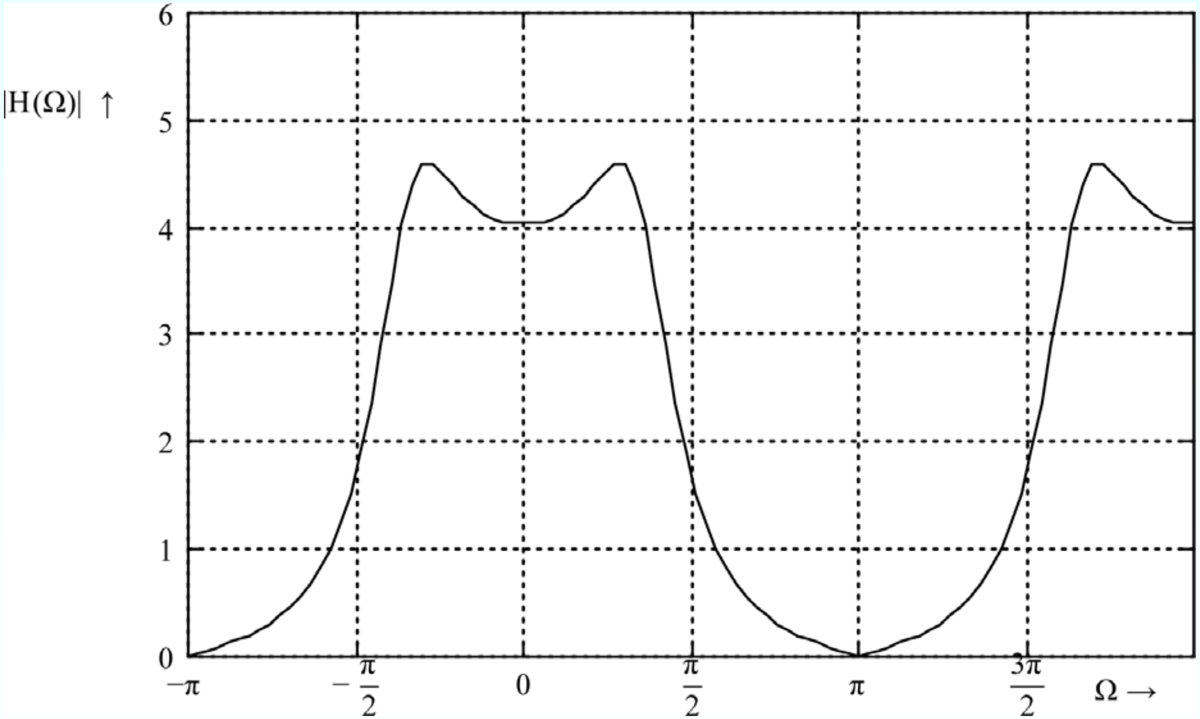
$$\left|H_{Na}(\Omega)\right| = \left|H_{Na}(-\Omega)\right| \qquad \varphi_{Na}(-\Omega) = -\varphi_{Na}(\Omega)$$

$$\tau_{Nga}(-\Omega) = \tau_{Nga}(\Omega)$$

4.2.5 Behaviour of $H_z(z)$ on the unit circle

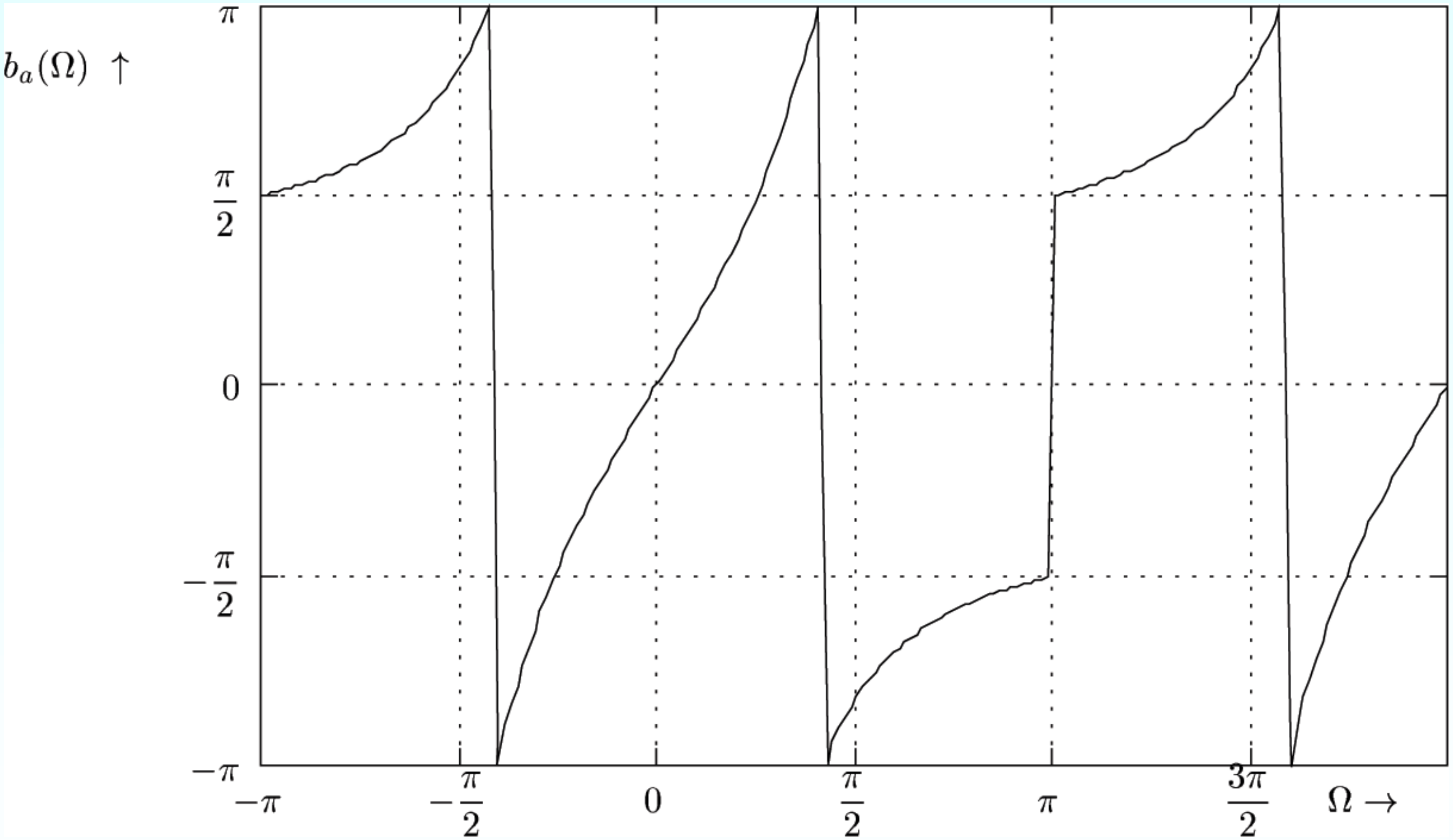
Example: 3rd order system with the discrete transfer function

$$H(z) = \frac{(z+1)(z+0.2)}{(z-0.3)(z-0.3-j0.6)(z-0.3+j0.6)}$$



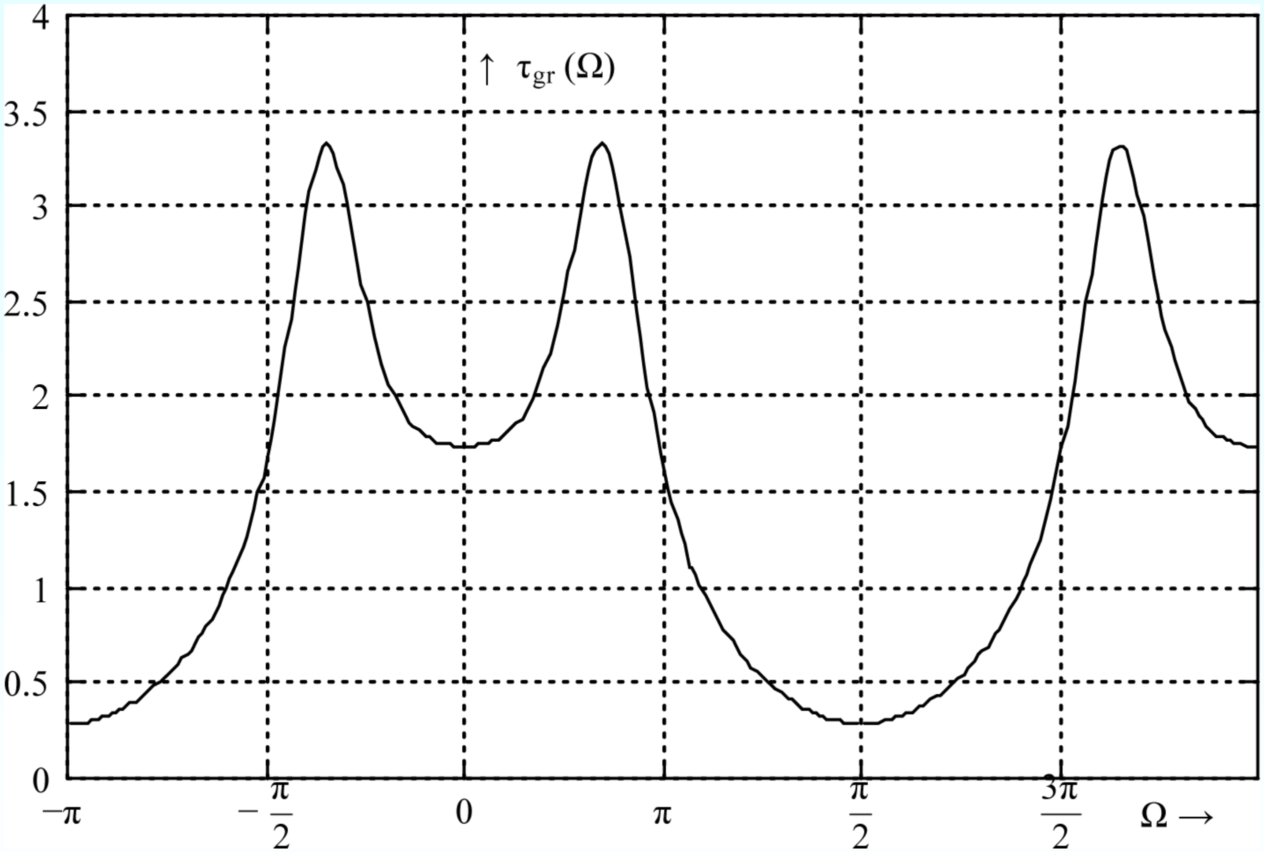
Magnitude of discrete transfer function of 3rd order system

