Experimental Studies “Signals and Systems 1”			
Experiment No. 1:	Linear, Time-Invariant Systems		
Department:	Nachrichtentechnik		
Name:		Matr.-Nr.:	
Supervisor:		Date:	

The preparatory exercises must be solved prior to the date of experiment

Contents

1 Introduction.

The transfer behavior of linear time-invariant systems (LTI-systems) is completely determined with a single time-function (for example the pulse response $h(t)$) or alternatively with the corresponding Fourier-transform $H(\omega) \stackrel{\mathcal{F}}{\bullet\!\!-\!\!\circ} h(t)$ in the frequency domain. It is not necessary to know anything about the more or less complicated electronic circuits of the system to determine the system response for arbitrary input signals $s(t)$.

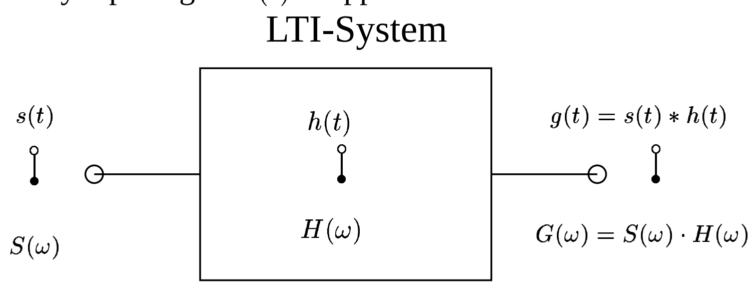
The output signal is calculated from the input signal $s(t)$ and the pulse response $h(t)$ with the convolution integral:

$$g(t) \;=\; s(t) * h(t) = \int_{-\infty}^{\infty} s(\tau) \cdot h(t-\tau)\, d\tau \tag{1}$$

With $G(\omega) \stackrel{\mathcal{F}}{\bullet\!\!-\!\!\circ} g(t)$, $S(\omega) \stackrel{\mathcal{F}}{\bullet\!\!-\!\!\circ} s(t)$, $H(\omega) \stackrel{\mathcal{F}}{\bullet\!\!-\!\!\circ} h(t)$, the Fourier-transforms of the corresponding time-functions $g(t)$, $s(t)$, $h(t)$ the same relation is described in the frequency domain as follows:

$$G(\omega) \;=\; S(\omega) \cdot H(\omega) \tag{2}$$

These fundamental relations are summarized in the following figure. It is important to note that these relations are valid only on the condition that the system is linear, time-invariant and that the system is in the "zero state" before any input signal $s(t)$ is applied.



While dealing with LTI-systems we often use idealizations which do not correspond at all or only approximately with reality. (For example: we use Dirac-pulses as input signals, rectangular transfer-functions (ideal low-pass or bandpass filters), we are dealing with non-causal systems, we often ignore the strong relation between real- and imaginary part of the transfer function which establishes causality of the system, we assume unlimited linearity ranges of the systems under consideration, etc.).

Anyhow, these idealizations render a good understanding of the overall system behavior with comparatively simple calculations even in case of more complicated systems, composed of several blocks of different subsystems.

With theoretical considerations and experiments in this practices we will show - at least in some aspects - additional conditions to be met in experiments with realizable systems. Moreover we will show that System-Theory is a suitable tool for experimental laboratory work in communication engineering.

The system under test is a "black box" of which nothing else than the input and output terminals are known. With suitably designed experiments we will classify the system and determine the transfer function of the system.

The Exercises in chapter 2 must be solved prior to the date of the experiments to achieve and prove an adequate preparation for the practices. The results of these exercises must be registered properly in this document.