4.2.5 Behaviour of $H_z(z)$ on the unit circle



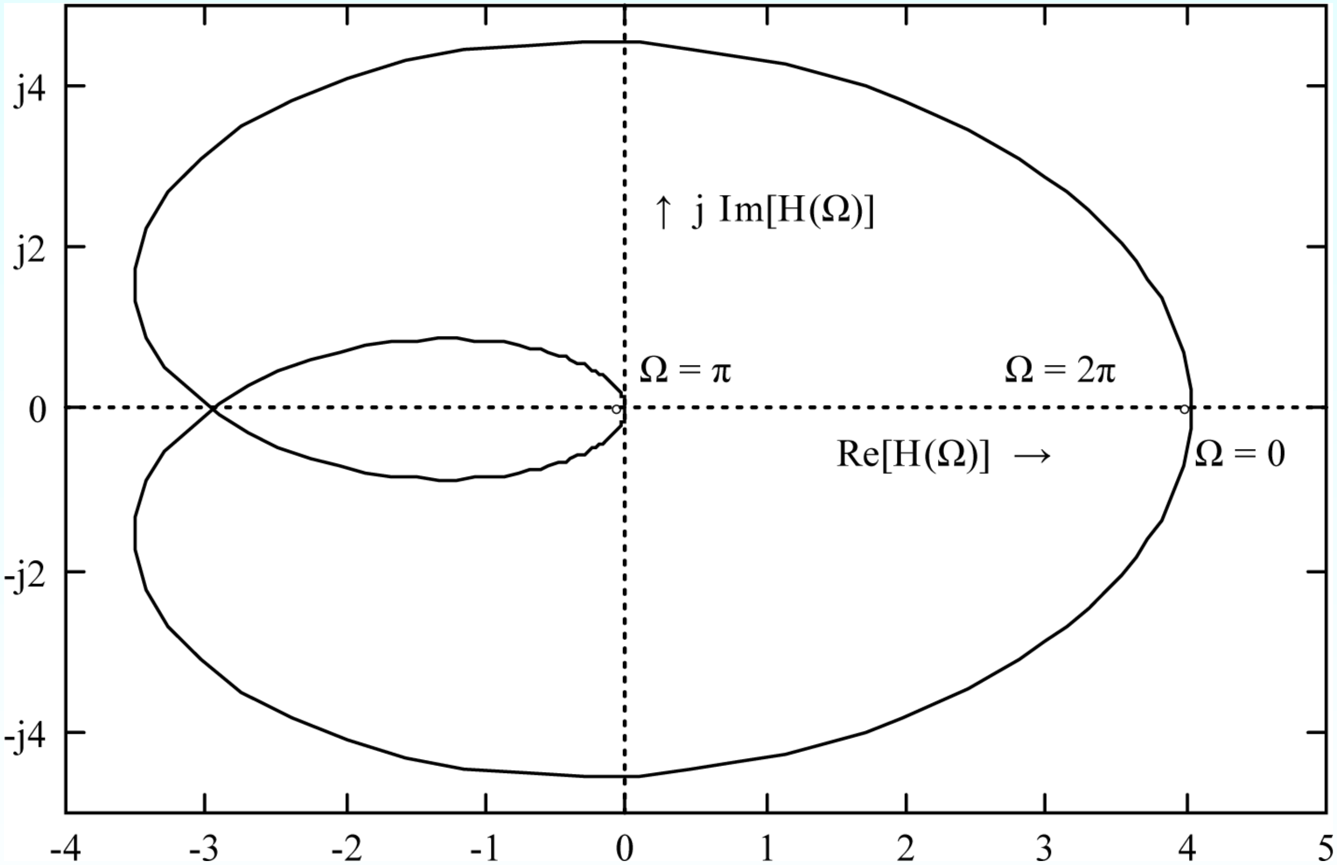
Phase of discrete transfer function of 3rd order system

4.2.5 Behaviour of $H_z(z)$ on the unit circle



Envelope delay of a 3rd order system

4.2.5 Behaviour of $H_z(z)$ on the unit circle



Locus of the discrete transfer function of 3rd order system

4.2.5 Behaviour of $H_z(z)$ on the unit circle

If the regarded system are causal and stable then:

1. $H_L(p)$ is analytically regular for $\operatorname{Re} p > 0$ and $H_Z(z)$ is analytically regular for $|z| < 1$
2. $H_L(j\omega) = H(\omega)$ shows the frequency behaviour of analog filter

   $\mathrm{H}_Z(e^{j\omega t}) = H_Z(e^{j\Omega}) = H_a(\omega)$ shows the frequency behaviour of digital filter.

with
$$H_z(e^{j\Omega}) = \operatorname{Re}\{H_z(e^{j\Omega})\} + j\operatorname{Im}\{H_z(e^{j\Omega})\} \quad \text{or}$$

$$H_{Na}(\Omega) = \operatorname{Re}\{H_{Na}(\Omega)\} + j\operatorname{Im}\{H_{Na}(\Omega)\}$$

It results:
$$\operatorname{Re}\{H_{Na}(\Omega)\} = \lim_{z\to\infty} H_z(z) - \frac{1}{2\pi}\int_{-\pi}^{+\pi} \operatorname{Im}\{H_{Na}(\eta)\}\cot\frac{\eta-\Omega}{2}\,d\eta$$

$$\operatorname{Im}\{H_{Na}(\Omega)\} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \operatorname{Re}\{H_{Na}(\eta)\}\cot\frac{\eta-\Omega}{2}\,d\eta$$

4.2.5 Behaviour of $H_z(z)$ on the unit circle

Note: For minimum-phase systems, it applies:

$$\ln\left|H_{Na}(\Omega)\right| = \lim_{z\to\infty} \ln\left|H_z(z)\right| - \frac{1}{2\pi}\int_{-\pi}^{+\pi} \varphi_a(\eta)\cot\frac{\eta-\Omega}{2}\,d\eta$$

$$\varphi_{Na}(\Omega) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \ln\left|H_{Na}(\eta)\right|\cot\frac{\eta-\Omega}{2}\,d\eta$$

4.2.6 All-Pass Filters

An all-pass filter is defined according to:

$$|H_a(\Omega)| = |H_z(e^{j\Omega})| = \text{const. } \forall\Omega$$

For the discrete transfer fuction of an all-pass: $H_z(z) = \dfrac{P(z)}{Q(z)}$

one observes:

$$\frac{1}{Q(z)} = \frac{1}{c_n \prod_{\nu=1}^{n} (z - z_{\infty\nu})}$$

and looks for a:

$$P(z) = b_m \prod_{\mu=1}^{m} (z - z_{0\mu})$$

In such a way that equation is fulfilled.

4.2.6 All-Pass Filters

The following possibilities are given:

- Case $z_{\infty\nu} = 0 \quad , \forall \nu$

$$\left|H_{Na}(\Omega)\right| = \frac{1}{\left|c_n\right|} \text{ where } P(z) = 1, \text{ because } \left|z\right| = 1 \text{ where } z = e^{j\Omega}$$

- Other case it results:

$$\left|H_{Na}(\Omega)\right| = \left|\frac{d_m}{c_n}\right| \frac{\prod_{\mu=1}^{m} \sqrt{1 - 2\left|z_{0\mu}\right|\cos(\Omega - \psi_{0\mu}) + \left|z_{0\mu}\right|^2}}{\prod_{\nu=1}^{n} \sqrt{1 - 2\left|z_{\infty\nu}\right|\cos(\Omega - \psi_{\infty\nu}) + \left|z_{\infty\nu}\right|^2}}$$

So every pole makes a contribution $\sqrt{1 - 2\left|z_{\infty\nu}\right|\cos(\Omega - \psi_{\infty\nu}) + \left|z_{\infty\nu}\right|^2}$ to $\left|H_{Na}(\Omega)\right|$

4.2.6 All-Pass Filters

$\left|H_{Na}(\Omega)\right| =$ const. results in case the contribution of every pole is compensated by :

$$\sqrt{1-2\left|z_{0\mu}\right|\cos(\Omega-\psi_{0\mu})+\left|z_{0\mu}\right|^2}$$