2 Explanations, preliminary tasks and exercises.

2.1 Theoretical Calculations

The theoretical test of linearity and time-invariance of a given system is achieved with the following schemes:

2.1.1 Test of Linearity:

1. trial: $s_1(t) \rightarrow g_1(t) = T\{s_1(t)\}$ (3)

2. trial: $s_2(t) \rightarrow g_2(t) = T\{s_2(t)\}$ (4)

3. trial: $s_3(t) = as_1(t) + bs_2(t) \rightarrow g_{3_{actual}}(t) = T\{s_3(t)\}$ (5)

   Condition for a system to be linear:

   $g_{3_{required}}(t) = ag_1(t) + bg_2(t)$ (6)

$T\{s_i(t)\}$ describes the transform of the input signal $s_i(t)$ to the output signal $g_i(t)$ performed by the system. The arrow always indicates the input $\rightarrow$ output transfer of a system.
The actual response $g_{3_{actual}}(t) = T\{s_3(t)\}$ of the system to the input signal $s_3(t)$ is given in the 3rd trial. This must be compared with the required response of a LTI-system $g_{3_{required}}(t)$.
To achieve this comparison it is necessary to recalculate $g_{3_{Ist}}(t)$ as well as $g_{3_{required}}(t)$ to obtain comparable functions which is possible if we use the given equations in the above scheme in conjunction with some adequate substitutions.

2.1.2 Test of Time-Invariance:

1. trial: $s_1(t) \rightarrow g_1(t) = T\{s_1(t)\}$ (7)

2. trial: $s_2(t) = s_1(t - t_0) \rightarrow g_{2_{Ist}}(t) = T\{s_2(t)\}$ (8)

   Condition for a time-invariant system:

   $g_{2_{required}}(t) = g_1(t - t_0)$ (9)

Using the equations in the scheme above and adequate substitutions we can recalculate $g_{2_{actual}}(t)$ and $g_{2_{required}}(t)$ in terms of s_1. Hence we can decide whether the actual and required responses are equal or not.

Exercise 2.1:

Test whether the system $s(t) \rightarrow g(t) = s(a \cdot t)$ is time invariant or not!

1. trial: $$s_1(t) \rightarrow g_1(t) = s_1(a \cdot t) \tag{10}$$
2. trial: $$s_2(t) = s_1(t - t_0) \rightarrow g_{2_{actual}}(t) = s_2(a \cdot t) = \tag{11}$$

   Condition for the time-invariance of the system:

   $$g_{2_{required}}(t) = g_1(t - t_0) = \tag{12}$$

In this scheme $s_2(a \cdot t)$ as well as $g_1(t - t_0)$ can be rewritten in terms of $s_1(..)$ if the equations above are used with adequate substitutions.

Complete the missing terms in the above scheme!

Is the system $s(t) \rightarrow g(t) = s(a \cdot t)$ time-invariant?

In practice experiments according to the above schemes would require much effort and they are possible only with some restrictions.

If we test for example the linearity of a system according to the above scheme we carry out 3 trials at the same time which is impossible in practice with a single system. Only if we can assume the time invariance of the system we could as well carry out these experiments sequentially.

Each $s_i(t)$ represents in the schemes above any function of a certain class. In practical experiments we can only carry out each trial with a determined function.

t_0 represents any positive or negative time shift; in practice t_0 is restricted to comparatively short positive time shifts.

These are some reasons to search for a more simple linearity test to be used in practical laboratory work even if this test is not so rigurous. The alternative test is based on a special feature of LTI-systems explained in the following subsection.

2.1.3 Response of LTI-systems on sinusoidal inputs:

LTI - systems **always** respond to sinusoidal inputs with sinusoidal output signals with the same frequency. Usually the output amplitude is different from the input amplitude and the output signal shows a different phase angle. Both the amplitude and phase angle of the output signal are dependent on the frequency.

$$s(t) = A \cdot \cos(\omega_0 t) \rightarrow g(t) = A \cdot |H(\omega_0)| \cdot \cos(\omega_0 t + \varphi(\omega_0)) \tag{13}$$

This relation - which describes the behavior of an LTI-system in the time domain - is dependent on the terms $|H(\omega_0)|$ and $\varphi(\omega_0)$ which are characteristics of the transfer function
$H(\omega_0) = |H(\omega_0)| \cdot e^{j\varphi(\omega_0)}$ and defined in the frequency domain. The index at ω_0 simply indicates a fixed frequency.

Hence it is possible to determine all parameters of the transfer function $H(\omega_0)$ at a certain frequency with measurements in the time domain.