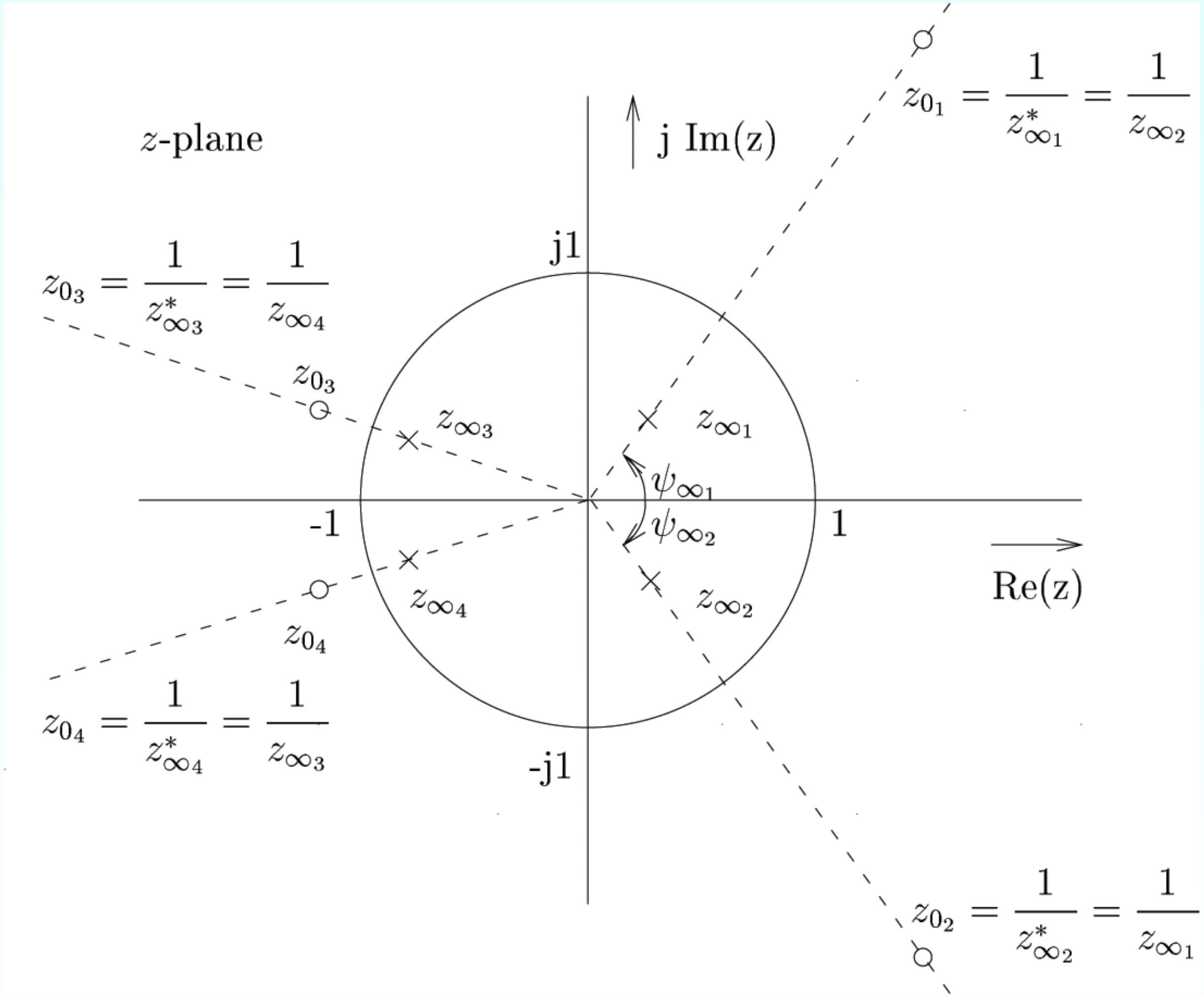
This is possible, if m=n and:

$$z_{0\mu}=\left|z_{0\mu}\right|e^{j\psi_{0\mu}}=\frac{1}{\left|z_{\infty\nu}\right|}e^{j\psi_{\infty\nu}}=\frac{1}{\left|z_{\infty\nu}\right|e^{-j\psi_{\infty\nu}}}=\frac{1}{z^{*}_{\infty\nu}}$$

so that:

$$\frac{\left|e^{j\Omega}-z_{0\mu}\right|}{\left|e^{j\Omega}-z_{\infty\nu}\right|}=\frac{\sqrt{1-2\frac{1}{\left|z_{\infty\nu}\right|}\cos(\Omega-\Psi_{\infty\nu})+\frac{1}{\left|z_{\infty\nu}\right|^2}}}{\sqrt{1-\left|z_{\infty\nu}\right|\cos(\Omega-\Psi_{\infty\nu})+\left|z_{\infty\nu}\right|^2}}=\frac{1}{\left|z_{\infty\nu}\right|}$$

4.2.6 All-Pass Filters



Pole-Zero diagram of an All-pass in the z-plane

4.2.6 All-Pass Filters

One gets in case of:

$$|z_{\infty\nu}| = \frac{1}{|z_{0\nu}|} \qquad \text{and} \qquad \Psi_{\infty\nu} = \Psi_{0\nu}$$

phase delay:

$$\varphi_{Na}(\Omega) = \sum_{\nu=1}^{n} \arctan \frac{\left(1-|z_{\infty\nu}|^2\right)\sin(\Omega-\Psi_{\infty\nu})}{(1+|z_{\infty\nu}|^2)\cos(\Omega-\Psi_{\infty\nu})-2|z_{\infty\nu}|}$$

envelope delay:

$$\tau_{Nga}(\Omega) = \sum_{\nu=1}^{n} \frac{1-|z_{\infty\nu}|^2}{1-2|z_{\infty\nu}|\cos(\Omega-\Psi_{\infty\nu})+|z_{\infty\nu}|^2} \qquad \text{where} \quad |z_{\infty\nu}|<1 \quad \text{for all } \nu$$

4.2.7 Minimum-phase Systems

Discrete system can also be devided into all-passes and minimum-phase systems

Let‘s assume an stable LTI discrete system with the transfer funsction:

$$H_z(z) = \frac{d_m}{c_n} \frac{\prod_{\mu=1}^{m} (z - z_{0\mu})}{\prod_{\nu=1}^{n} (z - z_{\infty\nu})} = \frac{d_m}{c_n} \frac{\prod_{\mu=1}^{m_1} (z - z^{(1)}{}_{0\mu})}{\prod_{\nu=1}^{n} (z - z_{\infty\nu})} \prod_{\mu=m_1+1}^{m} (z - z^{(2)}{}_{0\mu})$$

with $\left| z^{(1)}{}_{0\mu} \right| \leq 1$ for $\mu = 1, \ldots, m_1$ the first m_1 zeros in the unit-circle

$\left| z^{(2)}{}_{0\mu} \right| > 1$ for $\mu = m_1 + 1, \ldots, m$ other zeros outside the unit-circle

4.2.7 Minimum-phase Systems

With $\prod_{\mu=m_1+1}^{m} (z^{(2)}{}_{0\mu}.z-1) = \prod_{\mu=m_1+1}^{m} z^{(2)}{}_{0\mu}(z-\frac{1}{z^{(2)}{}_{0\mu}})$, it yields:

$$H_z(z) = \underbrace{\frac{b_m}{c_n} \frac{\prod_{\mu=1}^{m_1}(z-z^{(1)}{}_{0\mu}) \prod_{\mu=m_1+1}^{m}(z^{(2)}{}_{0\mu}.z-1)}{\prod_{\nu=1}^{n}(z-z_{\infty\nu})}}_{H_{zM}(z)} \cdot \underbrace{\frac{\prod_{\mu=m_1+1}^{m}(z-z^{(2)}{}_{0\mu})}{\prod_{\mu=m_1+1}^{m} z^{(2)}{}_{0\mu}(z-\frac{1}{z^{(2)}{}_{0\mu}})}}_{H_{z_{Allp.}}}$$

Or in a short form:

$$H_z(z) = H_{ZM}(z).H_{z_{Allp.}}(z)$$

4.2.7 Minimum-phase Systems

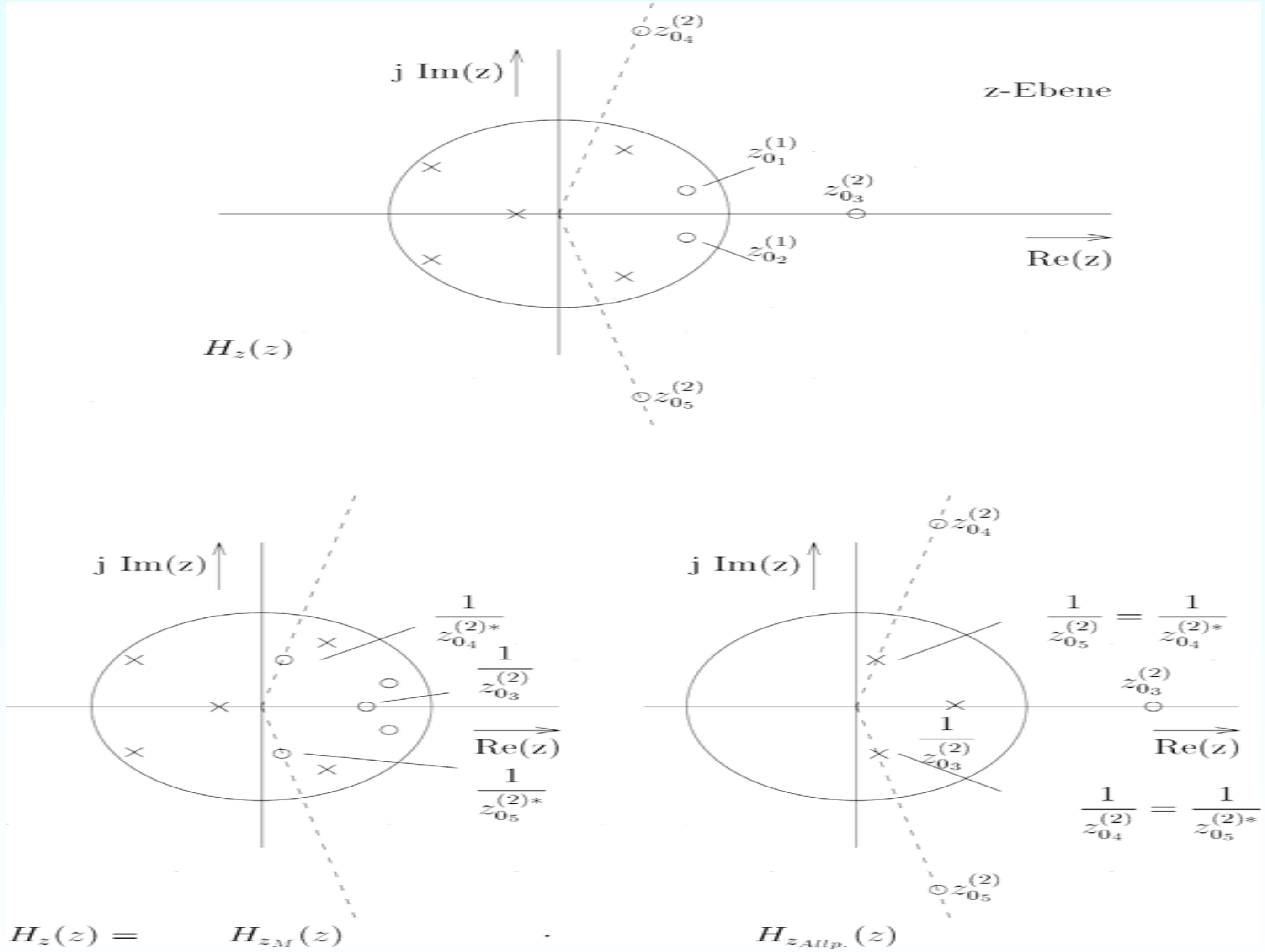
With $z_{0\mu}{}^{(2)} = \frac{1}{z_{\infty\mu}}$, one obtains for the all-pass:

$$H_{z_{Allp.}}(z) = \underbrace{\frac{1}{\prod_{\mu=m_1+1}^{m} z_{0\mu}{}^{(2)}}}_{\text{real constant}} \cdot \frac{\prod_{\mu=m_1+1}^{m} \left(z - \frac{1}{z_{\infty\mu}}\right)}{\prod_{\mu=m_1+1}^{m} (z - z_{\infty\mu})}$$

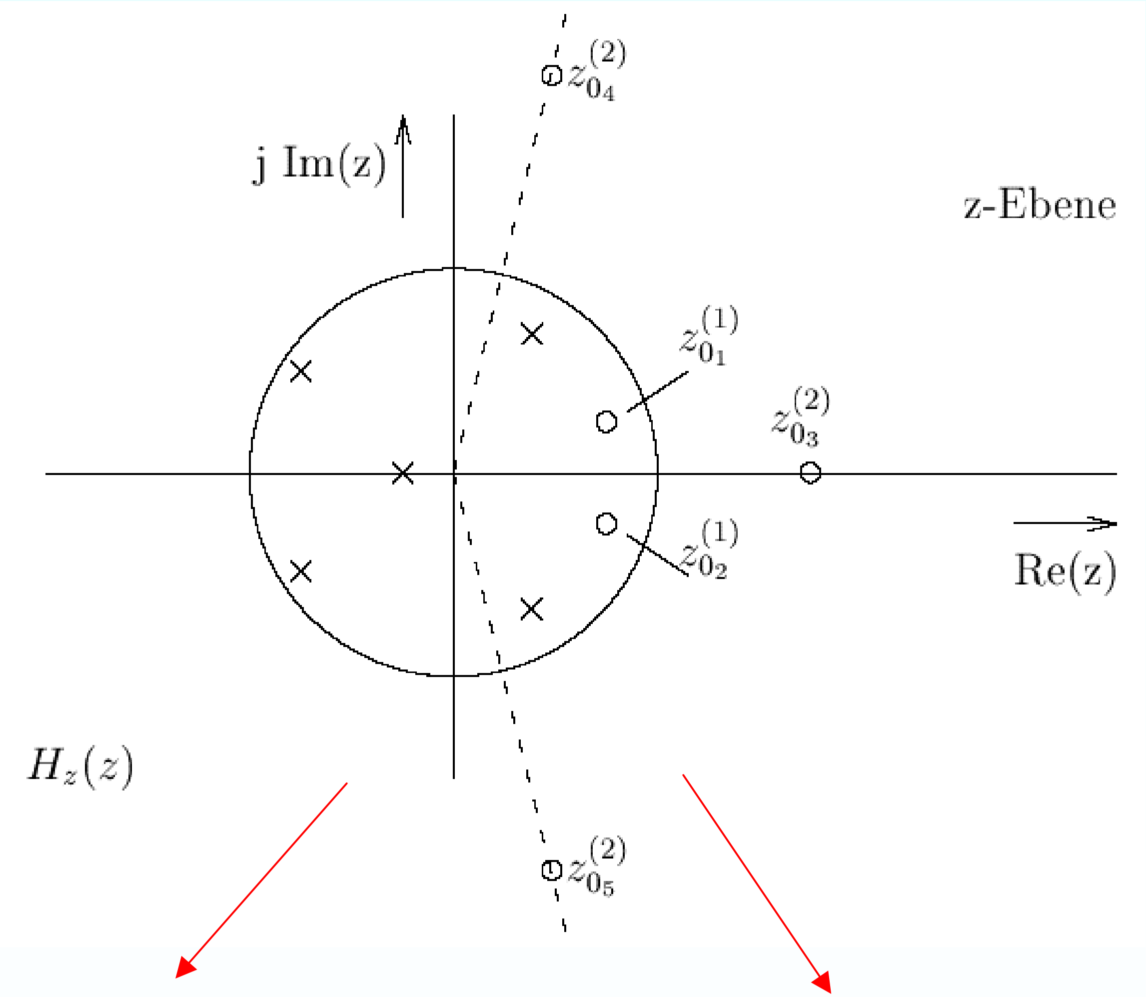
Because $\left|z_{0\mu}{}^{(2)}\right| > 1 \quad \Leftrightarrow \quad \left|z_{\infty\mu}\right| = \frac{1}{\left|z_{0\mu}{}^{(2)}\right|} < 1$

The poles of this all-pass lie inside the unit-circle

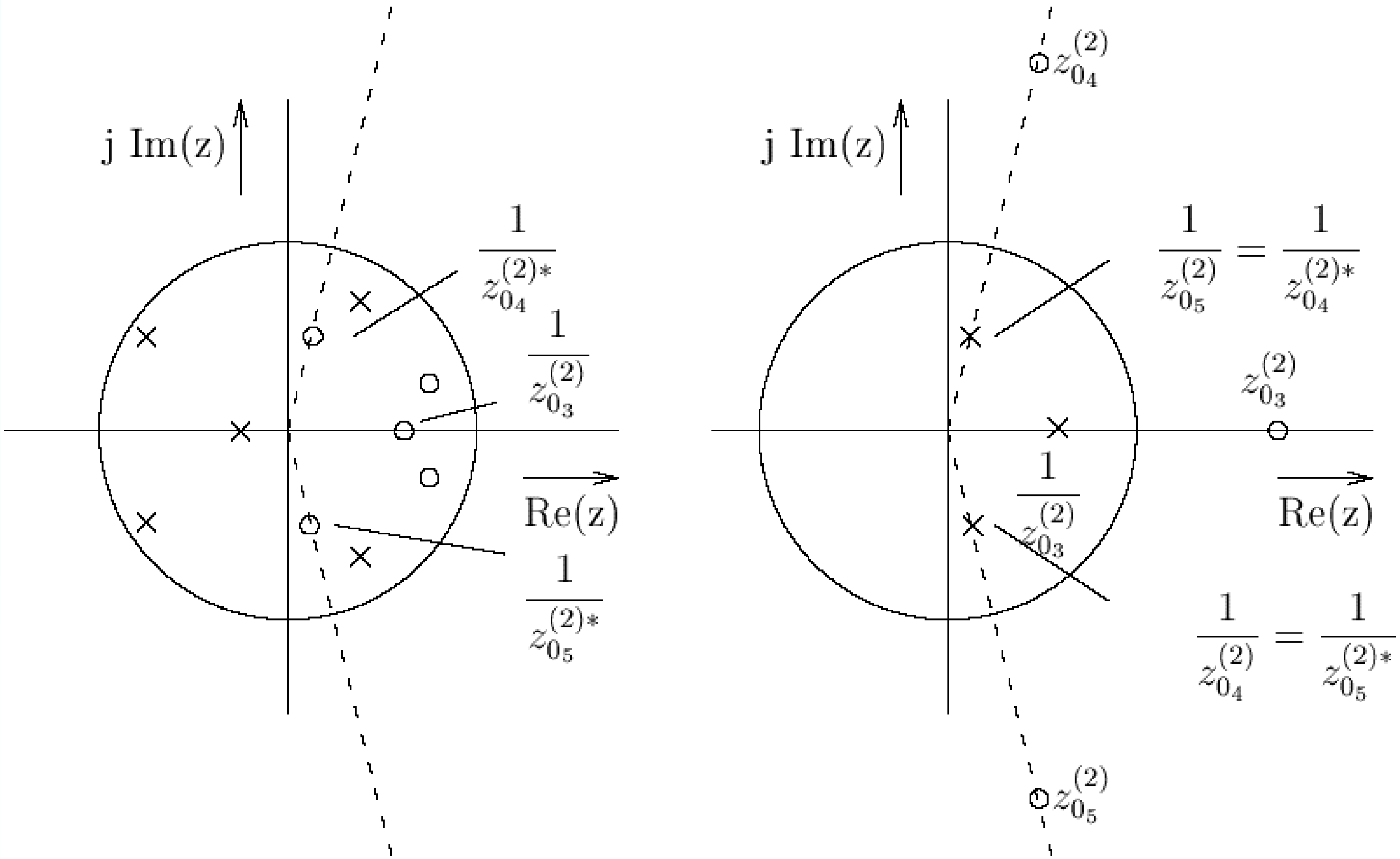
4.2.7 Minimum-phase Systems



4.2.7 Minimum-phase Systems



4.2.7 Minimum-phase Systems



$$H_z(z) = H_{z_M}(z) \cdot H_{z_{Allp.}}(z)$$

Minimum phase system **All pass system**

4.2.8 Systems with Linear Phase

Linear phase means a constant envelope delay with the frequency

$$\tau_{Nga}(\Omega) \neq f(\Omega)$$

where:

$$\tau_{Nga}(\Omega) = \sum_{\nu=1}^{n} \frac{1-\left|z_{\infty\nu}\right|\cos(\Omega-\Psi_{\infty\nu})}{1-2\left|z_{\infty\nu}\right|\cos(\Omega-\Psi_{\infty\nu})+\left|z_{\infty\nu}\right|^2} - \sum_{\mu=1}^{m} \frac{1-\left|z_{0\mu}\right|\cos(\Omega-\Psi_{0\mu})}{1-2\left|z_{0\mu}\right|\cos(\Omega-\Psi_{0\mu})+\left|z_{0\mu}\right|^2}$$