Exercise 2.2:

Proof the validity of the transfer feature of LTI-systems with sinusoidal inputs as given above. Start your calculation with the attempt below:

$$g(t) = A \cdot \cos(\omega_0 t) * h(t) = A \cdot \frac{e^{j\omega_0 t} + e^{-j\omega_0 t}}{2} * h(t) \tag{14}$$

Carry out the convolution and simplify the corresponding integrals using the definition of the Fourier-Transform:

$$H(\omega_0) = \int_{-\infty}^{\infty} h(\tau) \cdot e^{-j\omega_0 \tau} \, d\tau \tag{15}$$

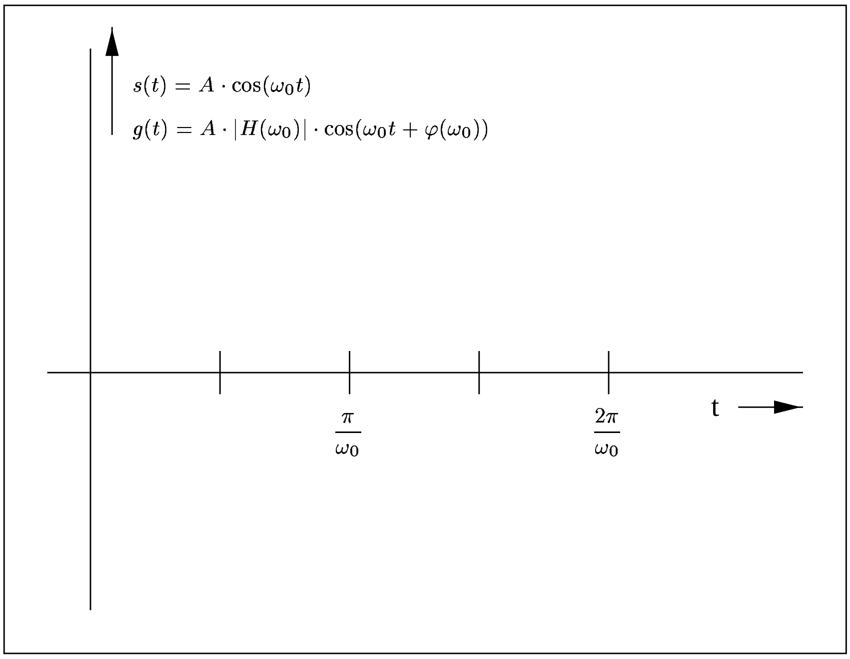
and the **features of real systems**: $H(-\omega) = H^*(\omega) = |H(\omega)| \cdot e^{-j\varphi(\omega)}$.

$$g(t) = \tag{16}$$

Exercise 2.3:

Describe a method to determine the amplitude characteristic and phase angle (or real- and imaginary part) of the transfer function at a certain frequency from a double oscillogram, where input- and output-sinusoidal signals are displayed simultaneously.

Use the following diagram, plot at least one period of the input- and output-signals according to eqn. 13 and complete the abscissa and ordinate scales with all significant parameters.



Exercise 2.4:

Determine the Fourier Transforms of the input- and output signals given below.

$t_{ph} = -\dfrac{\varphi(\omega_0)}{\omega_0}$ is the phase delay time of a sinusoidal signal of fixed frequency ω_0.

$$s(t) = A \cdot \cos(\omega_0 t) \overset{\mathcal{F}}{\circ\!\!-\!\!\bullet} S(\omega) = \tag{17}$$

$$g_{LZI}(t) = A \cdot |H\omega_0)| \cdot \cos(\omega_0 t + \varphi(\omega_0)) = A \cdot |H(\omega_0)| \cdot cos\left(\omega_0\left[t + \underbrace{\frac{\varphi(\omega_0)}{\omega_0}}_{-t_{ph}}\right]\right) \tag{18}$$

$$\mathcal{F} \updownarrow$$

$$G_{LZI}(\omega) = \tag{19}$$

2.1.4 Response of Nonlinear Time-Invariant Systems on sinusoidal input signals.

Nonlinear Time-Invariant systems (NTI-Systems) respond to sinusoidal inputs $s(t) = A \cdot \cos(\omega_0 t)$ with periodical, but nonlinear distorted output signals $g_{NZI}(t)$. (Examples: Amplitude clipping, rectification etc.).

The most suitable series expansion for periodic signals is the Fourier-Series-Expansion with the following representation:

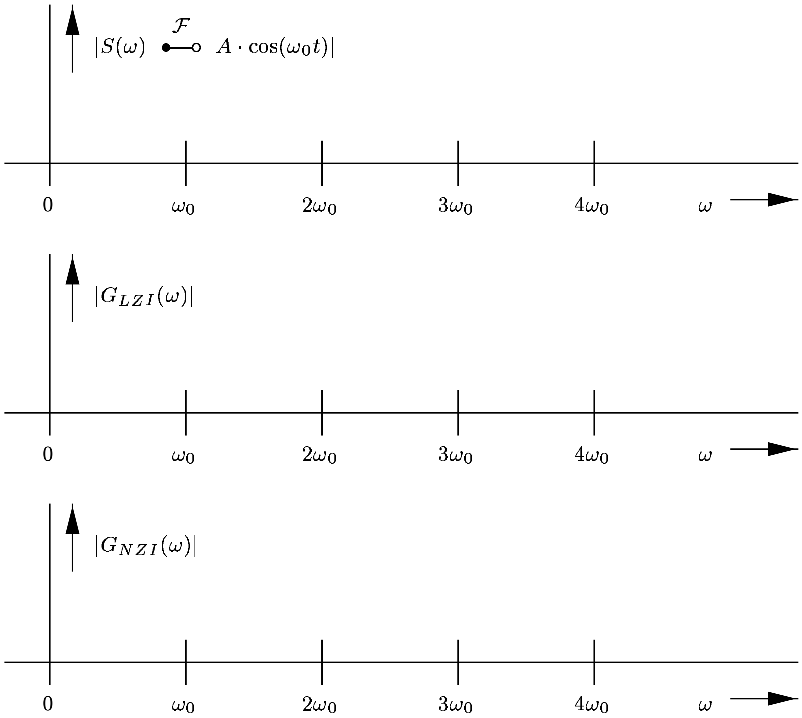
$$g_{NZI}(t) = \sum_{k=0}^{\infty} a_k \cdot \cos(k \cdot \omega_0 t) + b_k \cdot \sin(k \cdot \omega_0 t) = \sum_{k=0}^{\infty} c_k \cdot \cos(k \cdot \omega_0 t + \varphi_k) \quad (20)$$

Exercise 2.5:

Determine the Fourier Transform $G_{NZI}(\omega)$ of the time function $g_{NZI}(t)$,

$$G_{NZI}(\omega) = \quad (21)$$

and plot the corresponding amplitude spectra using the following diagrams !



LTI-systems respond to spectral clean inputs (with only 1 frequency component) with spectral clean outputs at the same frequency. From the calculations above and the 3rd plot we conclude that Nonlinear-Time-Invariant systems respond to spectral clean inputs with Line Spectra of equidistant spectral components at frequencies $k \cdot \omega_0$. The degree of nonlinearity is characterized with the "Distortion factor" (see definition in the lectures).

In general $G_{NZI}(\omega)$ shows a spectral component at $\omega = 0$.

Which signal component is characterized by this spectral line?

Realizable systems are never linear and time-invariant in a strong sense because the dynamic operating range is always limited and aging of electronic components and environmental influences (temperature etc.) lead to slowly varying transfer characteristics in time.
In practice we need to know the linear input-/output dynamic ranges and we accept extremely slow varying time variances.