The condition is fulfilled in cases:

1. $n = m$ and $z_{0\mu} = z_{\infty\nu}$ for all $\nu = \mu \Rightarrow$ poles and zeros exhibit the same location

2. $\left|z_{0\mu}\right| = 1 \quad \forall \mu$ and $\left|z_{\infty\nu}\right| = 1 \quad \forall \nu \Rightarrow$ leads to only conditional stable systems and therefore is ignored in the following:

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4.2.8 Systems with Linear Phase

A) $|z_{\infty\nu}|<1 \;\; \forall \nu$ must be fulfilled, $|z_{\infty\nu}|=0 \;\; \forall \nu$ ensures that a frequncy independent contribution from the poles is introduced in the formula for the group delay.

B) The zeros $z_{0\mu}$ have to be located in such a way, that:

$$\sum_{\mu=1}^{m} \frac{1-|z_{0\mu}|\cos(\Omega-\Psi_{0\mu})}{1-2|z_{0\mu}|\cos(\Omega-\Psi_{0\mu})+|z_{0\mu}|^2} \;\; \rightarrow \;\; const.$$

To fulfill B) condition, it is required:

1) $$H_z(z)=\frac{b_m}{c_n}\frac{\prod_{\mu=1}^{m}(z-z_{0\mu})}{\prod_{\nu=1}^{n}(z-z_{\infty\nu})}=\frac{b_m}{c_n}\frac{1}{z^n}\prod_{\mu=1}^{m}(z-z_{0\mu}) \Rightarrow$$ non-recursive system.

4.2.8 Systems with Linear Phase

2) The zeros $z_{0\mu}$ have to be located pair-wise symmetrically to each other:

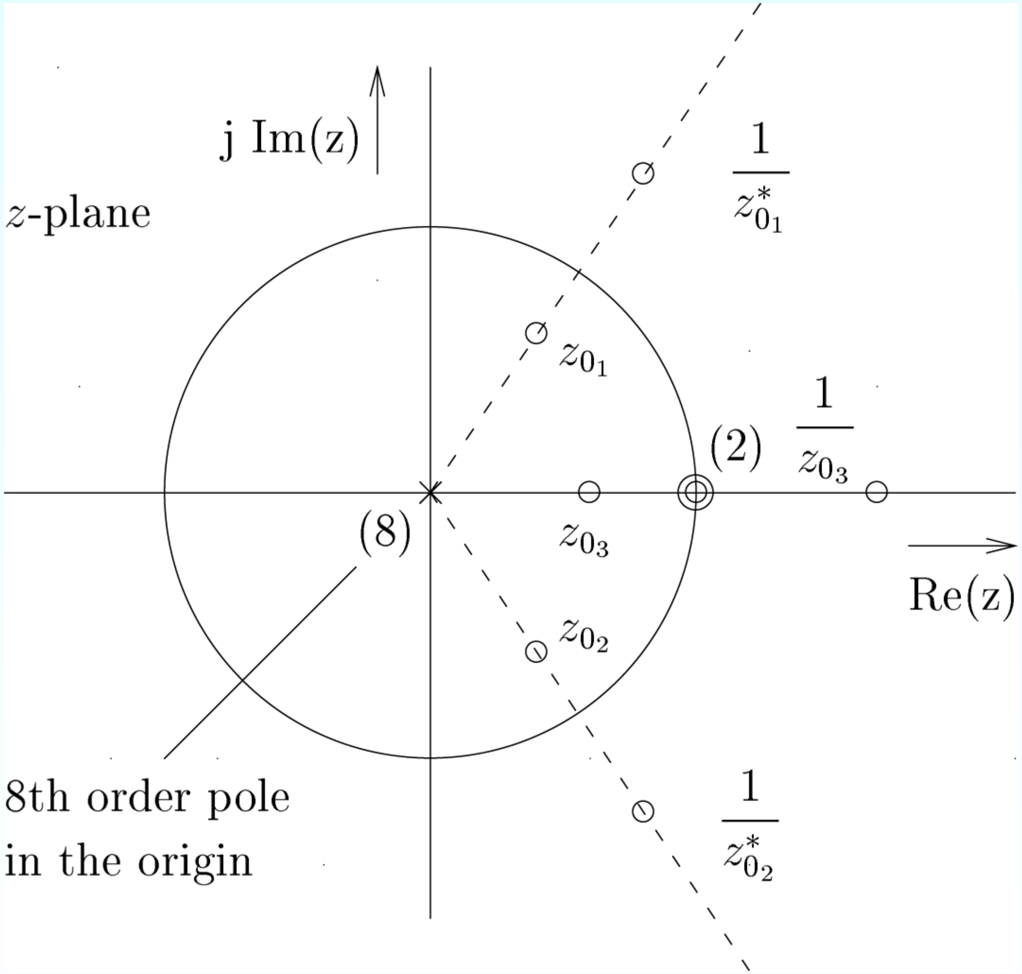
$$z_{0_a} = \left|z_{0_a}\right| e^{j\Psi_{0_a}} = \frac{1}{z_{0_b}^{\;*}} = \left|\frac{1}{z_{0_b}}\right| e^{j\Psi_{0_b}}$$

Due to this relation, the total group delay gives:

$$\frac{1-\left|z_{0_a}\right|\cos(\Omega-\Psi_{0_a})}{1-2\left|z_{0_a}\right|\cos(\Omega-\Psi_{0_a})+\left|z_{0_a}\right|^2}+\frac{1-\left|z_{0_b}\right|\cos(\Omega-\Psi_{0_b})}{1-2\left|z_{0_b}\right|\cos(\Omega-\Psi_{0_b})+\left|z_{0_b}\right|^2}$$

$$=\frac{1-\left|z_{0_a}\right|\cos(\Omega-\Psi_{0_a})}{1-2\left|z_{0_a}\right|\cos(\Omega-\Psi_{0_a})+\left|z_{0_a}\right|^2}+\frac{1-\frac{1}{\left|z_{0_a}\right|}\cos(\Omega-\Psi_{0_a})}{1-\frac{2}{\left|z_{0_a}\right|}\cos(\Omega-\Psi_{0_{\lambda_1}})+\frac{1}{\left|z_{0_a}\right|^2}}=1$$

4.2.8 Systems with Linear Phase



Pole-Zero diagram of Non-Recursive System with Linear Phase

4.2.8 Systems with Linear Phase

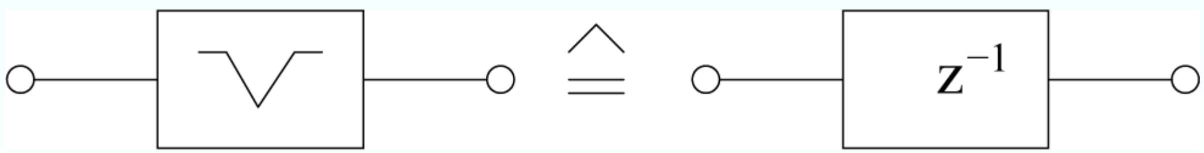
Therefore discrete systems with linear phase have always a transfer function :

$$H_z(z) = \frac{b_m}{c_n}\frac{1}{z^n}(z-1)^{m_3}(z+1)^{m_2}\prod_{\mu=1}^{m_1}(z-z_{0\mu})(z-\frac{1}{z_{0\mu}^{*}})$$

where $2m_1 + m_2 + m_3 = m \le n$

Example: for $n = 1 \Rightarrow m = 0$, i.e., $m_1 = m_2 = m_3 = 0$

and $H_Z(z) = \frac{b_m}{c_n} z^{-1}$ ●—○ $h(k) = \frac{b_m}{c_n}\gamma_0(k-1)$



Delay element with $\frac{b_m}{c_n} = 1$

4.2.9 Non-Recursive Systems (FIR-Filters)

Non-recursive systems have been defined by:

$$g(k) = \frac{1}{b_0} \sum_{\alpha=\varepsilon}^{n} a_\alpha s(k-\alpha) \quad \circ\!\!-\!\!\!-\!\!\bullet \quad G(z) = \frac{1}{b_0} \sum_{\alpha=\varepsilon}^{n} a_\alpha S(z) z^{-\alpha}$$

or

$$H_z(z) = \frac{G(z)}{S(z)} = \frac{1}{b_0} \sum_{\alpha=0}^{n} a_\alpha z^{-\alpha}$$

Properties:

1. Because of $H_z(z) = \sum_{\nu=0}^{m} h(\nu) z^{-\nu}$, the impulse response:

$$h(k) = \frac{a_k}{b_0} \quad \text{for} \quad k = 0 \ldots n \quad \text{with} \quad h(k) = 0 \quad \text{for} \quad k < 0 \quad \text{and} \quad k > n$$

→ finite duration ⇒ also called FIR (Finite Impulse Response) - systems.