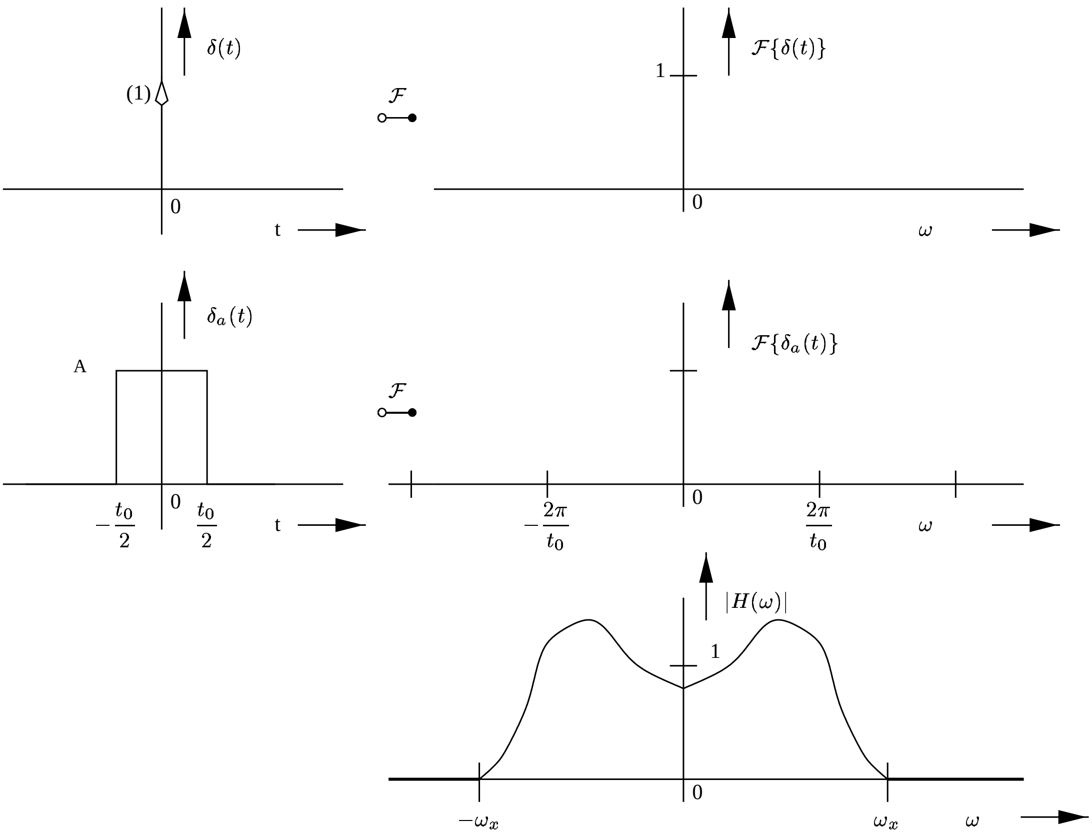
Recall that the transfer characteristic of LTI-Systems are completely determined with the pulse response $h(t)$ or the corresponding Fourier-Transform $H(\omega) \overset{\mathcal{F}}{\bullet\!\!-\!\!\circ} h(t)$ if the system is in the zero state before an input signal is applied.

2.2 Considerations concerning measurements with LTI-Systems.

2.2.1 The approximated Dirac-Pulse

From theory we know the response of an LTI-System to the input signal (pulse) $\delta(t)$ is the pulse response $h(t)$!
It is impossible to generate a Dirac-Pulse in a laboratory because it is infinitely small and infinitely high.
Since realizable systems are always limited in their linear dynamic ranges we only can use pulses of limited amplitudes and finite duration. A sufficiently good description of a pulse we approximately can realize in our laboratory is $\delta_a(t) = A \cdot \mathrm{rect}(\frac{t}{t_0})$ although we will never be in the position to generate rise- and fall times of zero duration.

Exercise 2.6:
Plot the spectra of an ideal pulse $\delta(t)$ and the approximated pulse $\delta_a(t) = A \cdot \mathrm{rect}(\frac{t}{t_0})$ with all characteristic parameters using the following diagrams.
In the lower right diagram a fancy amplitude characteristic $|H(\omega)|$ is plotted. We need to know that realizable transfer systems always show a band limited amplitude characteristic, e.g. independent of the specific shape of the transfer function there is always an upper cut off frequency limit $-\omega_x \leq \omega \leq \omega_x$ such that $|H(\omega)| \approx 0$.



Independent of the specific shape of the transfer function in the range $-\omega_x \leq \omega \leq \omega_x$ we would conclude that substitution of the ideal Dirac-pulse with the proposed approximation would yield errors in the pulse response.
With $G(\omega) = S(\omega) \cdot H(\omega)$ we have $S(\omega) = \mathcal{F}\{\delta(t)\} = 1 \Rightarrow G(\omega) = 1 \cdot H(\omega)$ in the ideal case, but with our approximation $S(\omega) = \mathcal{F}\{\delta_a(t)\}$ obviously the corresponding product
$\mathcal{F}\{\delta_a(t)\} \cdot H(\omega) \neq H(\omega)$.

Exercise 2.7:

Determine the required inequality between the width t_0 and ω_x which establishes a “sufficiently” constant spectrum of the approximated “Dirac-pulse” within the transfer range $-\omega_x \leq \omega \leq \omega_x$ of the given system.