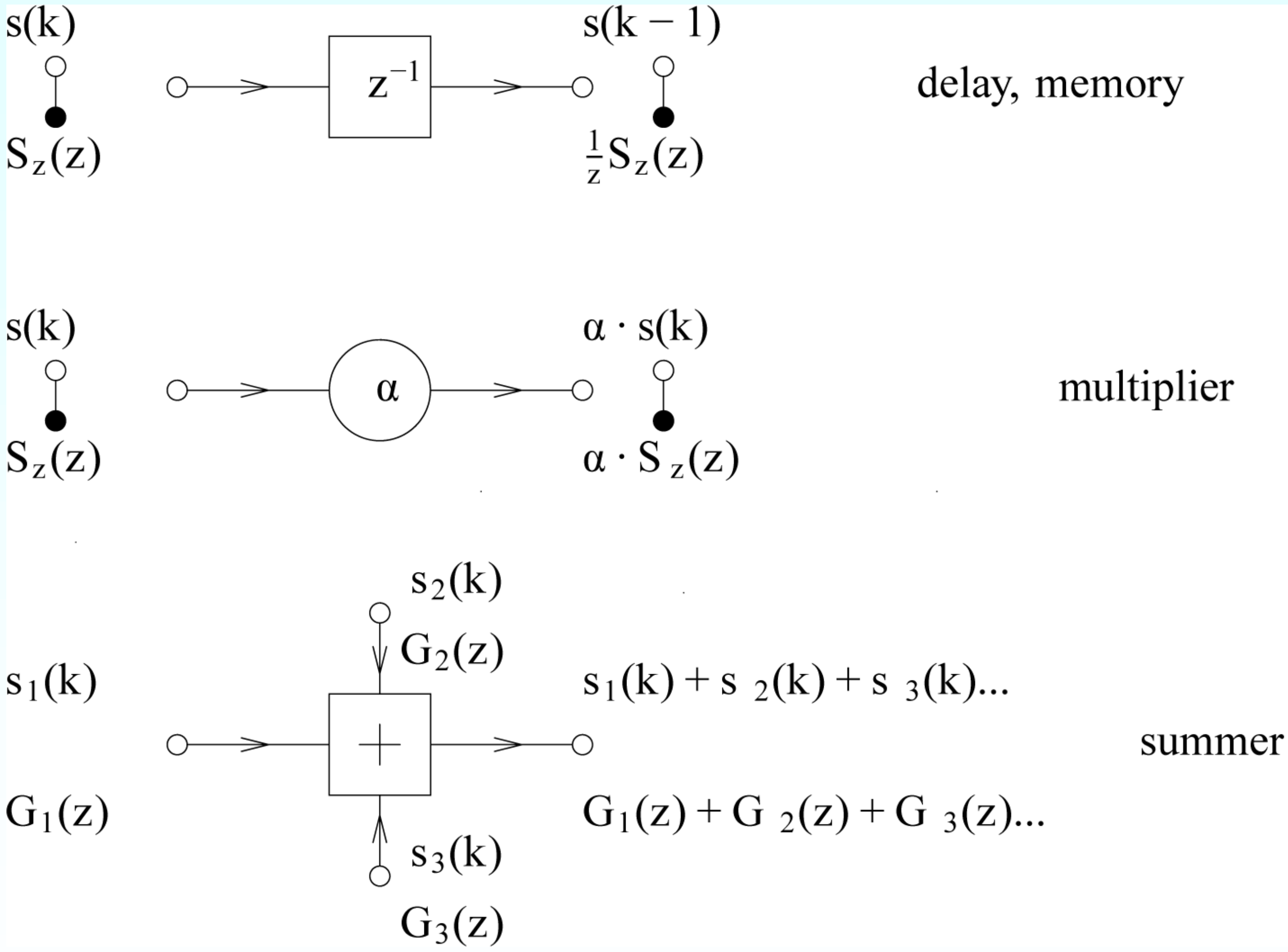
With the output signal: $g(k) = s(k) * h(k) = \sum_{\nu=0}^{n} h(\nu) s(k-\nu)$

4.2.9 Non-Recursive Systems (FIR-Filters)

2. From this results: $H_z(z) = \frac{1}{b_0} \sum_{\alpha=0}^{n} a_\alpha z^{-\alpha} = \frac{1}{b_0} \sum_{\alpha=0}^{n} \frac{a_\alpha z^{n-\alpha}}{z^n}$

non-recursive systems have just an nth order pole at z = 0 → always stable!

4.3 System Structures for Discrete LTI-Systems



4.3.1 The First Canonical Form of a Discrete System

A canonical form is a system structure with a minimized number of memories (delay elements).

From chapter 4.2.1: $g(k) = \frac{1}{b_0}\left[\sum_{\alpha=0}^{n} a_\alpha s(k-\alpha) - \sum_{\beta=1}^{n} b_\beta g(k-\beta)\right]$

by setting m = n one obtains:

$$g(k) = \frac{1}{b_0}\left[\sum_{\alpha=0}^{n} a_\alpha s(k-\alpha) - \sum_{\beta=1}^{n} b_\beta g(k-\beta)\right]$$

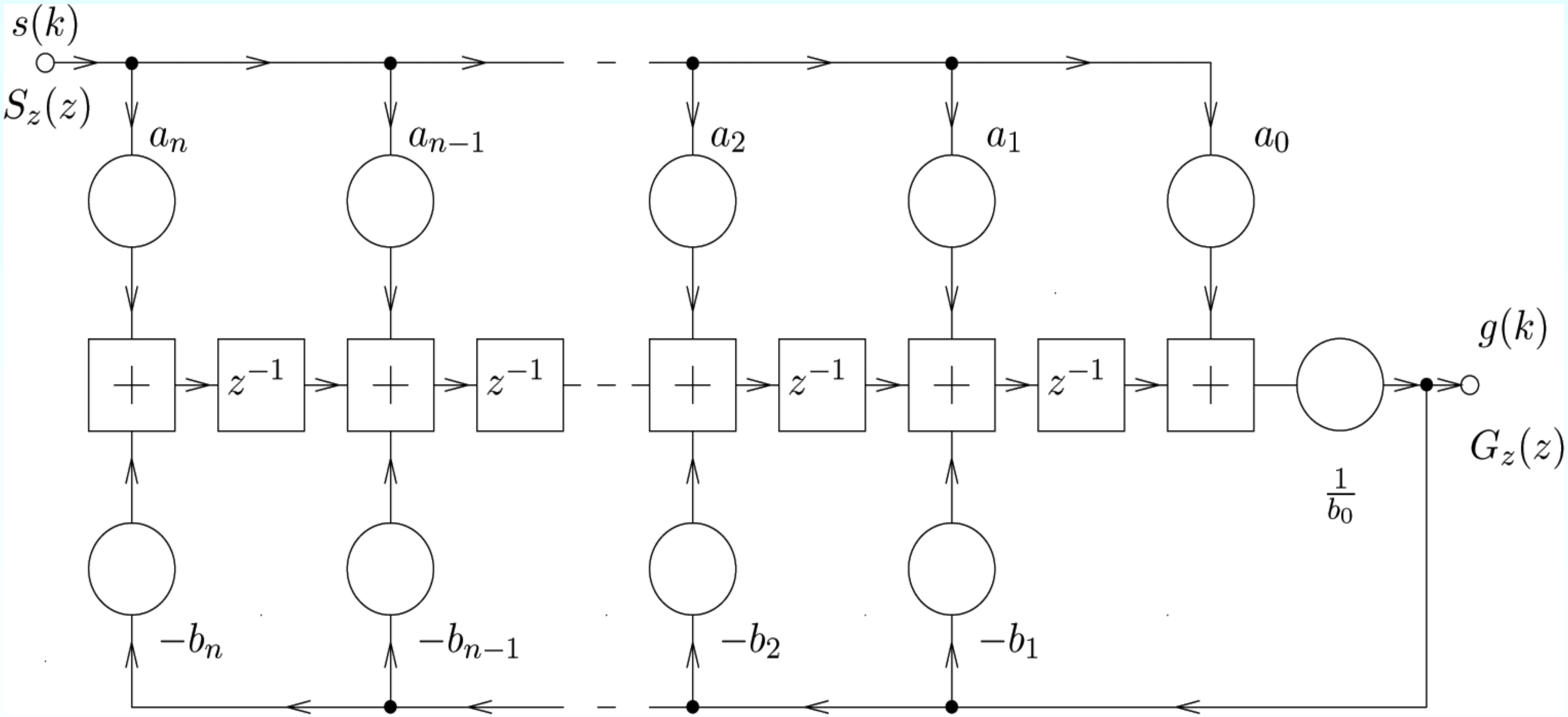
or

$$g(k) = \frac{a_0}{b_0} s(k) + \frac{1}{b_0}\left[\sum_{\gamma=1}^{n} \left\{ a_\gamma s(k-\gamma) - b_\gamma g(k-\gamma) \right\}\right]$$

$$\circ\!\!-\!\!\bullet$$

$$G_z(z) = \frac{a_0}{b_0} S_z(z) + \frac{1}{b_0}\left[\sum_{\gamma=1}^{n} \left\{ a_\gamma S_z(z) z^{-\gamma} - b_\gamma G_z(z) z^{-\gamma} \right\}\right]$$

4.3.1 The First Canonical Form of a Discrete System



First Canonical form of a digital filter

4.3.2 The Second Canonical Form of a Discrete Filter

A second canonical form results as follows:

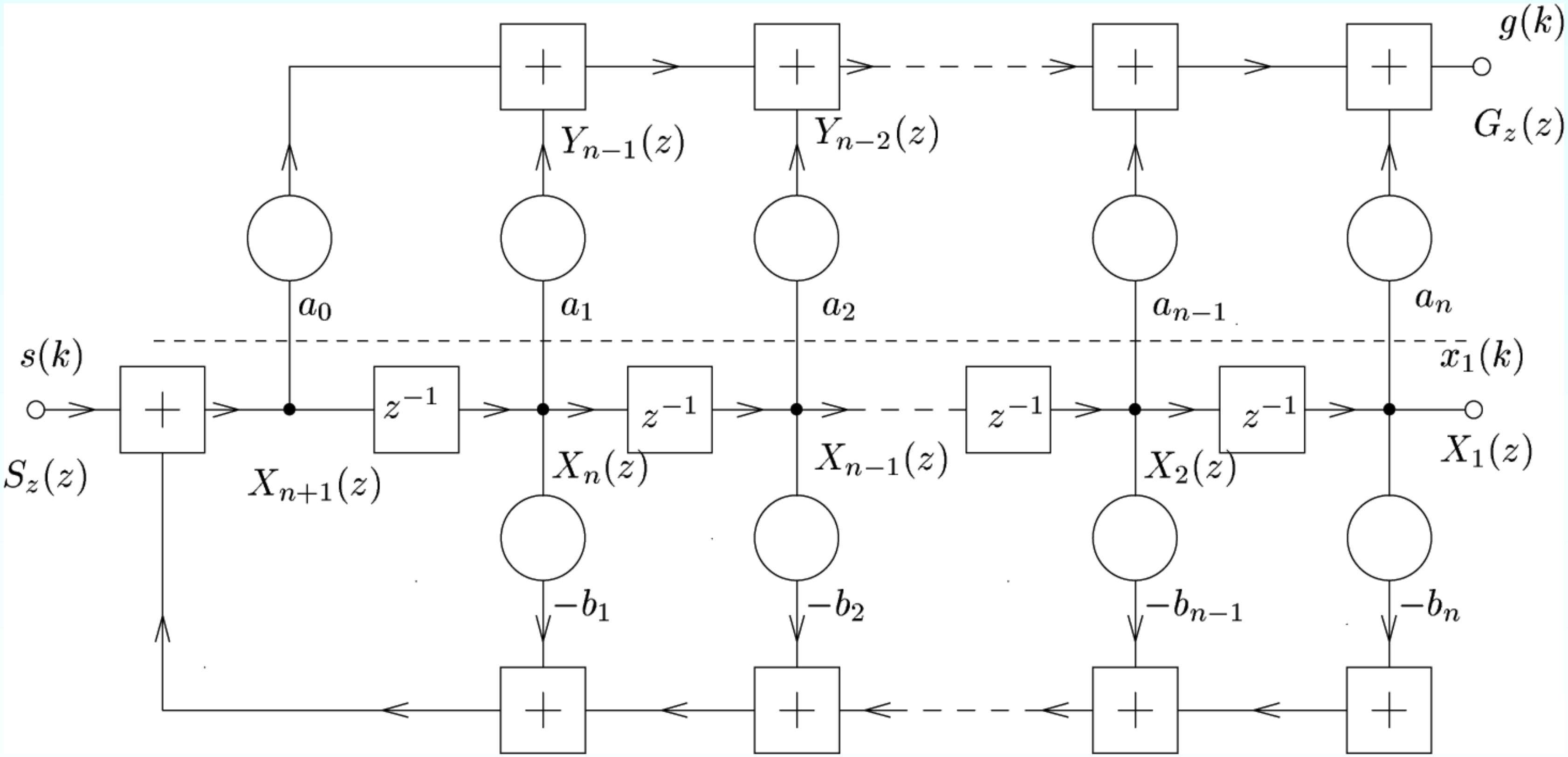
$$G_z(z) = \frac{\sum_{\mu=0}^{n} d_\mu z^\mu}{\sum_{\nu=0}^{n} c_\nu z^\nu} S_z(z)$$

The second form works equal to the first canonical form; this is proved in the following:

- The representation in following figure with

$$H_z(z) = \frac{G_z(z)}{S_z(z)} = \frac{\sum_{\mu=0}^{n} b_\mu z^\mu}{\sum_{\nu=0}^{n} c_\nu z^\nu}$$ equals the one in figure of the 1st Form

4.3.2 The Second Canonical Form of a Discrete Filter



The Second Canonical Form of a Digital Filter

4.3.2 The Second Canonical Form of a Discrete Filter

First, one takes a look at the part underneath the dashed line → this system part can obviously described by:

$$s(k) \rightarrow x_1(k) \quad \circ\!\!-\!\!\bullet \quad S_z(z) \rightarrow X_1(z)$$

Furthermore:

$$X_1(z) = z^{-n} X_{n+1}(z) \xrightarrow{\text{in general}} X_{\nu+1}(z) = z^{\nu} X_1(z)$$

Due to $x_{n+1}(k) = s(k) - \sum_{\nu=1}^{n} b_\nu x_{n-\nu+1}(k),$ we obtains:

$$X_{n+1}(z) = S_z(z) - \sum_{\nu=1}^{n} b_\nu X_{n-\nu+1}(z) \quad \Leftrightarrow \quad z^n X_1(z) = S_z(z) - \sum_{\nu=1}^{n} b_\nu X_{n-\nu+1}(z)$$

$$= S_z(z) - \sum_{\nu=1}^{n} b_\nu z^{n-\nu} X_1(z)$$

4.3.2 The Second Canonical Form of a Discrete Filter

$$\Leftrightarrow \quad X_1(z)\left[z^n + z^n \sum_{\nu=1}^{n-1} b_\nu z^{-\nu}\right] = S_z(z)$$

and with $b_0 = 1$ we obtain: $X_1(z) = \dfrac{S_z(z)}{z^n \sum_{\nu=0}^{n} b_\nu z^{-\nu}}$

For the upper part of the system the difference equation $g(k) = \sum_{\nu=0}^{n} a_\nu x_{n-\nu+1}(k)$ results:

$$G_z(z) = \sum_{\nu=0}^{n} a_\nu X_{n-\nu+1}(z) = \sum_{\nu=0}^{n} a_\nu z^{n-\nu} X_1(z)$$

$$= X_1(z) z^n \sum_{\nu=0}^{n} a_\nu z^{-\nu} = \frac{z^n \sum_{\nu=0}^{n} a_\nu z^{n-\nu}}{z^n \sum_{\nu=0}^{n} b_\nu z^{-\nu}} S_z(z)$$

4.3.2 The Second Canonical Form of a Discrete Filter

or:

$$G_z(z) = \frac{\sum_{\nu=0}^{n} a_\nu z^{-\nu}}{\sum_{\nu=0}^{n} b_\nu z^{-\nu}} S_z(z)$$

So the whole system is described by:

$$H_z(z) = \frac{G_z(z)}{S_z(z)} = \frac{\sum_{\nu=0}^{n} a_\nu z^{-\nu}}{\sum_{\nu=0}^{n} b_\nu z^{-\nu}}$$

4.3.3 The Third Canonical Form of a Digital System

3rd Form: cascade of the 1st and 2nd order system.

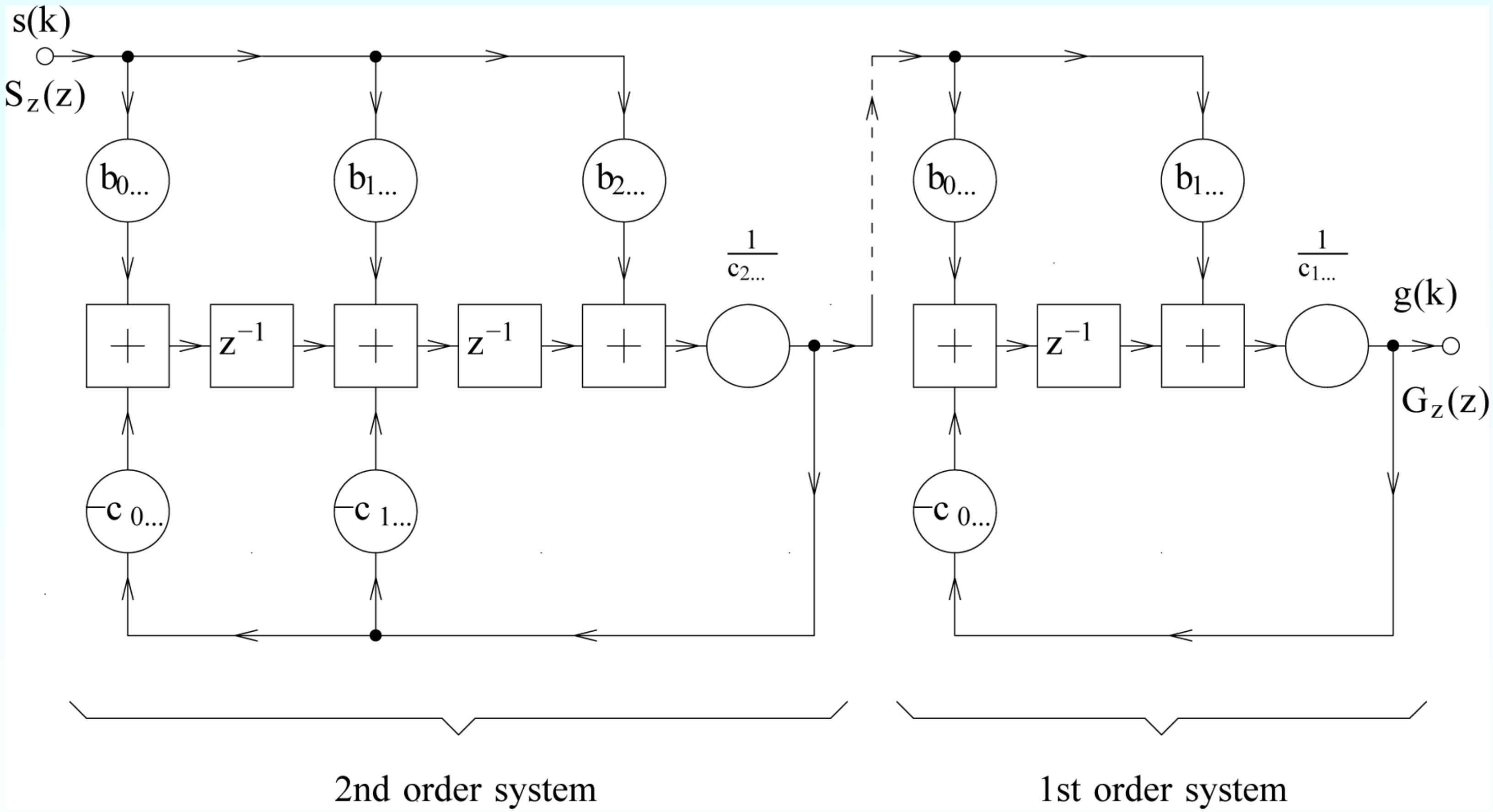
One can divide from:

$$H_z(z) = \frac{d_m}{c_n} \frac{\prod_{\mu=1}^{m}(z - z_{0\mu})}{\prod_{\nu=0}^{n}(z - z_{\infty\nu})} = \prod_{\gamma=1}^{\cdots} H_{z\gamma}(z)$$

into

$$H_{z\gamma}(z) = \frac{d_{1_\gamma} z + d_{0_\gamma}}{c_{1_\gamma} z + c_{0_\gamma}} \quad \text{and} \quad H_{z\gamma}(z) = \frac{d_{2_\gamma} z^2 + d_{1_\gamma} z + d_{0_\gamma}}{c_{2_\gamma} z^2 + c_{1_\gamma} z + c_{0_\gamma}}$$

4.3.4 The Fourth Canonical form of a Digital Filter



Cascade of a 1st and 2nd Order Digital Filter

4.3.4 The Fourth Canonical form of a Digital Filter

Another canonical structure can be obtained by converting the transfer function into a partional sum:

$$H_z(z) = \frac{\sum_{\mu=0}^{m} d_\mu z^\mu}{c_n \prod_{\nu=1}^{n} (z - z_{\infty\nu})} = R_\infty + \sum_{\nu=1}^{n} \frac{R_\nu}{(z - z_{\infty\nu})}$$

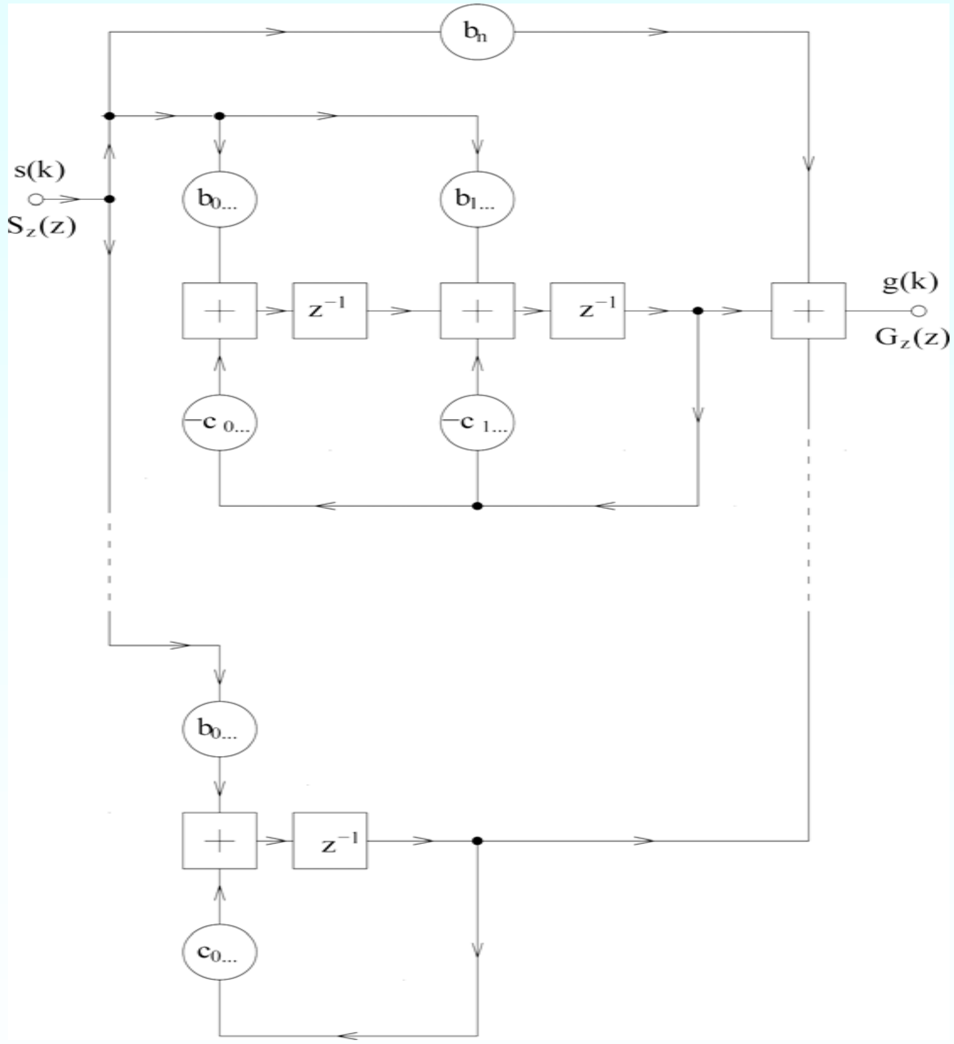
The residues in this simple case follow from (where $c_n = 1$):

$$R_\infty = \lim_{z\to\infty} H_z(z) = d_n \quad \text{for} \quad m = n$$

$$R_\nu = \lim_{z\to z_{\infty\nu}} \left\{(z - z_{\infty\nu}) H_z(z)\right\} \quad \text{for single poles}$$

➡ **This leads to a parallel connection of the parts.**

4.3.4 The Fourth Canonical form of a Digital Filter



4.3.4 The Fourth Canonical form of a Digital Filter

For real poles: $H_{z\gamma}(z) = \dfrac{d_{0_\gamma}}{z + c_{0_\gamma}}$ where $d_{0_\gamma} = R_\gamma$ and $c_{0_\gamma} = -z_\infty$

For conjugate complex poles, two terms must be combined:

$$H_{z\gamma}(z) = \frac{b_{1_\gamma} z + b_{0_\gamma}}{z^2 + c_{1_\gamma} z + c_{0_\gamma}} \quad \text{where}$$

$$d_{0_\gamma} = -2\,\mathrm{Re}\left\{R_\gamma z^*_{\infty\gamma}\right\},\ d_{1_\gamma} = 2\,\mathrm{Re}\left\{R_\gamma\right\},\ c_{0_\gamma} = \left|z_{\infty_\gamma}\right|^2 \text{ and } c_{1_\gamma} = -2\,\mathrm{Re}\left\{z_{\infty_\gamma}\right\}$$

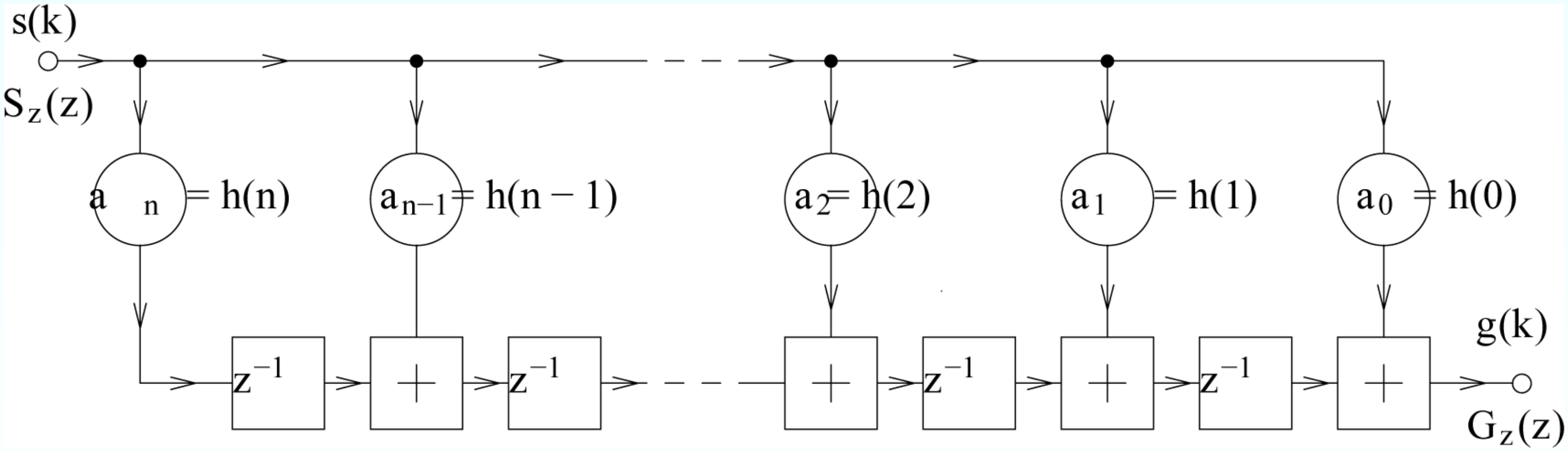
So in completion:

$$H_z(z) = d_n + \sum_{\gamma=1}^{\dots} H_{z\gamma}(z) \quad \text{where} \quad c_n = 1$$

4.3.5 System Structures for Non-Recursive Systems (FIR-Filters)

For any of the first three canonical forms, an appropriate non-recursive system can be directly derived from equation

$$g(k) = \frac{1}{b_0} \sum_{\alpha=0}^{n} a_\alpha s(k-\alpha)$$



First Canonical Form of a FIR - Filter