Hint: It is not necessary here to caculate a distinct deviation from the required constant within the range $-\omega_x \leq \omega \leq \omega_x$.
As long as the inequality holds sufficiently good the system response $h_a(t)$ on $\delta_a(t)$ corresponds in its shape with the ideal pulse response $h(t)$ except of a vertical scale factor.

Exercise 2.8:

Determine the required vertical scale factor which rescales $h_a(t)$ to $h(t)$!

Hint: Deduce this scale factor from the vertical scales of the ideal- and the approximated spectrum.

$$h(t) \quad = \quad h_a(t)\cdot \tag{22}$$

2.2.2 Relation between rise time and cut-off-frequency of an ideal Low-pass-System

The rise time of a Low-pass-System is defined as the rise time of its step response: $A\cdot\varepsilon(t) \rightarrow A\cdot w(t)$

The most common procedures used in practice for rise time measurements are either:

- the elapsed time within which the step response rises from 10% to 90% of the steady state output value. or alternatively
- (if we plot a tangent to the steepest point of ascent of the step response) the elapsed time between the zero state crossing and the steady state crossing of this tangent.

Exercise 2.9:

Calculate the rise time t_e of an ideal Low-Pass filter with transfer function $H(\omega) = H(0)\cdot \text{rect}(\frac{\omega}{2\omega_g})$ in terms of its cut-off frequency f_g!

Hint: Calculate in a general form in terms of $h(.)$, **without** using the special transform $h(t) \overset{\mathcal{F}}{\circ\!\!-\!\!\bullet} H(\omega) = H(0)\cdot \text{rect}(\frac{\omega}{2\omega_g})$, the step-response $A\cdot w(t) = A\cdot\varepsilon(t) * h(t)$.

$$A\cdot w(t) \quad = \quad A\cdot\varepsilon(t) * h(t) = A\cdot\int_{-\infty}^{\infty}\varepsilon(t-\tau)\cdot h(\tau)\,d\tau = \tag{23}$$

Determine from the following formula:

$$A\cdot w(\infty) \quad = \quad A\cdot\int_{-\infty}^{\infty} h(\tau)\,d\tau = A\cdot\int_{-\infty}^{\infty} h(\tau)\cdot\underbrace{e^{j\omega\tau}\big|_{\omega=0}}_{1}\,d\tau = A\cdot H(0) \tag{24}$$

$H(0)$ such, that the steady state value of the step-response reaches the same height as the input step height.

$$H(0) \quad = \tag{25}$$

Calculate from the general form of the step-response its ascent $A\cdot\dot{w}(t)$ and determine the maximum ascent $A\cdot\dot{w}(t)|_{max}$ using the specific $h(t) = \overset{\mathcal{F}}{\circ\!\!-\!\!\bullet} H(\omega) = H(0)\cdot \text{rect}(\frac{\omega}{2\omega_g})$.

Determine from the straight line equation of the tangent with maximum ascent $A\cdot\dot{w}(t)|_{max}$ the rise time t_e as defined above.

$$\left.\frac{\partial w(t)}{\partial t}\right|_{max} \quad = \tag{26}$$

$$t_e \quad = \tag{27}$$

This relation has been determined using an idealized input signal $\varepsilon(t)$ and an idealized transmission model (ideal Low-Pass"-filter), which is of course not realizable because it is a non-causal system. Anyhow we can use the result as an approximate $\approx$ relation which serves to estimate the bandwidth of a real Low-Pass filter from the measured rise-time and vice versa.

In measuring the rise time we must make sure in the laboratory that the rise time of the step function, produced with a real pulse generator as well as the self rise time of of the used oscilloscope are much shorter than the measured rise time of the system under test!

3 Experiments:

The system at disposal is a "black box" with access only to the input/output terminals. Hence we know nothing about the internal electronic circuits of the system, which indeed could be built with various completely different electronic components and circuitry to achieve the same transfer function.

3.1 Time-Invariance and Linearity tests.

Experiment 3.1:
Test whether the system is time-invariant using different time limited input signals!

Experiment 3.2:
Check the linearity range of the system using a sinusoid input signal!

Hint: From theory we know that a linear system always responds to a sinusoid input with a sinusoid output. If we increase the input amplitude the output amplitude increases accordingly as long as the system is in the linearity range. Outside this range we will observe nonlinear distortions of the output signal as long as the frequency of the input $\sin$-function is sufficiently low.

The latter condition is due to the fact, that nonlinear distortion is connected with line spectra of the output signal (see example spectra on page 7). At least two of the spectral components must be inside the transfer range of the system, which is more likely to occur with a low frequency of the sinusoidal input signal.

$$\leq \widehat{u_{in}} \leq$$

3.2 Measurement of the Transfer Function.

Experiment 3.3:

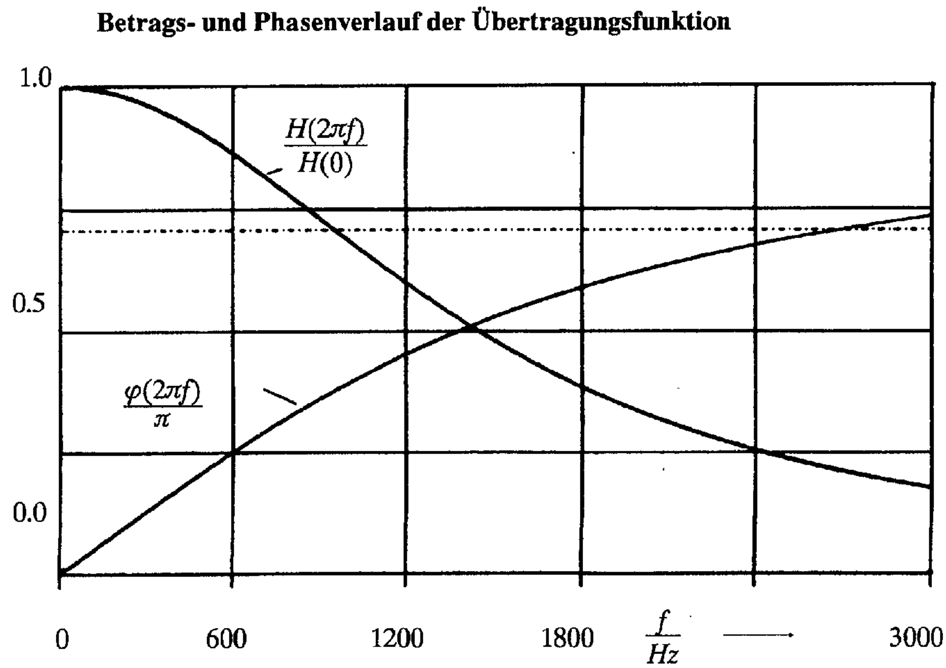
Determine at some frequencies given in the table below the phase delay time t_{ph}, the phase angle φ and $|H(2\pi f)|$!

$\frac{f}{Hz}$	0	600	1200	1800	2400
$t_{ph}(2\pi f)$	0				
$\varphi(2\pi f)$	0				
$\frac{\lvert H(2\pi f)\rvert}{H(0)}$	1				

Experiment 3.4:

Compare the above measured results with the corresponding values in the plot below which shows the amplitude characteristic and the normalized phase angle.

This plot has been generated automatically using a "Transfer-Function measurement system" equipped with a "Sweep Generator", "Phase-measurement device", "RMS-Voltage measurement device" and plotter.



3.3 Determination of the Bandwidth from the Pulse Response.

Experiment 3.5:

Use a periodic rectangular pulse sequence as input signal of the system and adjust the pulse width such, that the system response approaches steady state values and adjust the gap between subsequent pulses such, that the system returns to the zero state.

With an adjustment like that we can determine the step response of a system using a periodic rectangular pulse sequence!

Reason why?!

(Redesign the periodic rectangular pulse sequence using superimposed (time delayed) step input functions and determine the corresponding response in conjunction with the linearity- and and time invariance feature of the system.)

Experiment 3.6:

Determine from the measured rise time of the step response approximately the bandwidth of the system using the formula which has been developed in exercise 2.9.

$f_g \approx$

and compare the result with the figure on page 12.

Do you consider this estimate to be sufficiently good and is this method to estimate the bandwidth of a Low-pass system more simple in comparison with the measurement method used in experiment 3.3?